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A Logical Investigation of Context Dependency

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by

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Abstract

Tolerance contextual logic (TCL) is designed to deal with "tolerant" predicates (such as "tall"). This is done by extending the language of classical logic by a new *context operator*, which is meant to express the context in which a certain claim is made. Thus in the solution of the Sorites paradox given in TCL, a predicate like "tall" is taken to be context dependent and tolerant (the latter in the sense that its corresponding truth value is not affected by small changes). TCL has been introduced in [Ga-2010], where it is characterized both semantically and proof theoretically. However, here we show that the deductive system and semantics presented there do not match.

Taking TCL as our starting point, in this work we suggest three semantic frameworks for handling context dependency, and provide sound and complete proof systems for all of them. One of these semantic frameworks is equivalent to the original (from [Ga-2010]), while the original deductive system is sound and complete for another. The third is a more general framework, which is included (as a consequence relation) in the other two. After the presentation of these frameworks we investigate their logical characteristics, as well as their relations to each other and to classical first order logic. We end with a solution of the Sorites paradox which is based on the notions of context dependency and tolerance.

Contents

1 Introduction

1.1 Background

Artificial Intelligence is a multidisciplinary field of research. From the computer scientist view-point, it "concerns with designing ... systems that exhibit the characteristics we associate with intelligence in human behavior" [Barr-1981]. One of those characteristics is our ability to reason about vague predicates (e.g. "tall"). A naive formal treatment of our use of vague predicates leads to the famous Sorites paradox. One example of this paradox, which is a consequence of the use of the vague predicate "tall" is as follows: It is agreeable that a man of height 210 cm is tall, and that a man of height 100 cm isn't tall. It is also agreeable that 1 cm can't make the difference, i.e. that if a man of height x cm is tall, then so is a man of height $x - 1$ cm. From this we can conclude that a man of height 209 cm is tall. Repeating this process 110 times provides the contradictory conclusion that a man of height 100 cm is tall. The fact that the naive formalization of classical logic fails to capture our ordinary use of vague predicates in natural language doesn't mean that this use can't be modeled in a formalized way. Indeed, many formal models for the use of vague predicates, and in particular solutions to the Sorites paradox, were proposed in the literature. This work is mainly concerned with one of these approaches, namely *the contextual approach.*[1] More specifically, most of this work concerns with that of [Ga-2010], where it is claimed that a main characteristic of certain vague predicates is the fact that they are tolerant to small changes (according to some natural corresponding method of measuring) in the objects they classify (e.g. the tall example above). Tolerant predicates are context dependent, which means that their interpretation depends on the contexts in which they are being used. These observations lead to the construction of a new logical framework, namely Tolerance Contextual Logic (TCL), an extension of classical first order logic, in which it is possible to express the context in which a sentence is uttered. This thesis investigates the properties of the proposed framework and generalizes it to a more general framework of context dependency.

1.2 Contributions

Soundness and Completeness In his work, Gaifman presents a formal language, a semantics for it which captures the idea of tolerance, and a deductive system. Although the deductive system is intended to provide a sound and complete system for the semantics, we show below that it doesn't. Therefore, We present a revised deductive system which is obtained from the original one by the addition of a single axiom scheme. The new system is sound and complete for the original semantics. In addition, we provide a new semantics for which the original system is sound and complete, a semantics that has a value of its own. This new semantics is then easily modified to get an alternative semantics to the original (semantic) consequence relation. It is of course equivalent to the original semantics, but has a simpler form.

Expressive Power The new frameworks extend the language of classical logic by a new operator. However, we present a reduction between them to classical logic.

Types of Context Dependency TCL is a general tool for dealing with context dependency. One important application of it is modeling of tolerant predicates. However, tolerance and context dependency are not the same. It is true that context dependency is a common property of tolerant predicates, however the converse need not hold - there are context dependent predicates which are not tolerant. For example, consider the predicate "stranger". A man might be considered a stranger in the company of a particular group of people, and familiar in the company of another group of

[1] A survey of some contextual approaches to vagueness may be found in [Th].

people. Hence the predicate "Stranger" is context dependent. However, this predicate is not connected with any natural measuring method and hence it isn't tolerant. The new semantics and deductive systems that we present are constructed in a modular way: We start from a basic semantics which correspond to a basic deductive system, and then we add semantical constraints and axioms, hence constructing other new logics. Each axiom or semantical constraint has a specific role in the meaning of the induced consequence relation.

Generalization [Ga-2010] introduces a language in which there is only a single unary context-dependent predicate and all the rest are classical. In this work we consider more general languages with arbitrarily many context-dependent predicates of arbitrary arities.[2] More importantly, *all* predicates in these languages are potentially context dependent, but it is possible to add special axioms and corresponding semantical constraints that force a predicate to be context-independent (i.e. classical).

The Sorites Paradox As an example of the usage of contextual logic, we formalize a solution to the Sorites paradox which is based on it.

2 Preliminaries

2.1 Semantics

In all definitions, L is a first order language.

2.1.1 Classical Logic - The Substitutional Approach

There are two main approaches to classical logic semantics: The objectual approach and the substitutional approach. The first is based on assignments of objects to variables, while the second is based on naming the objects and then substituting those names for variables. We chose to use the substitutional approach as a basis rather than the more common objectual approach because of technical difficulties that the latter causes in the present context. Now we provide a short survey of the substitutional approach, based on [Sh-1967].

Definition 1.

1. A *first order structure* M for L consists of a domain $|M|$ and an interpretation function I^M which interprets all constant symbols, function symbols and predicate symbols.

2. Let M be a first order structure. $L(M)$ is the first order language obtained from L by adding for each individual $a \in |M|$ a constant symbol $\overline{a}$ which will be called the *name* of that individual. We extend I^M to be defined on names, so that for every individual $a \in |M|$, $I(\overline{a}) = a$.

3. Let φ be a formula of L with the free variables $x_1, ..., x_n$. Let M be a first order structure for L. An M-instance of φ is a closed formula of the form $\varphi\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\}$ in $L(M)$, with $a_1, ..., a_n$ being individuals of $|M|$. If φ is a sentence (closed formula), the only M-instance of it is itself.

Remark 1.

1. Since most of our first-order structures will be denoted by M, we will use I rather than I^M to denote the interpretation function of M.

[2] The possibility of doing this is partially noted in [Ga-2010].

2. We will use the word "instance" with two different meanings. The first is for an *M-instance* as was defined in Definition 1. The second is for the usual notion of an *instance of an axiom scheme.* To ensure that there is no confusion, we shall clarify in every use of the word "instance" which of these two notions is meant.

Definition 2. Let $M = \langle |M|, I \rangle$ be a first order structure and let φ be an $L(M)$-sentence. The satisfaction relation between M and φ (in symbols: $M \models \varphi$) is defined as follows:

1. $M \models P(t_1, ..., t_n)$ if $\langle I(t_1), ..., I(t_n) \rangle \in I(P)$.

2. $M \models \neg\psi$ if $M \not\models \psi$.

3. $M \models \psi_1 \to \psi_2$ if either $M \models \psi_2$ or $M \not\models \psi_1$.

4. $M \models \forall x\psi$ if for every M-instance ψ' of ψ $M \models \psi'$.

Definition 3. Let $M = \langle |M|, I \rangle$ be a first order structure and let φ be an L-formula. The satisfaction relation between M and φ (in symbols $M \models \varphi$) is defined as follows: $M \models \varphi$ if for every M-instance φ' of φ $M \models \varphi$.

Definition 4. Let T be a theory and φ be a formula. $T \vdash_{FOL} \varphi$ if for every M, if $M \models T$ then $M \models \varphi$.

Lemma 1. *Let M be a* first-order structure *for L, t a ground-term of $L(M)$ and $\overline{a}$ the name of $I(t)$.*

1. *If b is a term of $L(M)$ in which no variable except x occurs, then $I(b\{t/x\}) = I(b\{\overline{a}/x\})$.*

2. *If φ is a formula of $L(M)$ in which no variable except x is free, then $M \models \varphi\{t/x\}$ iff $M \models \varphi\{\overline{a}/x\}$.*

2.1.2 Fuzzy Logic

Later in this work, when we present a solution to the Sorites paradox using contextual logic, we will also present a solution given by fuzzy logic. We chose to use Łukasiewicz fuzzy logic, since it seems to provide the simplest solution to the paradox among the main three fuzzy logics.

Definition 5. An $Ł$-structure M for L is a first order structure $\langle |M|, I \rangle$ such that I is defined differently over predicate symbols: For every predicate symbol P of arity n, $I(P) : |M|^n \to [0,1]$. The definition of names and M-instances are the same as in first-order classical structures. For every sentence φ, $M(\varphi) \in [0,1]$ and is defined as follows:[3]

1. $M(P(t_1, ..., t_n)) = I(P)(\langle I(t_1), ..., I(t_n) \rangle)$.

2. $M(\psi_1 \to \psi_2) = min(1, 1 - M(\psi_1) + M(\psi_2))$.

3. $M(\neg\psi) = 1 - M(\psi)$.

4. $M(\psi_1 \wedge \psi_2) = min(M(\psi_1), M(\psi_2))$.

5. $M(\psi_1 \vee \psi_2) = max(M(\psi_1), M(\psi_2))$.

[3]We use the definitions from pages 179, 263 and 265 of [Be-2008], but note that we have changed them to fit the substitutional approach.

6. $M(\forall x\psi) = inf\{M(\psi\{\overline{a}/x\}) :\ a \in M\}$.

7. $M(\exists x\psi) = sup\{M(\psi\{\overline{a}/x\}) :\ a \in M\}$.

$M \models \varphi$ if $M(\varphi) = 1$.

Remark 2. The above definition can easily be generalized for formulas, but this is not necessary for the present work.

2.2 Proof Theory

2.2.1 Classical Logic - $HFOL$

Any of the standard Hilbert-style deductive systems for classical logic may serve as a basis for the sequel. Here we use the succinct and convenient system that is presented on pages 7-8 and 13 of [Ha-1998].

Rules of inference

- Modus Ponens (MP) $\frac{A \quad A \to B}{B}$
- Generalization (Gen) $\frac{A}{\forall x A}$

Axiom Schemes

- $\varphi \to (\psi \to \varphi)$.
- $(\varphi \to (\psi \to \chi)) \to ((\varphi \to \psi) \to (\varphi \to \chi))$.
- $(\neg\varphi \to \neg\psi) \to (\psi \to \varphi)$.
- $\forall x\varphi \to \varphi\{t/x\}$ where t is substitutable for x in φ.
- $\forall x(\varphi \to \psi) \to (\varphi \to \forall x\psi)$ where x isn't free in φ.

3 Gaifman's Framework

In this section we present the framework of TCL (tolerance contextual logic), which was introduced in [Ga-2010]. Note that we have made some modifications to the original presentation. At the end of this section these modifications will be noted. In addition, we include a new result regarding the semantics, which shows that one of its original requirements is redundant.

3.1 Language

Definition 6. A C-language (C for context) is a first order language, extended by the following:

1. There are two new symbols: '[' and ']'.
2. There is a new unary operator, called the *context operator*. It is obtained by using '[' and ']' with a set of terms (which is called the *context set*) between them.

Definition 7. Atomic formulas in L are formulas of the form $R(t_1, ..., t_n)$, where R is any predicate of any arity n.

Well-Formed Formulas (wffs) in L are defined recursively:

1. Every atomic formula is a wff.

2. If φ and ψ are wffs and x is a variable symbol then $(\neg\varphi)$, $(\varphi \to \psi)$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \leftrightarrow \psi)$, $(\forall x\varphi)$ and $(\exists x\varphi)$ are wffs.[4]

3. If φ is a wff, $n > 0$ and $t_1, ..., t_n$ are terms then $[t_1, ..., t_n](\varphi)$ is a wff. All occurrences of variables in $t_1, ..., t_n$ are free in $[t_1, ..., t_n]\varphi$ for each φ.

Remark 3. We also allow formulas of the form $[\,]\varphi$, and identify $[\,]\varphi$ with φ.

3.1.1 Conventions

Let L be a C-language.

1. The complexity of an L-formula is the number of operators (connectives, quantifiers and nonempty context operators) in it. [5]

2. We omit brackets where there is no danger for confusion.

3. We shall denote the set of predicate symbols of L by $PRED(L)$.

4. For every term or formula A, $Fv(A)$ is the set of free variables occurring in A.

5. $A(x_1, ..., x_n)$ ranges over terms or formulas (depending on context) in which $Fv(A) \subseteq \{x_1, ..., x_n\}$.

6. If $C = \{t_1, ..., t_n\}$ and $C' = \{s_1, ..., s_m\}$ then $[C, D] = [t_1, ..., t_n, s_1, ..., s_m]$.

3.2 Semantics

3.2.1 The Original Framework

Definition 8. Let X be any set.

