Towards High Quality Gradient Estimation on Regular Lattices

SIMON FRASER UNIVERSITY
THINKING OF THE WORLD

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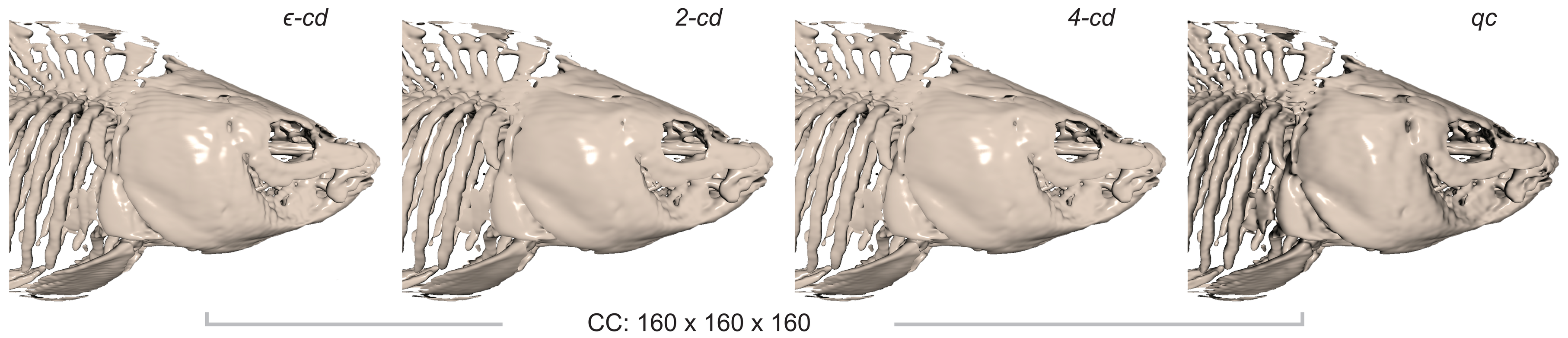
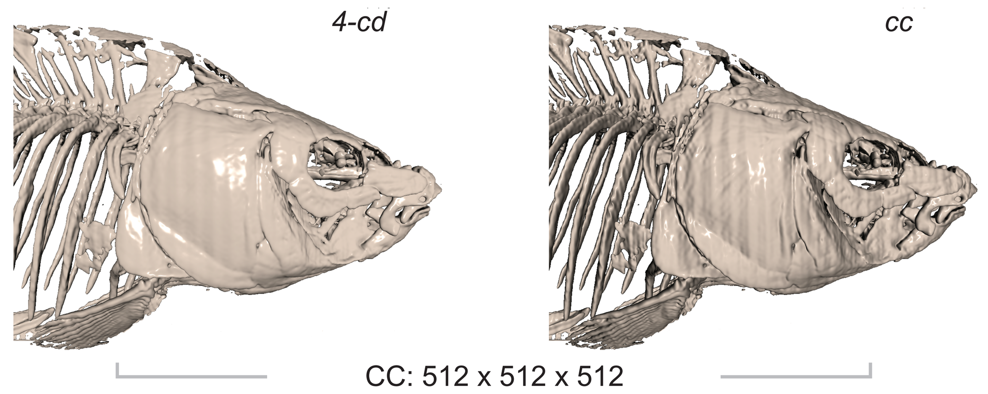
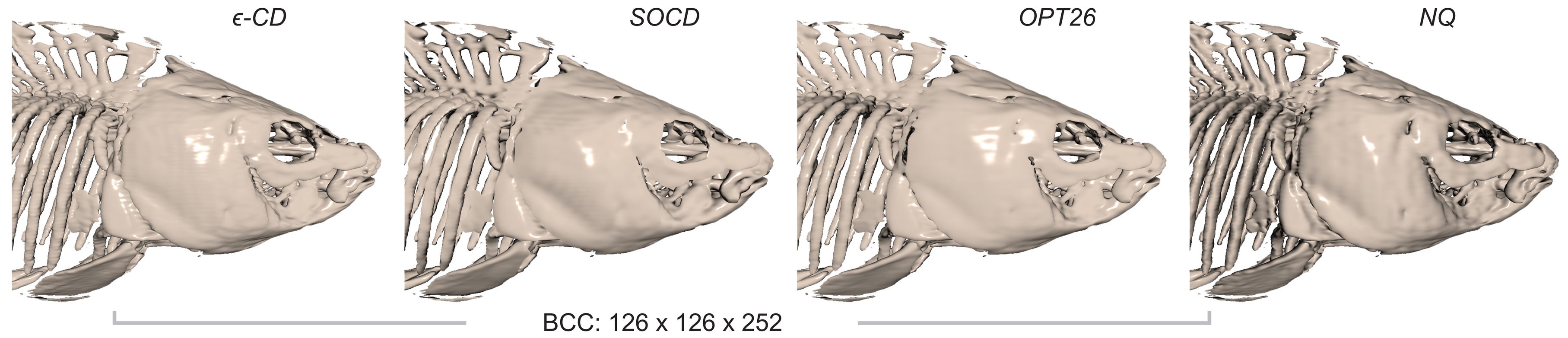






