Towards High Quality Gradient Estimation on Regular Lattices

Zahid Hossain
zha13@sfu.ca
Usman Raza Alim
ualim@cs.sfu.ca
Dr. Torsten Möller
torsten@cs.sfu.ca

Taylor Series Framework

- Convolution of discrete samples at scaled lattice sites, i.e., $f(\ell Lk)$ with a continuous filter $w$ can be written as:
  \[ f^\ell(x) = \sum_k f(\ell Lk) \cdot w\left(x - \frac{\ell Lk}{\ell}\right) \]
- Using Taylor Series we can expand $f(\ell Lk)$ about $x$ in the above equation to yield
  \[ f^\ell(x) = \sum_m D^m f(x) \cdot \omega^m(x) \]
  where
  \[ D^m(\cdot) := \frac{\partial^m y(x)}{\partial x_1^n \partial x_2^n \cdots \partial x_m^n}(\cdot) \]
  and
  \[ \omega^m(x) := \frac{(x - \ell Lk)^m}{m! \ell^n} \cdot \frac{1}{(x - \ell Lk)^m} \]
- This is a multidimensional analogue of the 1D analysis of Möller et al., (1997).

For a given discrete derivative filter $\Delta$, we derived a system of linear equations which is given by

\[ \omega^m = \frac{1}{m!} \sum_{m} \frac{L_m}{L_m} \Delta (L_m - m), \quad m, n \in \mathbb{N}^3 \]

Here $\Delta (L_m - m)$ are the unknown filter weights. Size of the set $m$, i.e., the support of the discrete filter, decides the number of unknowns while the size of the set $n$ decides the number of equations.

Therefore to compute the first derivative along $x$ for a 4th order polynomial we impose the following conditions:

\[ \omega^0 = 0, m, n \in \mathbb{N}^3 : |m|_1 \leq 4 \] and \[ n \neq [1, 0, 0] \]

\[ \omega^1 = 1, m = [1, 0, 0] \]

However, linear system formed above usually has an infinite solution space.

- We define an error metric $E$ as follows and seek to minimize $E$ within the infinite solution space to yield a discrete filter.

\[ E = \sum_m \omega^m \]

Orthogonal Projection Framework

Approximation Space

\[ V_2(\psi) := \left\{ \psi(x) = \sum_{k \in \mathbb{Z}^2} c_k \psi_k(x) : c_k \in \ell_2(\mathbb{Z}^2) \right\}, \quad \text{where} \quad \psi_k(x) := \psi(x - Lk) \]

Space spanned by the shifts of a generating function $\psi$ on a lattice characterized by $L$.

\[ P_{V_2(\psi)}(f) = \sum_{k \in \mathbb{Z}^2} (f \cdot \psi_k) \psi_k \]

gives the minimum error approximation of $f$.

Two Stage Gradient Approximation

1. Approximate $f$ in an auxiliary approximation space $V_2(\psi)$
   \[ f(x) \approx f_1(x) = \sum_{k} (f(x + p_1 k)) \psi_k(x) \]
   $p_1$ ensures that the interpolation constraint is satisfied.

2. Orthogonally project $f_1(x)$ onto the target space $V_2(\psi)$
   \[ f_2(x) := P_{V_2(\psi)}(f_1(x)) = \sum_{k} (f_1(x + p_1 k)) \psi_k(x) \]

where the discrete derivative filter $\hat{d}_i$ is given by

\[ \hat{d}_i \psi_k := (\partial_i \psi_k \psi_k) \]