1. $FIN(X)$ is the set of all finite subsets of X.

2. $FIN^+(X) := FIN(X) \setminus \{\emptyset\}$.

Definition 9. A C-structure for L is a pair (M, F) such that:

1. $M = \langle |M|, I\rangle$ is a first order structure such that I interprets only constant symbols and function symbols.

2. $F : PRED(L) \to \left(FIN(|M|) \to \bigcup_{k=1}^{\infty} P\left(|M|^k\right)\right)$ is a function such that for every predicate P of arity n, $F(P) : FIN(|M|) \to P(|M|^n)$. $F(P)$ is called the *context dependency function* of P, or in short, the *cdf* of P.

Definition 10. Let (M, F) be a C-structure and let C be a set of ground terms. $I(C) = \{I(t) : t \in C\}$.

[4] Similarly to many texts of classical logic, we take only $\neg$, $\to$ and $\forall$ as primitives.

[5] The reason we only count nonempty context operators (i.e. don't count $[\,]$) is that $[\,]\varphi$ is identified with φ, and hence should have the same complexity measure.

Definition 11. The GS-satisfaction relation (G for Gaifman, S for subsets) between C-structures (M, F) and sentences φ of $L(M)$ (in symbols: $(M, F) \models_{GS} \varphi$) is defined as follows:

1. $(M, F) \models_{GS} P(t_1, ..., t_n)$ if $\langle I(t_1), ..., I(t_n)\rangle \in F(P)(\{I(t_1), ..., I(t_n)\})$.
2. $(M, F) \models_{GS} \psi_1 \to \psi_2$ if either $(M, F) \models_{GS} \psi_2$ or $(M, F) \not\models_{GS} \psi_1$.
3. $(M, F) \models_{GS} \neg\psi$ if $(M, F) \not\models_{GS} \psi$.
4. $(M, F) \models_{GS} \forall x\psi$ if for every name $\overline{a}$ of $L(M)$, $(M, F) \models_{GS} \psi\{\overline{a}/x\}$.
5. $(M, F) \models_{GS} [t_1, ..., t_m]\psi$ for $m > 0$ if $(M, F_Y) \models_{GS} \psi$ where:
   (a) $Y := \{I(t_1), ..., I(t_m)\}$.
   (b) $F_Y := \lambda P^n.\lambda X \in FIN(|M|).F(P^n)(X \cup Y) \cap X^n$ where P^n varies over predicates of arity n.[6]

Remark 4. For convenience, we shall sometimes denote $F(P)$ by f^P and $F_Y(P)$ by f_Y^P.

Definition 12. Let (M, F) be a C-structure and φ a formula. (M, F) GS-satisfies φ (notation: $(M, F) \models_{GS} \varphi$) if $(M, F) \models_C \varphi'$ for every M-instance φ' of φ.

Lemma 2. *Let* (M, F) *be a* C-structure *for* L*,* t *a ground-term of* $L(M)$ *and* $\overline{a}$ *the name of* $I(t)$*. If* b *is a term of* $L(M)$ *in which no variable except* x *occurs, then* $I(b\{t/x\}) = I(b\{\overline{a}/x\})$.

Proof. Since we didn't change the definition of I over terms and function symbols, this follows from the original lemma, whose proof can be found on pages 19-20 of [Sh-1967]. □

Lemma 3. *Let* (M, F) *be a* C-structure *for* L*,* t *a ground-term of* $L(M)$*, and* $\overline{a}$ *the name of* $I(t)$*. If* φ *is a formula of* $L(M)$ *in which no variable except* x *is free, then* $(M, F) \models_{GS} \varphi\{t/x\}$ *iff* $(M, F) \models_{GS} \varphi\{\overline{a}/x\}$.

Proof. We use induction on the complexity of φ.

1. If φ is $P(s_1, ..., s_n)$ then:
   $(M, F) \models_{GS} P(s_1\{t/x\}, ..., s_n\{t/x\})$
   iff $\langle I(s_1\{t/x\}), ..., I(s_n\{t/x\})\rangle \in f^P(\{I(s_1\{t/x\}), ..., I(s_n\{t/x\})\})$
   iff $\langle I(s_1\{\overline{a}/x\}), ..., I(s_n\{\overline{a}/x\})\rangle \in f^P(\{I(s_1\{\overline{a}/x\}), ..., I(s_n\{\overline{a}/x\})\})$ (by Lemma 2) iff $(M, F) \models_{GS} P(s_1\{\overline{a}/x\}, ..., s_n\{\overline{a}/x\})$.
2. If φ is $\neg\psi$, $\psi_1 \to \psi_2$, or $\forall x\psi$ then the proof is the same as on pages 19-20 of [Sh-1967].
3. If φ is $[t_1, ..., t_n]\psi$ for $n > 0$ then: $(M, F) \models_{GS} \varphi\{t/x\}$ iff
   $(M, F) \models_{GS} [t_1\{t/x\}, ..., t_n\{t/x\}]\psi\{t/x\}$ iff $(M, F_Y) \models_{GS} \psi\{t/x\}$ with
   $Y = \{I(t_1\{t/x\}), ..., I(t_n\{t/x\})\}$ iff $(M, F_{Y'}) \models_{GS} \psi\{t/x\}$ with
   $Y' = \{I(t_1\{\overline{a}/x\}), ..., I(t_n\{\overline{a}/x\})\}$ (by Lemma 2) iff $(M, F_{Y'}) \models_{GS} \psi\{\overline{a}/x\}$ with
   $Y' = \{I(t_1\{\overline{a}/x\}), ..., I(t_n\{\overline{a}/x\})\}$ (by the i.h.) iff
   $(M, F) \models_{GS} [t_1\{\overline{a}/x\}, ..., t_n\{\overline{a}/x\}]\psi\{\overline{a}/x\}$ iff $(M, F) \models_{GS} \varphi\{\overline{a}/x\}$.

□

Definition 13. An S-structure is a C structure (M, F) in which for every $n \in \mathbb{N}$, $P \in PRED(L)$ of arity n and for every $X \in FIN^+(|M|)$ $f^P(X) \subseteq X^n$.

[6]i.e. for every P of arity n and for every $X \in FIN(|M|)$ and $Y \in FIN^+(|M|)$, $F_Y(P)(X) = F(P)(X \cup Y) \cap X^n$.

Lemma 4. *Assume (M, F) is an S-structure. Let $Y \in FIN(|M|)$ and let F_Y be the function defined by the GS-satisfaction relation. Then (M, F_Y) is also an S-structure.*

Definition 14. Let T be a theory and φ be a formula. $T \vdash^S_{GS} \varphi$ if for every S-structure (M, F), if $(M, F) \models_{GS} T$ then $(M, F) \models_{GS} \varphi$.

$\vdash^S_{GS}$ is the consequence relation which is induced by the original semantics from [Ga-2010].

3.2.2 The Unimportance of Inclusion

In this section we show that the inclusion restriction (that $f^P(X) \subseteq X^n$) is actually redundant. The same satisfaction relation can be defined over general C-structures, with a slight change in the definition. This fact makes the semantics easier and simpler to use in some contexts, as it omits the need to check the discussed requirement of cdfs.

Theorem 1. *Let $\models_G$ be the satisfaction relation obtained from $\models_{GS}$ by changing the definition of f^P_Y from $\lambda X \in FIN(|M|) . f^P(X \cup Y) \cap X^n$ to $\lambda X \in FIN(|M|) . f^P(X \cup Y)$. Let $\vdash_G$ be the consequence relation which is induced by $\models_G$ over C-structures (i.e. $T \vdash_G \varphi$ if for every C-structure (M, F), if $(M, F) \models_G T$ then $(M, F) \models_G \varphi$). Then $\vdash^S_{GS} = \vdash_G$.*

Proof. We prove this theorem using some new definitions and lemmas:

Lemma. *Let (M, F) be a C-*structure *for L, t a ground-term of $L(M)$ and $\overline{a}$ the name of $I(t)$. If φ is a formula of $L(M)$ in which no variable except x is free, then $(M, F) \models_G \varphi\{t/x\}$ iff $(M, F) \models_G \varphi\{\overline{a}/x\}$.*

Proof. This is proved in a similar manner to the proof of Lemma 3.

Definition. Let (M, F) be a C-structure.

1. Let $P \in PRED(L)$ be a predicate of arity n.
   $(F(P))^* = \lambda X \in FIN(|M|) . (F(P))(X) \cap X^n$.

2. $F^* = \lambda P \in PRED(L) . (F(P))^*$.

Remark. Let (M, F) be a C-structure. Then (M, F^*) is an S-structure.

Lemma. *Let (M, F) be a C-structure, let φ be an $L(M)$ sentence and let $Y \in FIN^+(|M|)$. Then $(F_Y)^* = (F^*)_Y$ where F_Y is defined in Definition 11 (i.e. for $\models_{GS}$ and not for $\models_G$).*

Proof. Let $P \in PRED(L)$ be any predicate of arity n. We need to prove that $(F_Y(P))^* = (F^*)_Y(P)$. Let $X \in FIN(|M|)$. We need to prove that $((F_Y(P))^*)(X) = ((F^*)_Y(P))(X)$. $(F_Y(P))^*(X) = (F_Y(P))(X) \cap X^n = F(P)(X \cup Y) \cap X^n \cap X^n = F(P)(X \cup Y) \cap X^n$. $((F^*)_Y(P))(X) = F^*(P)(X \cup Y) \cap X^n = F(P)(X \cup Y) \cap (X \cup Y)^n \cap X^n = F(P)(X \cup Y) \cap X^n$.

Lemma. *Let φ be a sentence. Then for every C-structure (M, F), $(M, F) \models_G \varphi$ iff $(M, F^*) \models_{GS} \varphi$.*

Proof. We prove this by induction on the complexity of φ.

1. If $\varphi = P(t_1, ..., t_n)$ then: For every C-structure (M, F), $(M, F) \models_G P(t_1, ..., t_n)$
   iff $\langle I(t_1), ..., I(t_n)\rangle \in F(P)(\{I(t_1), ..., I(t_n)\})$
   iff $\langle I(t_1), ..., I(t_n)\rangle \in F(P)(\{I(t_1), ..., I(t_n)\}) \cap \{I(t_1), ..., I(t_n)\}^n$
   $= (F(P))^*(\{I(t_1), ..., I(t_n)\})$ iff $(M, F^*) \models_{GS} P(t_1, ..., t_n)$.

2. If φ is $\neg\psi$, $\psi \to \phi$ or $\forall x\psi$ then this is done as usual. As an example, let us assume that $\varphi = \forall x\psi$: For every C-structure (M, F), $(M, F) \models_G \forall x\psi$ iff for every $a \in |M|$ $(M, F) \models_G \psi\{\overline{a}/x\}$ iff for every $a \in |M|$, $(M, F^*) \models_{GS} \psi\{\overline{a}/x\}$ (by the i.h.) iff $(M, F^*) \models_{GS} \forall x\psi$.

3. If $\varphi = [C]\,\psi$ then: let (M, F) be a C-structure. (M, F_Y) with $Y = I(C)$ is also a C-structure. Now: $(M, F) \models_G [C]\,\psi$ iff $(M, F_Y) \models_G \psi$ iff $(M, (F_Y)^*) \models_{GS} \psi$ (by the i.h.) iff $(M, (F^*)_Y) \models_{GS} \psi$ (by the previous lemma) iff $(M, F^*) \models_{GS} [C]\,\psi$.

Corollary. *Let φ be a formula. Then for every C-structure (M, F), $(M, F) \models_G \varphi$ iff $(M, F^*) \models_{GS} \varphi$.*

End of proof of Theorem 1. Let T be a theory and let φ be a formula. We First assume $T \vdash_G \varphi$ and prove $T \vdash^S_{GS} \varphi$. Let (M, F) be an S-structure such that $(M, F) \models_{GS} T$. Since (M, F) is an S-structure, $(M, F) = (M, F^*)$. By the above corollary, $(M, F) \models_G T$. By our assumption, $(M, F) \models_G \varphi$, and again by the previous corollary, $(M, F) \models_{GS} \varphi$. For the converse, we assume $T \vdash^S_{GS} \varphi$ and prove $T \vdash_G \varphi$. Let (M, F) be a C-structure such that $(M, F) \models_G T$. By the above corollary, $(M, F^*) \models_{GS} T$. Since $T \vdash^S_{GS} \varphi$, $(M, F^*) \models_{GS} \varphi$, and again by the above corollary, $(M, F) \models_G \varphi$.

□

3.3 Deductive System

Now we present the deductive system given in [Ga-2010].