Taylor Series Framework

- Convolution of discrete samples at scaled lattice sites, i.e. $f(h\boldsymbol{Lk})$ with a continuous filter w can be written as:

$$f_r^w(\boldsymbol{x}) = \sum_{\boldsymbol{k}} f(h\boldsymbol{Lk}) \cdot w(\frac{\boldsymbol{x} - h\boldsymbol{Lk}}{h})$$

- Using Taylor Series we can expand $f(h\boldsymbol{Lk})$ about $\boldsymbol{x}$ in the above equation to yield

$$f_r^w(\boldsymbol{x}) = \sum_{\boldsymbol{n}} D^{\boldsymbol{n}} f(\boldsymbol{x}) \cdot a_{\boldsymbol{n}}^w(\boldsymbol{x})$$

where

$$D^{\boldsymbol{n}}(\cdot) := \left(\frac{\partial^{n_1}}{\partial {x_1}^{n_1}} \frac{\partial^{n_2}}{\partial {x_2}^{n_2}} \cdots \frac{\partial^{n_d}}{\partial {x_d}^{n_d}} \right)(\cdot)$$

and

$$a_{\boldsymbol{n}}^w(\boldsymbol{x}) := \sum_{\boldsymbol{k}} \frac{(h\boldsymbol{Lk} - \boldsymbol{x})^{\boldsymbol{n}}}{\boldsymbol{n}!} \cdot w(\frac{\boldsymbol{x} - h\boldsymbol{Lk}}{h})$$

- This is a multidimensional analogue of the 1D analysis of Möller et al. (1997).

- For a given discrete derivative filter Δ, we derived a system of linear equations which is given by

$$a_{\boldsymbol{n}}^{\Delta} = \frac{h^{\boldsymbol{n}}}{\boldsymbol{n}!} \sum_{\boldsymbol{m}} (\boldsymbol{Lm})^{\boldsymbol{n}} \cdot \Delta\left(\boldsymbol{L}\left(-\boldsymbol{m}\right)\right), \qquad \boldsymbol{m}, \boldsymbol{n} \in \mathbb{N}^3$$

Here $\Delta\left(\boldsymbol{L}\left(-\boldsymbol{m}\right)\right)$ are the unknown filter weights. Size of the set $\boldsymbol{m}$, i.e. the support of the discrete filter, decides the number of unknowns while the size of the set $\boldsymbol{n}$ decides the number of equations.