Definition 15. HCT is the deductive system obtained from $HFOL$ (Section 2.2.1) by the addition of the following axiom schemes:

- (CC) $[C]\,[C']\,\varphi \leftrightarrow [C, C']\,\varphi$.
- $(C_\neg)$ $[C]\,\neg\varphi \leftrightarrow \neg\,[C]\,\varphi$.
- $(C_\to)$ $[C]\,(\varphi \to \psi) \leftrightarrow ([C]\,\varphi \to [C]\,\psi)$.
- $(C_\forall)$ $[C]\,\forall x\varphi \leftrightarrow \forall x\,[C]\,\varphi$ if x doesn't occur in C.
- (T) $P(x_1, ..., x_n) \leftrightarrow [x_1, ..., x_n]\,P(x_1, ..., x_n)$.

Remark 5. Note that axiom T holds only for atomic formulas, i.e. it doesn't hold that $\vdash_{HCT} \varphi(x_1, ..., x_n) \leftrightarrow [x_1, ..., x_n]\,\varphi(x_1, ..., x_n)$ for every $\varphi(x_1, ..., x_n)$. This property will be discussed in Section 6.2.

3.4 Context Independent Predicates

Similarly to to the possible existence of crisp predicates in fuzzy logic, there can exist context independent predicates in contextual logics. In fuzzy logic, the crisp predicates have an interpretation function whose range is $\{0, 1\}$ (rather than $[0, 1]$ for fuzzy predicates). In this section we provide a similar semantical condition and an equivalent proof-theoretical condition for context independent predicates.

The axiom scheme which best describes the context independence of a predicate R is now defined:

Definition 16. IND_R is the following axiom Scheme:

$$(IND_R)\quad R(x_1, ..., x_n) \leftrightarrow [C]\,R(x_1, ..., x_n)$$

for every context set C.

Hence, if one would like to construct a theory in which R is considered context independent, every instance of (IND_R) should be a theorem of that theory. Now we provide a characterization of the S-structures which models the instances of (IND_R).

Definition 17. Let (M, F) be an S-structure and let R be any n-ary predicate. We say that R is context independent in (M, F) if for every $\langle a_1, ..., a_n \rangle \in |M|^n$ and for every $X \in FIN(|M|)$, $\langle a_1, ..., a_n \rangle \in f^R(X \cup \{a_1, ..., a_n\})$ iff $\langle a_1, ..., a_n \rangle \in f^R(\{a_1, ..., a_n\})$.

Lemma 5. *Let (M, F) be an S-structure and let R be any n-ary predicate. R is context independent in (M, F) iff $f^R = \lambda X. \left\{\langle a_1, ..., a_n \rangle \in X^n : \langle a_1, ..., a_n \rangle \in f^R(\{a_1, ..., a_n\})\right\}$.*

Proof. Assume R is context independent in (M, F). Let $X \in FIN(|M|)$. We prove that $f^R(X) = \left\{\langle a_1, ..., a_n \rangle \in X^n : \langle a_1, ..., a_n \rangle \in f^R(\{a_1, ..., a_n\})\right\}$. Let $\langle a_1, ..., a_n \rangle \in f^R(X)$. We show that $\langle a_1, ..., a_n \rangle \in f^R(\{a_1, ..., a_n\}) \cap X^n$. Since $f^R(X) \subseteq X^n$, $\langle a_1, ..., a_n \rangle \in X^n$ and hence $X = X \cup \{a_1, ..., a_n\}$. This means that $\langle a_1, ..., a_n \rangle \in f^R(X \cup \{a_1, ..., a_n\})$. Therefore, since R is context independent in (M, F), $\langle a_1, ..., a_n \rangle \in f^R(\{a_1, ..., a_n\})$. For the other direction, let $\langle a_1, ..., a_n \rangle \in f^R(\{a_1, ..., a_n\}) \cap X^n$. We show that $\langle a_1, ..., a_n \rangle \in f^R(X)$. Since R is context independent in (M, F), $\langle a_1, ..., a_n \rangle \in f^R(X \cup \{a_1, ..., a_n\}) \cap X^n$. Since $\{a_1, ..., a_n\} \subseteq X$, $\langle a_1, ..., a_n \rangle \in f^R(X)$. □

Lemma 6. *Let (M, F) be an S-structure. Then R is context independent in (M, F) iff for every C $(M, F) \models_{GS} R(x_1, ..., x_n) \leftrightarrow [C] R(x_1, ..., x_n)$.*

Proof. Assume that R is context independent in (M, F). Let $R(\overline{a_1}, ..., \overline{a_n}) \leftrightarrow [t_1, ..., t_m] R(\overline{a_1}, ..., \overline{a_n})$ be an M-instance of $R(x_1, ..., x_n) \leftrightarrow [C] R(x_1, ..., x_n)$. $(M, F) \models_{GS} R(\overline{a_1}, ..., \overline{a_n})$ iff $\langle a_1, ..., a_n \rangle \in f^R(\{a_1, ..., a_n\})$ iff $\langle a_1, ..., a_n \rangle \in f^R(\{a_1, ..., a_n, I(t_1), ..., I(t_m)\}) \cap \{a_1, ..., a_n\}^n = f^R_Y(\{a_1, ..., a_n\})$ with $Y = \{I(t_1), ..., I(t_m)\}$ iff $(M, F_Y) \models_{GS} R(\overline{a_1}, ..., \overline{a_n})$ iff $(M, F) \models_{GS} [t_1, ..., t_m] R(\overline{a_1}, ..., \overline{a_n})$. Now assume $(M, F) \models_{GS} R(x_1, ..., x_n) \leftrightarrow [C] R(x_1, ..., x_n)$. Let $\{b_1, ..., b_m\} \in FIN(|M|)$ and let $a_1, ..., a_n \in |M|$. Then $(M, F) \models_{GS} R(\overline{a_1}, ..., \overline{a_n}) \leftrightarrow \left[\overline{b_1}, ..., \overline{b_m}\right] R(\overline{a_1}, ..., \overline{a_n})$. This means that $(M, F) \models_{GS} R(\overline{a_1}, ..., \overline{a_n})$ iff $(M, F) \models_{GS} \left[\overline{b_1}, ..., \overline{b_m}\right] R(\overline{a_1}, ..., \overline{a_n})$ iff $(M, F_Y) \models_{GS} R(\overline{a_1}, ..., \overline{a_n})$ with $Y = \{b_1, ..., b_m\}$. Therefore, $\langle a_1, ..., a_n \rangle \in f^R(\{a_1, ..., a_n\})$ iff $\langle a_1, ..., a_n \rangle \in f^R(\{a_1, ..., a_n\} \cup \{b_1, ..., b_m\}) \cap \{a_1, ..., a_n\}^n$ iff $\langle a_1, ..., a_n \rangle \in f^R(\{a_1, ..., a_n\} \cup \{b_1, ..., b_m\})$. □

3.5 Relation to The Original Framework

The framework presented in the previous sections is a modified version of the original framework from [Ga-2010]. Some of the modifications were introduced in order to provide a more general and modular framework, while others where introduced for simplicity. In this section the differences between the two frameworks are described.

1. In the original language, the set of predicates is divided into two subsets: Context dependent predicates and context independent predicates. The semantical interpretation of context independent predicates is classical (i.e. the interpretation function of a context independent predicate is a set of tuples), where the semantical interpretation of context dependent predicates is not. In addition, the original deductive system contains IND_R as an axiom scheme for every context independent predicate R. In our version, the set of predicates is not divided into two subsets, and all predicates are treated the same, semantically and proof theoretically. Whenever there is a need to declare a predicate R as context-independent, one simply adds IND_R as a special axiom. Obviously, this does not change anything proof theoretically, as it is possible to include IND_R for a special set of predicates and hence obtain the original version. Semantically, it makes our presentation more general, uniform

and succinct, as our satisfaction relation doesn't distinguish between types of predicates. The semantical characterization of context independent predicates (Section 3.4) ensures that they are handled properly in our version of the semantics.

2. In [Ga-2010], inside the context operator there are finite *lists* of terms while here we use finite *sets* of terms. This does not change the meaning of the context operator, both semantically and proof theoretically: Semantically, the interpretation functions are indifferent to the order of the terms inside the context operator (this holds for all interpretation functions, both in our version and in the original version). Proof theoretically, this is ensured by the inclusion of the following axiom scheme in the original deductive system:

$$[x_1, ...x_n]\, \varphi \to [x'_1, ..., x'_m]\, \varphi$$

where $\{x_1, ..., x_n\} = \{x'_1, ..., x'_m\}$.

3. In [Ga-2010], the semantics and deductive system are described for the special case where there exists a single context dependent predicate which is unary. It is noted there that a generalization to many predicates of different arities is possible, and some comments as for the way such a generalization could be achieved are given. In this work we use a generalized form of the framework from the beginning.

4. The semantics of [Ga-2010] is defined using the objectual approach, while we use the substitutional approach. One of the main contributions of this work is the introduction of a new framework for context dependency, which is based on the original framework. In our new framework, all our semantics are defined using the substitutional approach, because of technical difficulties that the objectual approach causes in our definitions. For the purpose of showing the connection between our framework and the original framework, we chose to present the latter using the substitutional approach as well. It is routine to see that using the substitutional approach in this semantics doesn't change anything in its induced consequence relation. A detailed proof of this fact can be found in the appendix (Section 10.1).

4 New Semantics

In this section we present new semantics for C-languages (Definition 6). We start by introducing a new satisfaction relation between C-structures (Definition 9) and formulas. Then, we present several kinds of C-structures, each induces (along with the new satisfaction relation) a different consequence relation. We shall generally call these semantical frameworks "C-semantics" (as opposed to Gaifman's "G-semantics").

The C-satisfaction relation is based on classical first-order semantics, with two exceptions: the interpretation of the context operator (which does not exist in classical logic) and the interpretation of predicates.

In the C-satisfaction relation, a predicate's interpretation isn't a set of tuples, but rather a function which assigns to every context (finite set of objects) a special set of tuples. This means that the interpretation may be different for every context. Classical predicates (or context independent predicates) form a special case - their interpretation is constant, and is not affected by context.

Throughout this section, L is a C-language.

4.1 C-Structures

4.1.1 Defining $\vdash_C$

Definition 18. The C-satisfaction relation between C-structures (M, F) and $L(M)$-sentences φ (in symbols: $(M, F) \models_C \varphi$) is defined as follows, for $m, k \geq 0$ and $n > 0$:

1. $(M, F) \models_C [s_1, ..., s_m] P(t_1, ..., t_n)$ if $\langle I(t_1), ..., I(t_n)\rangle \in f^P(\{I(s_1), ..., I(s_m)\})$.
2. $(M, F) \models_C [s_1, ..., s_m](\psi_1 \to \psi_2)$ if either $(M, F) \models_C [s_1, ..., s_m]\psi_2$ or $(M, F) \not\models_C [s_1, ..., s_m]\psi_1$.
3. $(M, F) \models_C [s_1, ..., s_m]\neg\psi$ if $(M, F) \not\models_C [s_1, ..., s_m]\psi$.
4. $(M, F) \models_C [s_1, ..., s_m]\forall x\psi$ if for every name $\overline{a}$ of $L(M)$,
$(M, F) \models_C [s_1, ..., s_m](\psi\{\overline{a}/x\})$.[7] [8]
5. $(M, F) \models_C [s_1, ..., s_m][t_1, ..., t_k]\psi$ if $(M, F) \models_C [s_1, ..., s_m, t_1, ..., t_k]\psi$.

Definition 19. Let (M, F) be a C-structure and φ a formula. (M, F) C-satisfies φ (notation: $(M, F) \models_C \varphi$) if $(M, F) \models_C \varphi'$ for every M-instance φ' of φ.

Now we prove that our structures interpret names as expected.

Lemma 7. *Let (M, F) be a C-*structure *for L, t a ground-term of $L(M)$ and $\overline{a}$ the name of $I(t)$. If φ is a formula of $L(M)$ in which no variable except x is free, then $(M, F) \models_C \varphi\{t/x\}$ iff $(M, F) \models_C \varphi\{\overline{a}/x\}$.*

Proof. We use induction on the complexity of φ. Throughout this proof, $m, n > 0$.

1. If φ is $P(s_1, ..., s_n)$ then:
$(M, F) \models_C P(s_1\{t/x\}, ..., s_n\{t/x\})$
iff $\langle I(s_1\{t/x\}), ..., I(s_n\{t/x\})\rangle \in f^P(\emptyset)$
iff $\langle I(s_1\{\overline{a}/x\}), ..., I(s_n\{\overline{a}/x\})\rangle \in f^P(\emptyset)$ (by Lemma 2)
iff $(M, F) \models_C P(s_1\{\overline{a}/x\}, ..., s_n\{\overline{a}/x\})$.
2. If φ is $\neg\psi$, $\psi_1 \to \psi_2$, or $\forall x\psi$ then the proof is the same as on pages 19-20 of [Sh-1967].
3. If φ is $[s_1, ..., s_m] P(t_1, ..., t_n)$ then:
$(M, F) \models_C [s_1\{t/x\}, ..., s_m\{t/x\}] P(t_1\{t/x\}, ..., t_n\{t/x\})$
iff $\langle I(t_1\{t/x\}), ..., I(t_n\{t/x\})\rangle \in f^P(\{I(s_1\{t/x\}), ..., I(s_m\{t/x\})\})$
iff $\langle I(t_1\{\overline{a}/x\}), ..., I(t_n\{\overline{a}/x\})\rangle \in f^P(\{I(s_1\{\overline{a}/x\}), ..., I(s_m\{\overline{a}/x\})\})$ (by Lemma 2) iff $(M, F) \models_C [s_1\{\overline{a}/x\}, ..., s_m\{\overline{a}/x\}] P(t_1\{\overline{a}/x\}, ..., t_n\{\overline{a}/x\})$
4. If φ is $[s_1, ..., s_m](\psi_1 \to \psi_2)$ then:
$(M, F) \models_C [s_1\{t/x\}, ..., s_m\{t/x\}](\psi_1\{t/x\} \to \psi_2\{t/x\})$
iff either $(M, F) \models_C [s_1\{t/x\}, ..., s_m\{t/x\}]\psi_2\{t/x\}$
or $(M, F) \not\models_C [s_1\{t/x\}, ..., s_m\{t/x\}]\psi_1\{t/x\}$
iff either $(M, F) \models_C [s_1\{\overline{a}/x\}, ..., s_m\{\overline{a}/x\}]\psi_2\{\overline{a}/x\}$
or $(M, F) \not\models_C [s_1\{\overline{a}/x\}, ..., s_m\{\overline{a}/x\}]\psi_1\{\overline{a}/x\}$ (by the i.h. and Lemma 2)
iff $(M, F) \models_C [s_1\{\overline{a}/x\}, ..., s_m\{\overline{a}/x\}](\psi_1\{\overline{a}/x\} \to \psi_2\{\overline{a}/x\})$.
5. If φ is $[s_1, ..., s_m]\neg\psi$ then the proof is similar.

[7] Note that since this definition concerns only the cases where φ is a sentence, x doesn't occur in $[s_1, ..., s_m]$.

[8] It seemed very difficult to provide a definition for the satisfaction of formulas of this form using the more standard objectual approach. This was the main reason for preferring the substitutional approach.

6. If φ is $[s_1, ..., s_m] \forall y\psi$ then: Assume first $x = y$. In this case, $(M, F) \models_C \varphi\{t/x\}$ iff $(M, F) \models_C [s_1\{t/x\}, ..., s_m\{t/x\}] \forall x\psi$ (since x isn't free in $\forall x\psi$) iff for every name $\overline{b}$ $(M, F) \models_C [s_1\{t/x\}, ..., s_m\{t/x\}] \left(\psi\left\{\overline{b}/x\right\}\right)$ iff for every name $\overline{b}$ $(M, F) \models_C [s_1\{\overline{a}/x\}, ..., s_m\{\overline{a}/x\}] \left(\psi\left\{\overline{b}/x\right\}\right)$ (by the i.h. and Lemma 2) iff $(M, F) \models_C [s_1\{\overline{a}/x\}, ..., s_m\{\overline{a}/x\}] (\forall x\psi)$ iff $(M, F) \models_C \varphi\{\overline{a}/x\}$.
   Now assume $x \neq y$. $(M, F) \models_C \varphi\{t/x\}$ iff for every name $\overline{b}$ $(M, F) \models_C [s_1\{t/x\}, ..., s_m\{t/x\}] \psi\left\{t/x, \overline{b}/y\right\}$ iff for every name $\overline{b}$ $(M, F) \models_C [s_1\{\overline{a}/x\}, ..., s_m\{\overline{a}/x\}] \left(\psi\left\{\overline{a}/x, \overline{b}/y\right\}\right)$ (by the i.h. and Lemma 2) iff $(M, F) \models_C [s_1\{\overline{a}/x\}, ..., s_m\{\overline{a}/x\}] \forall y\psi\{\overline{a}/x\}$ iff $(M, F) \models_C \varphi\{\overline{a}/x\}$.

7. If φ is $[s_1, ..., s_m][t_1, ..., t_n]\psi$ then:
   $(M, F) \models_C [s_1\{t/x\}, ..., s_m\{t/x\}][t_1\{t/x\}, ..., t_n\{t/x\}]\psi\{t/x\}$
   iff $(M, F) \models_C [s_1\{t/x\}, ..., s_m\{t/x\}, t_1\{t/x\}, ..., t_n\{t/x\}]\psi\{t/x\}$
   iff $(M, F) \models_C [s_1\{\overline{a}/x\}, ..., s_m\{\overline{a}/x\}, t_1\{\overline{a}/x\}, ..., t_n\{\overline{a}/x\}]\psi\{\overline{a}/x\}$ (by the i.h.)
   iff $(M, F) \models_C [s_1\{\overline{a}/x\}, ..., s_m\{\overline{a}/x\}][t_1\{\overline{a}/x\}, ..., t_n\{\overline{a}/x\}]\psi\{\overline{a}/x\}$.

□

Lemma 8. *Let (M, F) be a C-*structure*, let φ_1, φ_2 be sentences and let $t_1, ..., t_n$ be ground terms. then:*

1. $(M, F) \models_C [t_1, ..., t_n] \neg\varphi_1$ *iff* $(M, F) \models_C \neg[t_1, ..., t_n]\varphi_1$.
2. $(M, F) \models_C [t_1, ..., t_n](\varphi_1 \to \varphi_2)$ *iff* $(M, F) \models_C [t_1, ..., t_n]\varphi_1 \to [t_1, ..., t_n]\varphi_2$.
3. $(M, F) \models_C [t_1, ..., t_n]\forall x\varphi_1$ *iff* $(M, F) \models_C \forall x[t_1, ..., t_n]\varphi_1$.
4. $(M, F) \models_C [t_1, ..., t_n](\psi_1 \leftrightarrow \psi_2)$ *iff it holds that* $(M, F) \models_C [t_1, ..., t_n]\psi_1$ *iff* $(M, F) \models_C [t_1, ..., t_n]\psi_2$.

Definition 20. Let T be a theory and φ be a formula. $T \vdash_C \varphi$ if for every C-structure (M, F), if $(M, F) \models_C T$ then $(M, F) \models_C \varphi$.

$\vdash_C$ is our most basic semantic consequence relation. Restricting it to special kinds of C-structures yields more useful consequence relations. The importance of $\vdash_C$ lies in its generality, and it will serve as a basis for our modular construction of other consequence relations.

4.2 T-Structures

$\vdash_C$ is neither equivalent to the consequence relation induced by the original semantics from [Ga-2010], nor to the one induced by the original deductive system. In this section we restrict $\models_C$ to a subclass of C-structures. This induces the same consequence relation as the one induced by the deductive system from [Ga-2010]. There is also an intuitive motivation for this restriction. Since a context is a set of objects to be taken under consideration, it is reasonable (but not necessary) to demand that $P(c)$ would hold whenever $[c]P(c)$ holds. However, the above semantics doesn't require this restriction. This example is formalized in the next lemma:

Lemma 9. $\nvdash_C P(c) \leftrightarrow [c]P(c)$.

Proof. We present a C-structure (M, F) such that $(M, F) \not\models_C P(c) \leftrightarrow [c]P(c)$: $|M| = \{1\}$, $I(c) = 1$. $f^P(\emptyset) = \emptyset$, $f^P(\{1\}) = \{1\}$. Clearly, $(M, F) \models_C [c]P(c)$. However, $(M, F) \not\models_C P(c)$. □

4.2.1 Defining $\vdash_C^T$

We now present a semantics that is adequate for the above intuitive requirement.

Definition 21. A T-structure is a C-structure (M, F) in which for every $P \in PRED(L)$ of arity n and for every $a_1, ..., a_n \in |M|$, $\langle a_1, ..., a_n \rangle \in f^P(\emptyset)$ iff $\langle a_1, ..., a_n \rangle \in f^P(\{a_1, ..., a_n\})$.

Lemma 10. *Let (M, F) be a T-structure and let $t_1, ..., t_n$ be ground terms. $(M, F) \models_C P(t_1, ..., t_n)$ iff $\langle I(t_1), ..., I(t_n) \rangle \in f^P(\{I(t_1), ..., I(t_n)\})$.*

Definition 22. Let T be a theory and φ be a formula. $T \vdash_C^T \varphi$ if for every T-structure (M, F), if $(M, F) \models_C T$ then $(M, F) \models_C \varphi$.

Restricting the C-satisfaction relation to T-structures provides the rational requirement that $P(c)$ would hold whenever $[c] P(c)$ holds. In particular, the induced consequence relation has this restriction as a valid formula:

Lemma 11. $\vdash_C^T P(x_1, ..., x_n) \leftrightarrow [x_1, ..., x_n] P(x_1, ..., x_n)$.

Proof. Let (M, F) be a T-structure and let $P(\overline{a_1}, ..., \overline{a_n}) \leftrightarrow [\overline{a_1}, ..., \overline{a_n}] P(\overline{a_1}, ..., \overline{a_n})$ be an M-instance of $P(x_1, ..., x_n) \leftrightarrow [x_1, ..., x_n] P(x_1, ..., x_n)$. $(M, F) \models_C P(\overline{a_1}, ..., \overline{a_n})$ iff $\langle a_1, ..., a_n \rangle \in f^P(\emptyset)$ iff $\langle a_1, ..., a_n \rangle \in f^P(\{a_1, ..., a_n\})$ (since (M, F) is a T-structure) iff $(M, F) \models_C [\overline{a_1}, ..., \overline{a_n}] P(\overline{a_1}, ..., \overline{a_n})$. □

However, this equivalence need not hold in every context:

Lemma 12. $\nvdash_C^T [d] (P(c) \leftrightarrow [c] P(c))$.

Proof. We present a T-structure (M, F) such that $(M, F) \not\models_C [d] (P(c) \leftrightarrow [c] P(c))$: $|M| = \{1, 2\}$. $I(c) = 1$ and $I(d) = 2$. $f^P(\{2\}) = f^P(\emptyset) = \{2\}$, $f^P(\{1, 2\}) = \{1\}$, $f^P(\{1\}) = \emptyset$. $(M, F) \models_C [d] (P(c) \leftrightarrow [c] P(c))$ iff the following holds: $(M, F) \models_C [d] P(c)$ iff $(M, F) \models_C [d] [c] P(c)$. Now, $(M, F) \not\models_C [d] P(c)$ since $I(c) = 1 \notin \{2\} = f^P(\{2\}) = f^P(\{I(d)\})$. $(M, F) \models_C [d] [c] P(c)$ is equivalent to $(M, F) \models_C [d, c] P(c)$. $(M, F) \models_C [d, c] P(c)$ since $I(c) = 1 \in \{1\} = f^P(\{1, 2\}) = f^P(\{I(c), I(d)\})$. Hence, $(M, F) \not\models_C [d] (P(c) \leftrightarrow [c] P(c))$. □

T-structures provide a formal treatment to context dependent predicates which are not necessarily tolerant, i.e. predicates whose meaning depends on the context in which they are used, but aren't sensitive to small changes in the predicated objects. Now we provide two examples in which T-structures are useful.

Example 1. Consider the predicate "Stranger" which was discussed in the introduction and denote it as P. It is indeed possible that a will be considered a stranger in the context $[a_1, a_2, a_3]$, but will be considered a non-stranger in the context $[a, a_1, a_2, a_3]$. For example, imagine Gulliver arriving at Lilliput. Before he has arrived, he was a stranger. But after he stayed there for a while, he was no longer considered a stranger. $\vdash_C^T$ is appropriate for describing situations of this sort, as they allow the case where a is P in $[a_1, a_2, a_3]$ but not in $[a, a_1, a_2, a_3]$. We shall later return to the Guliver example.

Example 2. Consider the predicate "True" (for the set of English sentences) and denote it as P. Interpret a as the sentence "all sentences written on the board are false" and read $[C] \varphi$ as "when the elements of C are written on the board, φ holds". Consider the sentence $[a_1, ..., a_n] P(a)$ (for simplicity, assume $a_i \neq a$ for every $1 \leq i \leq n$). This sentence is true or false, depending on the truth values of $a_1, ..., a_n$. However, the sentence $[a, a_1, ..., a_n] P(a)$ provides us with a version of the liar paradox, and hence in this situation $[a_1, ..., a_n] P(a) \leftrightarrow [a, a_1, ..., a_n] P(a)$ doesn't have to hold.

Lemma 13. $\vdash_C \subsetneq \vdash_C^T$.

Proof. Obviously, $\vdash_C \subseteq \vdash_C^T$. By Lemmas 9 and 11 $\vdash_C \neq \vdash_C^T$. □

4.2.2 Another Characterization of $\vdash_C^T$

It is possible to obtain the same consequence relation as $\vdash_C^T$, without the restriction to T-structures. This is done by making a small modification to the definition of $\models_C$.