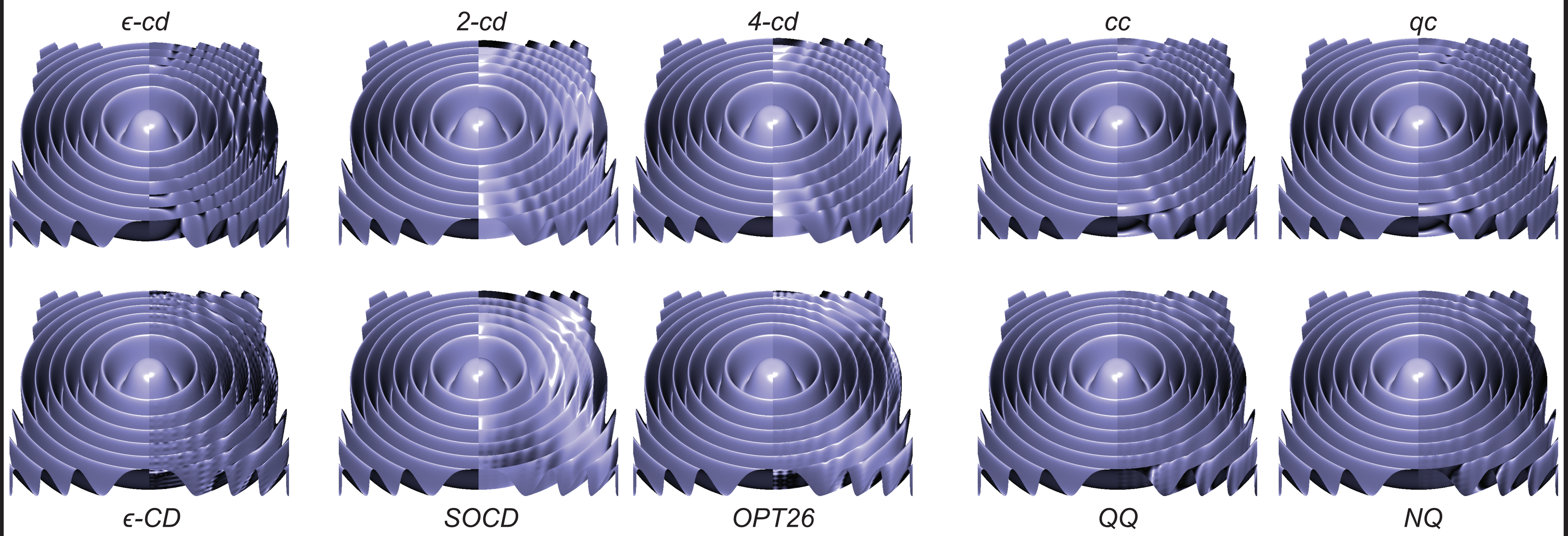
- Therefore to compute the first derivative along x for a 4th order polynomial we impose the following conditions

$$a_{\boldsymbol{n}}^{\Delta} = 0, \boldsymbol{n} \in \mathbb{N}^3 : \|\boldsymbol{n}\|_1 \leq 4 \text{ and } \boldsymbol{n} \neq [1,0,0]$$
$$a_{\boldsymbol{n}}^{\Delta} = 1, \boldsymbol{n} = [1,0,0]$$

- However, linear system formed above usually has an infinite solution space.
- We define an error metric E as follows and seek to minimize E within the infinite solution space to yield a discrete filter.

$$E = \sum_{\boldsymbol{n}} \left(a_{\boldsymbol{n}}^{\Delta}\right)^2$$

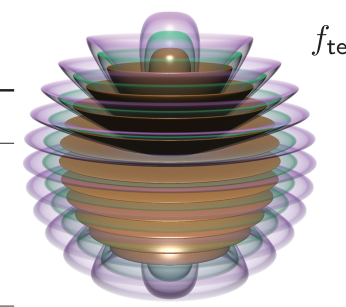
Results



Taylor Series Filters

Filter Type	Filter Name	Approx. Order (OF)	Filter Size
BCC	*SOCD*	2	2
	BCD	2	8
	OPT16	4	16
	OPT26	4	26
CC	*2-cd*	2	2
	4-cd	4	4

Second-Order Central Differencing (*SOCD*), Box Central Differencing (*BCD*), Support-Optimal-16 (*OPT16*) and Error-Optimal-26 (*OPT26*).



$$f_{\text{test}}(\boldsymbol{x}) := \gamma\|\boldsymbol{x}\| - \alpha \cos(2\pi f_m \frac{x_3}{\|\boldsymbol{x}\|})$$