Theorem 2. *Let $\models_{CT}$ be the satisfaction relation obtained from $\models_C$ by introducing the following change: $(M,F) \models_{CT} P(t_1,...,t_n)$ if $\langle I(t_1),...,I(t_n)\rangle \in f^P(\{I(t_1),...,I(t_n)\})$. Let $\vdash_{CT}$ be the consequence relation induced by $\models_{CT}$ over C-structures (i.e. $T \vdash_{CT} \varphi$ if for every C-structure (M,F), if $(M,F) \models_{CT} T$ then $(M,F) \models_{CT} \varphi$). Then $\vdash_C^T = \vdash_{CT}$.*

Proof. We introduce some new definitions and lemmas:

Lemma. *Let (M,F) be a C-*structure *for L, t a ground-term of $L(M)$ and $\overline{a}$ the name of $I(t)$. If φ is a formula of $L(M)$ in which no variable except x is free, then $(M,F) \models_{CT} \varphi\{t/x\}$ iff $(M,F) \models_{CT} \varphi\{\overline{a}/x\}$.*

Proof. This is proved in a similar manner to the proof of Lemma 7.

Definition.

1. Let D be a set, let $n \in \mathbb{N}$ and let $f : FIN(D) \to P(D^n)$.

$$f^* := \lambda X \in FIN(D) . \begin{cases} \{\langle a_1,...,a_n\rangle \in |M|^n : \langle a_1,...,a_n\rangle \in f^P(\{a_1,...,a_n\})\} & X = \emptyset \\ f^P(X) & otherwise \end{cases}$$

2. Let (M,F) be a C-structure. $F^* := \lambda P \in PRED(L) . \left(f^P\right)^*$.

Lemma. *Let (M,F) be a C-structure. Then (M,F^*) is a T-structure.*

Lemma. *Let (M,F) be a C-structure and let φ be a sentence. $(M,F) \models_{CT} \varphi$ iff $(M,F^*) \models_C \varphi$.*

Proof. By induction on the complexity of φ. We denote f^P by f.

1. If φ is $P(t_1,...,t_n)$ then: $(M,F) \models_{CT} P(t_1,...,t_n)$ iff $\langle I(t_1),...,I(t_n)\rangle \in f(\{I(t_1),...,I(t_n)\})$ iff $\langle I(t_1),...,I(t_n)\rangle \in f^*(\emptyset)$ iff $(M,F^*) \models_C P(t_1,...,t_n)$.

2. If φ is $[s_1,...,s_m]P(t_1,...,t_n)$ with $m > 0$ then: $(M,F) \models_{CT} [s_1,...,s_m]P(t_1,...,t_n)$
   iff $\langle I(t_1),...,I(t_n)\rangle \in f(\{I(s_1),...,I(s_m)\})$
   iff $\langle I(t_1),...,I(t_n)\rangle \in f^*(\{I(s_1),...,I(s_m)\})$
   iff $(M,F^*) \models_C [s_1,...,s_m]P(t_1,...,t_n)$.

3. Otherwise, the proof is trivial, by the definition of f^*, and by the fact that on other forms of sentences, $\models_C$ and $\models_{CT}$ are defined identically.

Corollary. *Let (M,F) be a C-structure and let φ be a formula. $(M,F) \models_{CT} \varphi$ iff $(M,F^*) \models_C \varphi$.*

End of proof of Theorem 2. Let T be a theory and φ be a formula. Assume $T \vdash_C^T \varphi$. Let (M,F) be a C-structure and assume $(M,F) \models_{CT} T$. By the above corollary, $(M,F^*) \models_C T$. By our assumption, $(M,F^*) \models_C \varphi$. Again, by the above corollary, $(M,F) \models_{CT} \varphi$. Now assume $T \vdash_{CT} \varphi$. Let (M,F) be a T-structure and assume $(M,F) \models_C T$. By the definition of $\left(f^P\right)^*$ and the fact that (M,F) is a T-structure, $(M,F) = (M,F^*)$. By the above corollary, $(M,F) \models_{CT} T$. By our assumption, $(M,F) \models_{CT} \varphi$. And again by the above corollary, $(M,F) \models_C \varphi$.

$\square$

4.3 R-Structures

A special kind of context dependent predicates is tolerant predicates. In this section we present a refinement of C-semantics for this kind. We also prove that this semantics is equivalent to the original semantics from [Ga-2010], $\vdash_{GS}^{S}$.

According to the definition of $\vdash_{C}^{T}$, $[\overline{a_1},...,\overline{a_n}]\, P(\overline{a})$ is not equivalent to $[\overline{a_1},...,\overline{a_n},\overline{a}]\, P(\overline{a})$. This means that $\{a_1,...,a_n\}$ and $\{a_1,...,a_n,a\}$ impose two different meanings on the predicate P, and those meanings may handle a differently. Let us consider the tolerant predicate "tall" and denote it by P. Does it make sense to consider a as tall in the context $\{a_1,...,a_n\}$ but not in $\{a_1,...,a_n,a\}$? Let us return to the Gulliver example. When all the tiny people look at Gulliver for the first time, they classify him as tall. It is more than reasonable to think that after he stays with them for a while, he will still be classified as tall. This is not what reasoning according to $\vdash_{C}^{T}$ implies, as $[a_1,...,a_n]\, P(a) \leftrightarrow [a_1,...,a_n,a]\, P(a)$ isn't C-valid in all T structures. In the Gulliver case, this means that when he has just arrived, he was considered tall, but when considering him among them, the tiny people changed their mind regarding his tallness.

Now we refine $\vdash_{C}^{T}$ to a new semantics which is adequate for this situation, and induces the same consequence relation as the original semantics from [Ga-2010]. Namely, we restrict the C-satisfaction relation to a special kind of T-structures. Before doing so, we introduce reflexive functions, which are essential for the definition of the refined consequence relation.

4.3.1 Reflexive Functions

Definition 23. Let D be any set and let $f : FIN(D) \to P(D^n)$ for some $n \in \mathbb{N}$. f is *reflexive if for every* $\langle a_1,...,a_n\rangle \in D^n$ *and* $X \in FIN(D)$, $\langle a_1,...,a_n\rangle \in f(X)$ *iff* $\langle a_1,...,a_n\rangle \in f(X \cup \{a_1,...,a_n\})$.

Example 3. Obviously, every $f : FIN(D) \to P(D^n)$ which is a constant function is reflexive.

Example 4. Assume $D = \mathbb{N}$ and $n = 1$. Define $f : FIN(\mathbb{N}) \to P(\mathbb{N})$ as $f = \lambda X \in FIN(\mathbb{N}) . \left\{x : x < \frac{maxX}{2}\right\}$. f is reflexive: Assume $x \in f(X)$. Then $x < \dfrac{maxX}{2}$ and hence $x < maxX$. Therefore, $maxX = max(X \cup \{x\})$ and hence $x < \dfrac{max(X \cup \{x\})}{2}$, which means that $x \in f(X \cup \{x\})$. Now assume $x \in f(X \cup \{x\})$. Then $x < \dfrac{max(X \cup \{x\})}{2} < max(X \cup \{x\})$. Therefore $\dfrac{max(X \cup \{x\})}{2} = \dfrac{maxX}{2}$. Therefore, $x \in f(X)$.

Definition 24. Let D be any set and let $n \in \mathbb{N}$.

1. $r_n = \lambda g : FIN(D) \to P(D^n) . \lambda X \in FIN(D) . \{\langle a_1,...,a_n\rangle : \langle a_1,...,a_n\rangle \in g(X \cup \{a_1,...,a_n\})\}$.
2. $r = \bigcup_{n=1}^{\infty} r_n$.

Lemma 14. *Let D be any set and let $f : FIN(D) \to P(D^n)$ be a reflexive function. Then $r(f) = f$.*

Proof. $\langle a_1,...,a_n\rangle \in r(f)(X)$ iff $\langle a_1,...,a_n\rangle \in f(X \cup \{a_1,...,a_n\})$ iff $\langle a_1,...,a_n\rangle \in f(X)$ (since f is reflexive). □

Lemma 15. *Let D be any set and let $n \in \mathbb{N}$. Then*
$r_n[FIN(D) \to P(D^n)] = \{f : FIN(D) \to P(D^n) \,:\, f\, is\, reflexive\}$.

Proof. Let $g : FIN(D) \to P(D^n)$. We show that $r_n(g)$ is reflexive. Let $\langle a_1, ..., a_n\rangle \in D^n$ and let $X \in FIN(D)$. Then $\langle a_1, ..., a_n\rangle \in r_n(g)(X)$ iff $\langle a_1, ..., a_n\rangle \in g(X \cup \{a_1, ..., a_n\})$ iff $\langle a_1, ..., a_n\rangle \in g(X \cup \{a_1, ..., a_n\} \cup \{a_1, ..., a_n\})$ iff $\langle a_1, ..., a_n\rangle \in r_n(g)(X \cup \{a_1, ..., a_n\})$. Now let f be a reflexive function. By Lemma 14, $r_n(f) = f$. Hence $f = r_n(f) \in r_n[FIN(D) \to P(D^n)]$. □

Corollary 1. $r\left[\bigcup_{n=1}^{\infty}(FIN(D) \to P(D^n))\right] = \{f \in \bigcup_{n=1}^{\infty}(FIN(D) \to P(D^n)) : f\, is\, reflexive\}$.

Example 5. Assume $D = \mathbb{N}$ and $n = 1$. Define $f : FIN(\mathbb{N}) \to P(\mathbb{N})$ as $f = \lambda X \in FIN(\mathbb{N}).\left\{x \in X : x < \frac{maxX}{2}\right\}$. f isn't reflexive. For example: $2 \notin f(\{1,5\})$, but $2 \in f(\{1,5\} \cup \{2\})$. Let us consider $r(f)$.

$$\begin{aligned} r(f) &= \lambda X \in FIN(\mathbb{N}).\{x : x \in f(X \cup \{x\})\} \\ &= \lambda X \in FIN(\mathbb{N}).\left\{x : x < \frac{max(X \cup \{x\})}{2}\right\} \\ &= \lambda X \in FIN(\mathbb{N}).\left\{x : x < \frac{maxX}{2}\right\} \end{aligned}$$

We obtained the reflexive function from example 4.

Definition 25. Let (M, F) be a C-structure. $r(F) = \lambda P \in PRED(L).r(F(P))$.

4.3.2 Defining $\vdash_C^R$

Definition 26. An R-structure is a C-structure in which for every predicate P, f^P is reflexive.

Lemma 16. *Let (M, F) be a C-structure. Then $(M, r(F))$ is an R-structure.*

Lemma 17. *Every R-structure is a T-structure.*

Definition 27. Let T be a theory and φ be a formula. $T \vdash_C^R \varphi$ if for every R-structure (M, F), if $(M, F) \models_C T$ then $(M, F) \models_C \varphi$.

The requirement that the cdfs must be reflexive captures the essence of the Gulliver problem. This requirement explicitly forces all interpretations of P to be indifferent to the addition of the examined objects to the context. In particular, the induced consequence relation has this requirement as a valid formula:

Lemma 18. $\vdash_C^R [y_1, ..., y_m](P(x_1, ..., x_n) \leftrightarrow [x_1, ..., x_n] P(x_1, ..., x_n))$.

Proof. Let (M, F) be an R-structure and let
$[\overline{a_1}, ..., \overline{a_m}]\left(P\left(\overline{b_1}, ..., \overline{b_n}\right) \leftrightarrow \left[\overline{b_1}, ..., \overline{b_n}\right] P\left(\overline{b_1}, ..., \overline{b_n}\right)\right)$ be an M-instance of
$[y_1, ..., y_n](P(x_1, ..., x_n) \leftrightarrow [x_1, ..., x_n] P(x_1, ..., x_n))$.
$(M, F) \models_C [\overline{a_1}, ..., \overline{a_m}] P\left(\overline{b_1}, ..., \overline{b_n}\right)$ iff $\langle b_1, ..., b_n\rangle \in f^P(\{a_1, ..., a_m\})$ iff
$\langle b_1, ..., b_n\rangle \in f^P(\{a_1, ..., a_m, b_1, ..., b_n\})$ (since f^P is reflexive) iff
$(M, F) \models_C \left[\overline{a_1}, ..., \overline{a_m}, \overline{b_1}, ..., \overline{b_n}\right] P\left(\overline{b_1}, ..., \overline{b_n}\right)$ iff
$(M, F) \models_C [\overline{a_1}, ..., \overline{a_m}]\left[\overline{b_1}, ..., \overline{b_n}\right] P\left(\overline{b_1}, ..., \overline{b_n}\right)$. □

Lemma 19. $\vdash_C^T \subsetneq \vdash_C^R$.

Proof. By Lemmas 12, 17 and 18. □

4.3.3 Another Characterization of $\vdash_C^R$

The requirement that all cdfs must be reflexive reflects the intended use of tolerant predicates, such as "tall". This definition is also useful when proving properties of the consequence relation. However, generating R-structures for specific purposes is not a trivial task, as it requires to check that all of the proposed cdfs are reflexive. One way to overcome this obstacle is to use the function r that was defined in Definition 24. Another way will be presented in this section, namely, introducing a small change in the definition of the C satisfaction relation.

Theorem 3. *Let $\models_{CR}$ be the satisfaction relation obtained from $\models_C$ by introducing the following change: $(M,F) \models_{CR} [s_1,...,s_m]\, P(t_1,...,t_n)$ if $\langle I(t_1),...,I(t_n)\rangle \in f^P(\{I(t_1),...,I(t_n),I(s_1),...,I(s_m)\})$ for $m \geq 0$. Let $\vdash_{CR}$ be the consequence relation induced by $\models_{CR}$ over C-structures (i.e. $T \vdash_{CR} \varphi$ if for every C-structure (M,F), if $(M,F) \models_{CR} T$ then $(M,F) \models_{CR} \varphi$). Then $\vdash_C^R = \vdash_{CR}$.*

Proof. We introduce some new definitions and lemmas:

Lemma. *Let (M,F) be a C-*structure *for L, t a ground-term of $L(M)$ and $\overline{a}$ the name of $I(t)$. If φ is a formula of $L(M)$ in which no variable except x is free, then $(M,F) \models_{CR} \varphi\{t/x\}$ iff $(M,F) \models_{CR} \varphi\{\overline{a}/x\}$.*

Proof. This is proved in a similar manner to the proof of Lemma 7.

Lemma. *Let (M,F) be a C-structure and let φ be a sentence. Then $(M,F) \models_{CR} \varphi$ iff $(M,r(F)) \models_C \varphi$.*

Proof. By induction on the complexity of φ.

1. If $\varphi = [s_1,...,s_m]\, P(t_1,...,t_n)$ (for $m \geq 0$) then $(M,F) \models_{CR} \varphi$
   iff $\langle I(t_1),...,I(t_n)\rangle \in f^P(\{I(s_1),...,I(s_m)\} \cup \{I(t_1),...,I(t_n)\})$
   iff $\langle I(t_1),...,I(t_n)\rangle \in r(f^P)(\{I(s_1),...,I(s_m)\})$ iff $(M,r(F)) \models_C \varphi$.
2. Otherwise, the definitions of the two satisfaction relation are defined identically.

Corollary. *Let (M,F) be a C-structure and let φ be a formula. Then $(M,F) \models_{CR} \varphi$ iff $(M,r(F)) \models_C \varphi$.*

End of proof of Theorem 3. Let T be a theory and φ be a formula. Assume that $T \vdash_C^R \varphi$. Let (M,F) be a C structure such that $(M,F) \models_{CR} T$. We prove that $(M,F) \models_{CR} \varphi$. By the above corollary, $(M,r(F)) \models_C T$, and hence by our assumption $(M,r(F)) \models_C \varphi$. Again, by the above corollary, $(M,F) \models_{CR} \varphi$. Now, assume that $T \vdash_{CR} \varphi$. Let (M,F) be an R-structure such that $(M,F) \models_C T$. We prove that $(M,F) \models_C \varphi$. Note that (M,F) is an R-structure, and hence for every P, f^P is reflexive. By Lemma 14, for every P, $r(f^P) = f^P$. This means that $r(F) = F$. Hence, by the above corollary, $(M,F) \models_{CR} T$. By our assumption, $(M,F) \models_{CR} \varphi$. Again, by the above corollary and by the fact that $(M,r(F)) = (M,F)$, $(M,F) \models_C \varphi$.

□

4.3.4 Relation with GS-Semantics

Our construction has led us to defining $\vdash_C^R$. In this section, we show that this consequence relation is the same consequence relation as the one presented in [Ga-2010] ($\vdash_{GS}^S$), hence providing another semantical characterization of the original semantics for tolerance. We start with a preliminary lemma:

Lemma 20. *Let (M,F) be an S-structure, $P \in PRED(L)$ of arity n, $X \in FIN(|M|)$ and $Y, Y' \in FIN^+(|M|)$. Then $\left(F_{Y_{Y'}}(P)\right)(X) = (F_{Y \cup Y'}(P))(X)$.*

Proof.

$$\begin{aligned}\left(F_{Y_{Y'}}(P)\right)(X) &= (F_Y(P))\left(X \cup Y'\right) \cap X^n = (F(P))\left(X \cup Y' \cup Y\right) \cap \left(X \cup Y'\right)^n \cap X^n \\ &= (F(P))\left(X \cup Y \cup Y'\right) \cap X^n = (F_{Y \cup Y'}(P))(X)\end{aligned}$$

□

Now we prove that $\vdash^S_{GS} \subseteq \vdash^R_C$.

Definition 28. Let $P \in PRED(L)$ be a predicate of arity n and let (M,F) be a C-structure. $F^*(P) = \lambda X \in FIN(|M|).F(P)(X) \cap X^n$.

Lemma 21. *(M,F) be an R-structure. Then (M,F^*) is an S-structure.*

Lemma 22. *Let φ be a sentence and let (M,F) be an R-structure. Then $(M,F) \models_C \varphi$ iff $(M,F^*) \models_{GS} \varphi$.*

Proof. By induction on the complexity of φ.

1. If φ is $P(t_1,...,t_n)$ then: $(M,F) \models_C P(t_1,...,t_n)$ iff $\langle I(t_1),...,I(t_n)\rangle \in F(P)(\{I(t_1),...,I(t_n)\})$ iff $\langle I(t_1),...,I(t_n)\rangle \in F(P)(\{I(t_1),...,I(t_n)\}) \cap \{I(t_1),...,I(t_n)\}^n$ iff $\langle I(t_1),...,I(t_n)\rangle \in (F^*(P))(\{I(t_1),...,I(t_n)\})$ iff $(M,F^*) \models_{GS} P(t_1,...,t_n)$.

2. If φ is $\psi \to \phi$, $\neg\psi$ or $\forall x\psi$ then the proof is usual.

3. If φ is $[s_1,...,s_m]\psi$ for $m > 0$, then the proof is by an inner induction on the complexity of ψ:

   (a) If ψ is $P(t_1,...,t_n)$ then: $(M,F) \models_C [s_1,...,s_m]P(t_1,...,t_n)$ iff $\langle I(t_1),...,I(t_n)\rangle \in F(P)(\{I(s_1),...,I(s_m)\})$ iff $\langle I(t_1),...,I(t_n)\rangle \in F(P)(\{I(s_1),...,I(s_m),I(t_1),...,I(t_n)\})$ (since $F(P)$ is reflexive) iff $\langle I(t_1),...,I(t_n)\rangle \in F(P)(\{I(s_1),...,I(s_m),I(t_1),...,I(t_n)\}) \cap \{I(s_1),...,I(s_m),I(t_1),...,I(t_n)\}^n$ iff $\langle I(t_1),...,I(t_n)\rangle \in F^*(P)(\{I(s_1),...,I(s_m),I(t_1),...,I(t_n)\})$ iff $\langle I(t_1),...,I(t_n)\rangle \in F^*(P)(\{I(s_1),...,I(s_m),I(t_1),...,I(t_n)\}) \cap \{I(t_1),...,I(t_n)\}^n = (F^*)_Y(P)(\{I(t_1),...,I(t_n)\})$ with $Y = \{I(s_1),...,I(s_m)\}$ iff $(M,(F^*)_Y) \models_{GS} P(t_1,...,t_n)$ iff $(M,F^*) \models_{GS} [s_1,...,s_m]P(t_1,...,t_n)$.

   (b) If ψ is $\psi_1 \to \psi_2$, $\neg\phi$ or $\forall x\phi$, then the proof is usual. We show the case where ψ is $\forall x\phi$ as an example: $(M,F) \models_C [s_1,...,s_m]\forall x\phi$ iff for every name $\overline{a}$ $(M,F) \models_C [s_1,...,s_m]\phi\{\overline{a}/x\}$ iff for every name $\overline{a}$ $(M,F^*) \models_{GS} [s_1,...,s_m]\phi\{\overline{a}/x\}$ (by the i.h.) iff for every name $\overline{a}$ $(M,(F^*)_Y) \models_{GS} \phi\{\overline{a}/x\}$ with $Y = \{I(s_1),...,I(s_m)\}$ iff $(M,(F^*)_Y) \models_{GS} \forall x\phi$ iff $(M,F^*) \models_{GS} [s_1,...,s_m]\forall x\phi$.

   (c) If ψ is $[t_1,...,t_n]\phi$ then: $(M,F) \models_C [s_1,...,s_m][t_1,...,t_n]\phi$ iff $(M,F) \models_C [s_1,...,s_m,t_1,...,t_n]\phi$ iff $(M,F^*) \models_{GS} [s_1,...,s_m,t_1,...,t_n]\phi$ (by the i.h.) iff $(M,(F^*)_{Y \cup Y'}) \models_{GS} \phi$ with $Y = \{I(s_1),...,I(s_m)\}$ and $Y' = \{I(t_1),...,I(t_n)\}$. By Lemma 20, this holds iff $\left(M,(F^*)_{Y_{Y'}}\right) \models_{GS} \phi$ iff $(M,(F^*)_Y)[t_1,...,t_n]\phi$ iff $(M,F^*) \models_{GS} [s_1,...,s_m][t_1,...,t_n]\phi$.

□

Corollary 2. *Let φ be a formula and let (M,F) be an R-structure. Then $(M,F) \models_C \varphi$ iff $(M,F^*) \models_{GS} \varphi$.*

Lemma 23. $\vdash^S_{GS} \subseteq \vdash^R_C$.

Proof. Let T be a theory and let φ be a formula. Assume $T \vdash^S_{GS} \varphi$. Let (M,F) be an R-structure and assume $(M,F) \models_C T$. Then by Corollary 2 $(M,F^*) \models_{GS} T$. By our assumption, $(M,F^*) \models_{GS} \varphi$, and again by Corollary 2, $(M,F) \models_C \varphi$. Hence $T \vdash^R_C \varphi$. □

Now we turn to prove the other direction.

Lemma 24. *Let φ be a sentence and let (M,F) be an S-structure. Then $(M,F) \models_{GS} \varphi$ iff $(M,r(F)) \models_C \varphi$.*

Proof. By induction on the complexity of φ.

1. If φ is $P(t_1,...,t_n)$ then: $(M,F) \models_{GS} P(t_1,...,t_n)$ iff $\langle I(t_1),...,I(t_n)\rangle \in F(P)(\{I(t_1),...,I(t_n)\})$ iff $\langle I(t_1),...,I(t_n)\rangle \in \{\langle a_1,...,a_n\rangle \mid \langle a_1,...,a_n\rangle \in F(P)(\{I(t_1),...,I(t_n)\} \cup \{a_1,...,a_n\})\}$ iff $\langle I(t_1),...,I(t_n)\rangle \in ((r(F))(P))(\{I(t_1),...,I(t_n)\})$ iff $(M,r(F)) \models_C P(t_1,...,t_n)$.

2. If φ is $\psi \to \phi$, $\neg\psi$ or $\forall x\psi$ then the proof is similar to the proof of Lemma 22.

3. If φ is $[s_1,...,s_m]\psi$ for $m > 0$ then we prove this by an inner induction on the complexity of ψ:

   (a) If ψ is $P(t_1,...,t_n)$ then: $(M,F) \models_{GS} [s_1,...,s_m]P(t_1,...,t_n)$ iff $(M,F_Y) \models_{GS} P(t_1,...,t_n)$ with $Y = \{I(s_1),...,I(s_m)\}$ iff $\langle I(t_1),...,I(t_n)\rangle \in F_Y(P)(\{I(t_1),...,I(t_n)\})$ iff $\langle I(t_1),...,I(t_n)\rangle \in F(P)(\{I(t_1),...,I(t_n),I(s_1),...,I(s_m)\}) \cap \{I(t_1),...,I(t_n)\}^n$ iff $\langle I(t_1),...,I(t_n)\rangle \in \{\langle a_1,...,a_n\rangle \mid \langle a_1,...,a_n\rangle \in F(P)(\{I(s_1),...,I(s_m)\} \cup \{a_1,...,a_n\})\}$ iff $\langle I(t_1),...,I(t_n)\rangle \in r(F)(P)(\{I(s_1),...,I(s_m)\})$ iff $(M,r(F)) \models_C [s_1,...,s_m]P(t_1,...,t_n)$.