$\alpha = 0.25$, $\gamma = 2$ and $f_m = 6$

Isovalues:
0.4 (opaque), 0.5 (green) and 0.6 (purple)

Orthogonal Projection Filters

B-splines b on CC

ψ \ φ	b^1	b^3
b^1	*ll* 18	
b^3	*cl* 100	*cc* 294
b^5	*ql* same as *cc*	*qc* 648

Box Splines Ξ on BCC

ψ \ φ	Ξ^2	Ξ^4
Ξ^2	*LL* 10	
Ξ^4	*QL* 52	*QQ* 150
Ξ^6	*NL* same as *QQ*	*NQ* 328

CC: *l*-trilinear, *c*-tricubic and *q*-triquintic. BCC: *L*-linear, *Q*-quintic and *N*-nonic. Filter sizes, in terms of the number of non-zero filter weights, are indicated.

Error Measures

RMS length of the error vector (l) and RMS angular deviation (θ in degrees) on the visible isosurface

CC

	f_{test} l	f_{test} θ	ML l	ML θ
2-cd	19.0	35.3	3.13	49.4
4-cd	17.9	31.8	2.79	44.1
ϵ-cd	11.9	22.4	1.52	25.7
ll	18.6	34.1	2.76	32.4
cl	17.6	32.2	2.20	29.8
ql	17.4	31.6	2.09	29.9
cc	16.1	27.7	1.78	23.8
qc	15.9	27.6	1.69	24.5

BCC

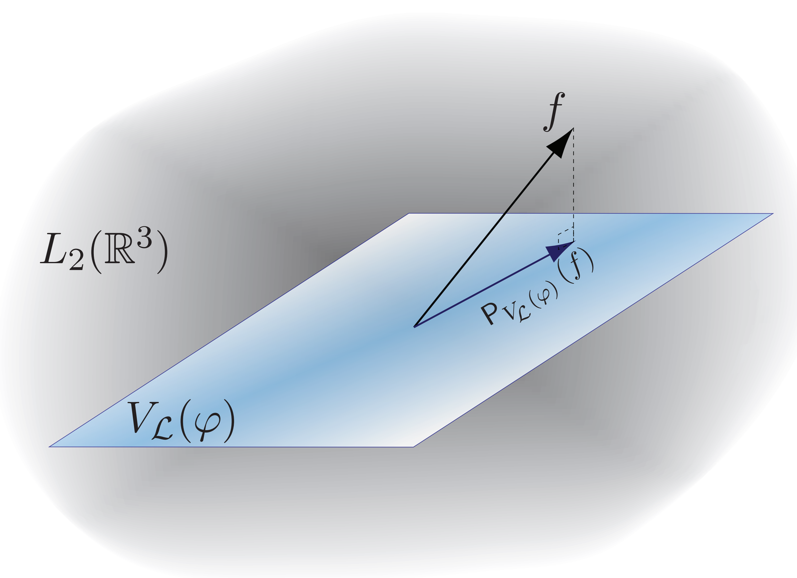
	f_{test} l	f_{test} θ	ML l	ML θ
SOCD	22.0	70.3	3.78	68.9
BCD	15.8	18.7	2.88	39.2
OPT16	13.7	17.0	2.42	28.7
OPT26	13.5	16.1	2.42	28.7
ϵ-CD	9.8	12.0	1.61	26.1
LL	15.1	22.9	2.50	29.3
QL	13.1	22.8	1.91	28.0
NL	12.5	22.9	1.80	29.1
QQ	10.5	13.0	1.38	19.4
NQ	9.7	12.5	1.29	21.9

The comparison is performed on the 0.4 *isosurface of* f_{test} *and the* 0.5 *isosurface of* ML. *For* f_{test}, $\epsilon = 0.005$ *and for* ML, $\epsilon = 0.003$.