   (b) If ψ is $\neg\phi$ then: $(M,r(F)) \models_C [s_1,...,s_m]\neg\phi$ iff $(M,r(F)) \not\models_C [s_1,...,s_m]\phi$ iff $(M,F) \not\models_{GS} [s_1,...,s_m]\phi$ (by the i.h.) iff $(M,F_Y) \not\models_{GS} \phi$ with $Y = \{I(s_1),...,I(s_m)\}$ iff $(M,F_Y) \models_{GS} \neg\phi$ iff $(M,F) \models_{GS} [s_1,...,s_m]\neg\phi$.

   (c) If ψ is $\psi_1 \to \psi_2$ then the proof is similar.

   (d) If ψ is $\forall x\phi$ then: $(M,r(F)) \models_C [s_1,...,s_m]\forall x\phi$ iff for every $a \in |M|$, $(M,r(F)) \models_C [s_1,...,s_m]\phi\{\overline{a}/x\}$ iff for every $a \in |M|$ $(M,F) \models_{GS} [s_1,...,s_m]\phi\{\overline{a}/x\}$ (by the i.h.) iff for every $a \in |M|$ $(M,F_Y) \models_{GS} \phi\{\overline{a}/x\}$ with $Y = \{I(s_1),...,I(s_m)\}$ iff $(M,F_Y) \models_{GS} \forall x\phi$ iff $(M,F) \models_{GS} [s_1,...,s_m]\forall x\phi$.

   (e) If ψ is $[t_1,...,t_k]\phi$ for $k > 0$ then: $(M,r(F)) \models_C [s_1,...,s_m][t_1,...,t_k]\phi$ iff $(M,r(F)) \models_C [s_1,...,s_m,t_1,...,t_k]\phi$ iff $(M,F) \models_{GS} [s_1,...,s_m,t_1,...,t_k]\phi$ (by the i.h.) iff $(M,F_{Y\cup Y'}) \models_{GS} \phi$ with $Y = \{I(s_1),...,I(s_m)\}$ and $Y' = \{I(t_1),...,I(t_k)\}$ iff $\left(M,F_{Y_{Y'}}\right) \models_{GS} \phi$ (by Lemma 20) iff $(M,F_Y) \models_{GS} [t_1,...,t_k]\phi$ iff $(M,F) \models_{GS} [s_1,...,s_m][t_1,...,t_k]\phi$.

□

Corollary 3. *Let φ be a formula and let (M,F) be an S-structure. Then $(M,F) \models_{GS} \varphi$ iff $(M,r(F)) \models_C \varphi$.*

Lemma 25. $\vdash^R_C \subseteq \vdash^S_{GS}$.

Proof. Similar to the proof of Lemma 23. □

Theorem 4. $\vdash_C^R = \vdash_{GS}^S$.

Proof. By Lemmas 23 and 25. □

4.4 Context Independent Predicates

Similarly to Section 3.4, Now we study context independent predicates in C-semantics.

Definition 29. Let R be a predicate and let (M, F) be a C-structure. We say that R is *context independent* in (M, F) if f^R is a constant function.

Lemma 26. *Let (M, F) be a C-structure. Then R is context independent in (M, F) iff for every C $(M, F) \models_C R(x_1, ..., x_n) \leftrightarrow [C] R(x_1, ..., x_n)$.* [9]

Proof. Assume R is context independent in (M, F). Let $R(\overline{a_1}, ..., \overline{a_n}) \leftrightarrow [t_1, ..., t_m] R(\overline{a_1}, ..., \overline{a_n})$ be an M-instance of $R(x_1, ..., x_n) \leftrightarrow [C] R(x_1, ..., x_n)$. $(M, F) \models_C R(\overline{a_1}, ..., \overline{a_n})$ iff $\langle a_1, ..., a_n \rangle \in f^R(\emptyset) = f^R(\{I(t_1), ..., I(t_m)\})$ (since f^R is constant) iff $(M, F) \models_C [t_1, ..., t_m] R(\overline{a_1}, ..., \overline{a_n})$.
Now assume $(M, F) \models_C R(x_1, ..., x_n) \leftrightarrow [C] R(x_1, ..., x_n)$.
Let $\{b_1, ..., b_m\} \neq \{c_1, ..., c_k\} \in FIN(|M|)$ and let $a_1, ..., a_n \in |M|$. Then $(M, F) \models_C R(\overline{a_1}, ..., \overline{a_n}) \leftrightarrow [\overline{b_1}, ..., \overline{b_m}] R(\overline{a_1}, ..., \overline{a_n})$ and $(M, F) \models_C R(\overline{a_1}, ..., \overline{a_n}) \leftrightarrow [\overline{c_1}, ..., \overline{c_k}] R(\overline{a_1}, ..., \overline{a_n})$. This means that $(M, F) \models_C [\overline{b_1}, ..., \overline{b_m}] R(\overline{a_1}, ..., \overline{a_n}) \leftrightarrow [\overline{c_1}, ..., \overline{c_k}] R(\overline{a_1}, ..., \overline{a_n})$. Therefore, $(M, F) \models_C [\overline{b_1}, ..., \overline{b_m}] R(\overline{a_1}, ..., \overline{a_n})$ iff $(M, F) \models_C [\overline{c_1}, ..., \overline{c_k}] R(\overline{a_1}, ..., \overline{a_n})$ and hence $\langle a_1, ..., a_n \rangle \in f^R(\{b_1, ..., b_m\})$ iff $\langle a_1, ..., a_n \rangle \in f^R(\{c_1, ..., c_k\})$. This is true for every $\{b_1, ..., b_m\} \neq \{c_1, ..., c_k\} \in FIN(|M|)$ and $a_1, ..., a_n \in |M|$ and hence f^R is a constant function, which means that R is context independent in (M, F). □

4.5 Summary

The main theme of this section was a modular construction of a new consequence relation, $\vdash_C^R$, which is equivalent to Gaifman's semantical consequence relation $\vdash_{GS}^S$. We started with the most basic consequence relation $\vdash_C$. This consequence relation is induced by the $\models_C$ satisfaction relation over C-structures. By restricting it to a special sub-class of C-structures (T-structures) we obtained a new, different consequence relation $\vdash_C^T$. This consequence relation will later be proved to be equivalent to the consequence relation which is induced by Gaifman's original deductive system. Finally, restricting $\models_C$ to a sub-class of T-structures (R-structures) induced $\vdash_C^R$. $\vdash_C^R$ was proved to be equivalent to $\vdash_{GS}^S$. Later on, we shall provide an adequate deductive system for it.

5 Corresponding Deductive Systems

In this section we present several deductive systems adequate for the semantics proposed in Section 4.

Throughout this section, L is a C-language.

[9] Recall that this is equivalent to IND_R (Definition 16) being valid in (M, F).

5.1 The Systems

All of the discussed deductive systems are obtained from $HFOL$ by the addition of some new axiom schemes. The new axiom schemes that will be used are:

- (CC) $[C]\,[C']\,\varphi \leftrightarrow [C,C']\,\varphi$.
- $(C_\neg)$ $[C]\,\neg\varphi \leftrightarrow \neg\,[C]\,\varphi$.
- $(C_\rightarrow)$ $[C]\,(\varphi \rightarrow \psi) \leftrightarrow ([C]\,\varphi \rightarrow [C]\,\psi)$.
- $(C_\forall)$ $[C]\,\forall x\varphi \leftrightarrow \forall x\,[C]\,\varphi$ if x doesn't occur in C.
- (T) $P\,(x_1,...,x_n) \leftrightarrow [x_1,...,x_n]\,P\,(x_1,...,x_n)$ for every P.
- (R) $[C]\,(P\,(x_1,...,x_n) \leftrightarrow [x_1,...,x_n]\,P\,(x_1,...,x_n))$ for every context set C for every predicate P.

Definition 30.

1. $HC = HFOL + CC + C_\neg + C_\rightarrow + C_\forall$.
2. $HCT = HC + T$.
3. $HCR = HC + R$.

Remark 6. HCT is the deductive system from [Ga-2010], that was introduced in Definition 15. It will be shown that it is complete for $\vdash^T_C$. HCR will be proved to be complete for $\vdash^S_{GS}$ (and $\vdash^R_C$). HC is the most basic deductive system, and it will proved to be equivalent to $\vdash_C$, which is the most basic semantical consequence relation.

Lemma 27. *Let C be a context set and let φ and ψ be formulas. Let $\# \in \{\wedge, \vee, \leftrightarrow\}$. Then $\vdash_{HC} [C]\,(\varphi\#\psi) \leftrightarrow ([C]\,\varphi\#\,[C]\,\psi)$.*

Proof. By axioms $C_\neg$ and $C_\rightarrow$. □

5.2 Soundness Theorems

In this section we prove the soundness of the above deductive systems with respect to the semantics which were presented in Section 4. Before doing so, we prove a lemma which will be useful for this purpose:

Lemma 28. *Let φ be one of the axioms of HC. Let (M,F) be a C-*structure. *Let φ' be an M-instance of φ. Then φ' is an instance of the same axiom scheme as φ.*

Proof. We assume that the free variables of φ are: $x_1,...,x_n$.

1. If φ is an instance of a propositional axiom, then this can be shown easily. For example, suppose $\varphi = \psi \rightarrow (\phi \rightarrow \psi)$. Then $\varphi' = (\psi \rightarrow (\phi \rightarrow \psi))\,\{\overline{a_1}/x_x,...,\overline{a_n}/x_n\}$ for some names $\overline{a_1},...,\overline{a_n}$, i.e. φ' is
$\psi\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\} \rightarrow (\phi\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\} \rightarrow \psi\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$, which is indeed an instance of the same axiom scheme.
2. If φ is an instance of a first-order axiom, then:

(a) If $\varphi = \forall x\psi \to \psi\{t/x\}$ where t is substitutable for x in ψ, then: Assume first that for every $1 \le i \le n$ $x_i \ne x$. In that case, φ' is $\forall x\psi\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\} \to \psi\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\}\{t\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\}/x\}$ for some $\overline{a_1}, ..., \overline{a_n}$, which is an instance of the same axiom scheme as φ. Now, if $x_k = x$ for some $1 \le k \le n$, then without loss of generality $k = 1$ and φ' is

$$\forall x\psi\{x, \overline{a_2}/x_2..., \overline{a_n}/x_n\} \to \psi\{t\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\}/x, \overline{a_2}/x_2, ..., \overline{a_n}/x_n\}$$

which is an instance of the same axiom scheme as φ. If there is more then one k for which $x_k = x$ then the proof is similar.

(b) If $\varphi = \forall x(\psi \to \phi) \to (\psi \to \forall x\phi)$ where x isn't free in ψ, then for every $1 \le i \le n$, $x_i \ne x$ (x isn't free in ψ, and therefore any occurrence of it in φ is bound). Let $\overline{a_1}, ..., \overline{a_n}$ be names. We denote $\psi\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\}$ by ψ^i and $\phi\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\}$ by ϕ^i. Then for some $\overline{a_1}, ..., \overline{a_n}$, φ' is $\forall x\left(\psi^i \to \phi^i\right) \to \left(\psi^i \to \forall x\phi^i\right)$, which is an instance of the same axiom scheme as φ.

3. If φ is an instance of one of the new axiom schemes of HC, then again, this can be shown easily. For example, suppose $\varphi = [C]\forall x\psi \leftrightarrow \forall x[C]\psi$ where x doesn't occur in C. Denote $C\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\}$ by C'. Then φ' is
$[C']\forall x\psi\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\} \leftrightarrow \forall x[C']\psi\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\}$ for some $\overline{a_1}, ..., \overline{a_n}$, since x doesn't occur in C'. Indeed φ' is an instance of the same axiom scheme as φ.

$\square$

Theorem 5. *Let T be a theory and let φ be a formula.*

1. *Assume $T \vdash_{HC} \varphi$. Then $T \vdash_C \varphi$.*
2. *Assume $T \vdash_{HCT} \varphi$. Then $T \vdash_C^T \varphi$.*
3. *Assume $T \vdash_{HCR} \varphi$. Then $T \vdash_C^R \varphi$.*

Proof.

1. By induction on φ's proof from T in HC.

   (a) If $\varphi \in T$ then by definition $T \vdash_{HC} \varphi$.

   (b) If φ is an instance of an axiom of the system then we need to prove that for every (M, F) and for every M-instance of φ φ',$(M, F) \models_C \varphi'$. Let (M, F) be a C-structure and let φ' be an M-instance of φ. By Lemma 28, we may assume that φ' is an instance of the same axiom scheme as φ. There are two possibilities:

   φ' is a propositional or first-order axiom In this case, this is shown similarly to the usual proof of the soundness theorem for $HFOL$.

   φ' is a new axiom In this case, this is true by Lemma 8.

   (c) If φ was derived by Modus Ponens or by the generalization rule, then this is done as in the proof of the soundness theorem for $HFOL$.

2. By Item 1, it is left to show that for every T-structure (M, F),
$(M, F) \models_C P(x_1, ..., x_n) \leftrightarrow [x_1, ..., x_n]P(x_1, ..., x_n)$. This immediately follows from Lemma 11.