Orthogonal Projection Framework

Approximation Space

$$V_{\mathcal{L}}(\varphi) := \left\{ s(\boldsymbol{x}) = \sum_{\boldsymbol{k} \in \mathbb{Z}^3} c[\boldsymbol{k}]\varphi_{\boldsymbol{k}}(\boldsymbol{x}) : c[\boldsymbol{k}] \in l_2(\mathbb{Z}^3) \right\}, \quad \text{where } \varphi_{\boldsymbol{k}}(\boldsymbol{x}) := \varphi(\boldsymbol{x} - \boldsymbol{Lk})$$



Space spanned by the shifts of a generating function φ on a lattice characterized by $\boldsymbol{L}$

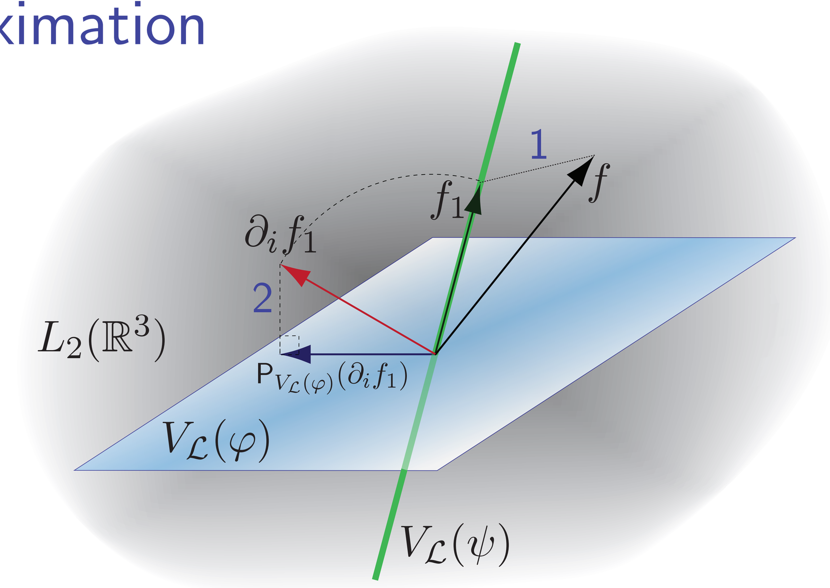
$$\mathsf{P}_{V_{\mathcal{L}}(\varphi)}(f) := \sum_{\boldsymbol{k} \in \mathbb{Z}^3} \langle f, \mathring{\varphi}_{\boldsymbol{k}} \rangle \varphi_{\boldsymbol{k}}$$

gives the *minimum error* approximation of f.

$\mathring{\varphi}_{\boldsymbol{k}}$ is the biorthogonal dual of $\varphi_{\boldsymbol{k}}$, i.e. $\langle \mathring{\varphi}_{\boldsymbol{k}}, \varphi_{\boldsymbol{l}} \rangle = \delta_{\boldsymbol{k},\boldsymbol{l}}$

Two Stage Gradient Approximation

Approximate $\boldsymbol{\nabla} f(\boldsymbol{x})$ from the sampled sequence $f[\boldsymbol{k}] = f(\boldsymbol{Lk})$



1. Approximate f in an auxiliary approximation space $V_{\mathcal{L}}(\psi)$

$$f(\boldsymbol{x}) \approx f_1(\boldsymbol{x}) = \sum_{\boldsymbol{k}} (f * p_1)[\boldsymbol{k}] \psi_{\boldsymbol{k}}(\boldsymbol{x}),$$

p_1 ensures that the interpolation constraint is satisfied.

2. Orthogonally project $\boldsymbol{\nabla} f_1$ onto the target space $V_{\mathcal{L}}(\varphi)$

$$f_{2,i}(\boldsymbol{x}) := \mathsf{P}_{V_{\mathcal{L}}(\varphi)}(\partial_i f_1) = \sum_{\boldsymbol{k}} \left((f * p_1) * \mathring{d}_i\right)[\boldsymbol{k}] \varphi_{\boldsymbol{k}}(\boldsymbol{x}),$$

where the discrete derivative filter $\mathring{d}_i$ is given by

$$\mathring{d}_i[\boldsymbol{n}] := \langle \partial_i \psi, \mathring{\varphi}_{\boldsymbol{n}} \rangle.$$

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