3. By Item 2, it is left to show that for every R-structure (M, F),
$(M, F) \models_C [C](P(x_1, ..., x_n) \leftrightarrow [x_1, ..., x_n]P(x_1, ..., x_n))$. This immediately follows from Lemma 18.

$\square$

5.3 Completeness Theorems

Theorem 6. *Let T be an L-theory and let φ be an L-formula.*

1. *Assume $T \vdash_C \varphi$. Then $T \vdash_{HC} \varphi$.*
2. *Assume $T \vdash_C^T \varphi$. Then $T \vdash_{HCT} \varphi$.*
3. *Assume $T \vdash_C^R \varphi$. Then $T \vdash_{HCR} \varphi$.*

We prove this theorem in a similar manner to the usual proof of the completeness theorem for $HFOL$. We will quote results from that proof and explicitly prove the required modifications.

Lemma 29. *Let $\vdash \in \left\{\vdash_{HC}, \vdash_{HCT}, \vdash_{HCR}, \vdash_C, \vdash_C^T, \vdash_C^R\right\}$. Then $T \vdash \varphi$ iff $\forall T \vdash \forall \varphi$.*

The inner structure of the formulas doesn't affect the classical proof of this lemma. Syntactically, this is justified by the generalization rule of inference and the axiom $\forall x A \rightarrow A\{t/x\}$.

Therefore, as in the usual proof of the completeness theorem, we may assume that $T \cup \{\varphi\}$ consists of sentences.

We continue with the proof:

Let $\vdash \in \{\vdash_{HC}, \vdash_{HCT}, \vdash_{HCR}\}$ and assume that in L, $T \nvdash \varphi$.

Let L' be the language obtained from L by the addition of $\aleph_0$ new constants (namely $d_1, d_2, ...$).

Lemma 30. *In L', $T \nvdash \varphi$ still holds.*

Lemma 31. *There exists an L' theory $T^{\vdash}$ which satisfies the following requirements:*

1. *$T \subseteq T^{\vdash}$.*
2. *$T^{\vdash}$ is a maximal theory in L' (in terms of inclusion) such that $T^{\vdash} \nvdash \varphi$, i.e. $T^{\vdash} \nvdash \varphi$ and for each $\psi \notin T^{\vdash}$ $T^{\vdash} \cup \{\psi\} \vdash \varphi$.*
3. *T' is a Henkin theory, i.e. if $T' \vdash \exists x \psi$ then there is a term t such that $T' \vdash \psi\{t/x\}$.*

We construct $T^{\vdash}$ in the usual way.

Lemma 32. *$T^{\vdash}$ is $\vdash$complete, i.e. for every sentence A, either $T^{\vdash} \vdash A$ or $T^{\vdash} \vdash \neg A$.*

Lemma 33. *For every sentence ψ, $T^{\vdash} \vdash \psi$ if and only if $\psi \in T^{\vdash}$.*

Definition 31. (Countermodel) Let $(M, F)^{\vdash}$ be the following C-structure:

1. $|M|$ is Herbrand space of L', i.e. $|M|$ consists of all ground terms of L'.
2. I is defined as in the usual proof.
3. For every P of arity m,

$$f^P \quad = \quad \lambda X \in FIN(|M|) \,.\, \left\{\langle s_1, ..., s_m\rangle : [X]\, P(s_1, ..., s_m) \in T^{\vdash}\right\}$$

Remark 7. $f^P(\emptyset) = \left\{\langle s_1, ..., s_m\rangle :\ P(s_1, ..., s_m) \in T^{\vdash}\right\}$.

Lemma 34. *For every sentence ϕ, $(M,F)^{\vdash} \models_C \phi$ if and only if $\phi \in T^{\vdash}$.*

Proof. For convenience, denote $(M,F)^{\vdash}$ by (M,F). The proof is by induction on the complexity of ϕ. Let $n \geq 0$. Recall that $[\,]\,\psi = \psi$ for every ψ.

1. If $\phi = [t_1,...,t_n]\,P(s_1,...,s_m)$ then: $(M,F) \models_C [t_1,...,t_n]\,P(s_1,...,s_m)$ iff $\langle I(s_1),...,I(s_m)\rangle \in f^P(\{I(t_1),...,I(t_n)\})$ (by definition) iff $\langle s_1,...,s_m\rangle \in f^P(\{t_1,...,t_n\})$ (by I's construction) iff $[t_1,...,t_n]\,P(s_1,...,s_m) \in T^{\vdash}$ (by f's construction).

2. If $\phi = [t_1,...,t_n]\,\neg\phi_1$ then: $(M,F) \models_C [t_1,...,t_n]\,\neg\phi_1$ iff $(M,F) \not\models_C [t_1,...,t_n]\,\phi_1$ iff $[t_1,...,t_n]\,\phi_1 \notin T^{\vdash}$ (by the i.h.) iff $\neg[t_1,...,t_n]\,\phi_1 \in T^{\vdash}$ (since $T^{\vdash}$ is complete and since $\vdash_{HC} \neg\neg A \rightarrow A$) iff $[t_1,...,t_n]\,\neg\phi_1 \in T^{\vdash}$ (by axiom $C_{\neg}$).

3. If $\phi = [t_1,...,t_n]\,(\phi_1 \rightarrow \phi_2)$ then the proof is similar, using axiom $C_{\rightarrow}$.

4. If $\phi = [t_1,...,t_n]\,(\forall x\phi_1)$ then: Assume $(M,F) \models_C [t_1,...,t_n]\,\forall x\phi_1$. Then for every ground term t of L', $(M,F) \models_C [t_1,...,t_n]\,\phi_1\{t/x\}$. By the i.h. it follows that for every such t, $[t_1,...,t_n]\,\phi_1\{t/x\} \in T^{\vdash}$. Now assume for contradiction that $[t_1,...,t_n]\,\forall x\phi_1 \notin T^{\vdash}$. By axiom $C_{\forall}$ and the fact that $t_1,...,t_n$ don't contain x (as they are ground terms), this means that $\forall x\,[t_1,...,t_n]\,\phi_1 \notin T^{\vdash}$. Since $T^{\vdash}$ is complete, $\neg\forall x\,[t_1,...,t_n]\,\phi_1 \in T^{\vdash}$, which means that $\exists x\neg\,[t_1,...,t_n]\,\phi_1 \in T^{\vdash}$. Since $T^{\vdash}$ is a Henkin theory, there exists a ground term t' such that $(\neg\,[t_1,...,t_n]\,\phi_1)\{t'/x\} \in T^{\vdash}$, which means that $\neg\,[t_1,...,t_n]\,\phi_1\{t'/x\} \in T^{\vdash}$. Hence $T^{\vdash}$ is inconsistent and in particular, $\varphi \in T^{\vdash}$, which is a contradiction. For the other direction, assume $[t_1,...,t_n]\,\forall x\phi_1 \in T^{\vdash}$. By $C_{\forall}$ (and the fact that x doesn't occur in $t_1,...,t_n$), $\forall x\,[t_1,...,t_n]\,\phi_1 \in T^{\vdash}$. Using the axiom $\forall xA \rightarrow A\{t/x\}$ for every ground term t of L', this means that for every such t, $([t_1,...,t_n]\,\phi_1)\{t/x\} \in T^{\vdash}$, and by the i.h., it holds that for every such t, $(M,F) \models_C [t_1,...,t_n]\,\phi_1\{t/x\}$. Since $|M|$ is the set of ground terms of L', $(M,F) \models_C \forall x\,[t_1,...,t_n]\,\phi_1$. By Lemma 8, $(M,F) \models_C [t_1,...,t_n]\,\forall x\phi_1$.

5. If $\phi = [t_1,...,t_n]\,[s_1,...,s_m]\,\phi_1$ then: $(M,F) \models_C [t_1,...,t_n]\,[s_1,...,s_m]\,\phi_1$ iff $(M,F) \models_C [t_1,...,t_n,s_1,...,s_m]\,\phi_1$ iff $[t_1,...,t_n,s_1,...,s_m]\,\phi_1 \in T^{\vdash}$ (by the i.h.) iff $[t_1,...,t_n]\,[s_1,...,s_m]\,\phi_1 \in T^{\vdash}$ (by axiom CC).

□

Corollary 4. *$(M,F)^{\vdash} \models_C T$ and $(M,F)^{\vdash} \not\models_C \varphi$.*

Lemma 35. *Let P be any predicate of arity k and let $t_1,...,t_n,s_1,...,s_k$ be terms. Then $\vdash_{HCR} [t_1,...,t_n]\,P(s_1,...,s_k) \leftrightarrow [t_1,...,t_n,s_1,...,s_k]\,P(s_1,...,s_k)$.*

Proof. $\vdash_{HCR} [t_1,...,t_n]\,(P(s_1,...,s_k) \leftrightarrow [s_1,...,s_k]\,P(s_1,...,s_k))$ (axiom CT).
$\vdash_{HCR} [t_1,...,t_n]\,P(s_1,...,s_k) \leftrightarrow [t_1,...,t_n]\,[s_1,...,s_k]\,P(s_1,...,s_k)$ (Lemma 27).
$\vdash_{HCR} [t_1,...,t_n]\,[s_1,...,s_k] \leftrightarrow [t_1,...,t_n,s_1,...,s_k]\,P(s_1,...,s_k)$ (axiom CC).
And hence $\vdash_{HCR} [t_1,...,t_n]\,P(s_1,...,s_k) \leftrightarrow [t_1,...,t_n,s_1,...,s_k]\,P(s_1,...,s_k)$. □

Lemma 36.

1. *$(M,F)^{\vdash_{HCT}}$ is a T-structure.*

2. *$(M,F)^{\vdash_{HCR}}$ is an R-structure.*

Proof. Let P be an n-ary predicate.

1. Denote $(M,F)^{\vdash_{HCT}}$ as (M,F) and $T^{\vdash_{HCT}}$ as T'. Let $t_1,...,t_n \in |M|$. $\langle t_1,...,t_n\rangle \in f^P(\{\emptyset\})$ iff $P(t_1,...,t_n) \in T'$ (by F's construction) iff $[t_1,...,t_n]\,P(t_1,...,t_n) \in T'$ (by axiom T) iff $\langle t_1,...,t_n\rangle \in f^P(\{t_1,...,t_n\})$ (by F's construction).

2. Denote $(M,F)^{\vdash_{HCR}}$ as (M,F) and $T^{\vdash_{HCR}}$ as T'. Let $t_1,...,t_n,s_1,...,s_m \in |M|$. $\langle t_1,...,t_n\rangle \in f^P(\{s_1,...,s_m\})$ iff $[s_1,...,s_m]\,P(t_1,...,t_n) \in T'$ (by F's construction) iff $[s_1,...,s_m,t_1,...,t_n]\,P(t_1,...,t_n) \in T'$ (by Lemma 35) iff $\langle t_1,...,t_n\rangle \in f^P(\{s_1,...,s_m,t_1,...,t_n\})$ (by F's construction).

□

End of proof of Theorem 6.

1. Assume $T \nvdash_{HC} \varphi$. By Corollary 4, there exists a C-structure, namely $(M,F)^{\vdash_{HC}}$ such that $(M,F)^{\vdash_{HC}} \models_C T$ and $(M,F)^{\vdash_{HC}} \not\models_C \varphi$. Hence $T \nvdash_C \varphi$.

2. Assume $T \nvdash_{HCT} \varphi$. By Corollary 4 and Lemma 36, there exists a T-structure, namely $(M,F)^{\vdash_{HCT}}$ such that $(M,F)^{\vdash_{HCT}} \models_C T$ and $(M,F)^{\vdash_{HCT}} \not\models_C \varphi$. Hence $T \nvdash_C^T \varphi$.

3. Assume $T \nvdash_{HCR} \varphi$. By Corollary 4 and Lemma 36, there exists an R-structure, namely $(M,F)^{\vdash_{HCR}}$ such that $(M,F)^{\vdash_{HCR}} \models_C T$ and $(M,F)^{\vdash_{HCR}} \not\models_C \varphi$. Hence $T \nvdash_C^R \varphi$.

□

6 Properties of the New Consequence Relations

In this section we discuss some properties of the presented consequence relations. All of the properties will be formulated in terms of semantics (e.g. for $\vdash_C^T$ rather than for $\vdash_{HCT}$). However, by the soundness and completeness theorems from Sections 5.2 and 5.3 those properties have proof theoretical counterparts as well.

Throughout this section, L is a C-language.

6.1 Rule N

In [Ga-2010], it is shown that for every first-order theorem of HCT φ and every context C, $[C]\,\varphi$ is also a theorem of HCT. In this section we study this property for all of our consequence relations and for all theorems of our systems (not just first order ones). We start with some preliminary definitions and lemmas before the actual results.

Definition 32. Rule N is the following rule of inference: $\frac{\varphi}{[C]\varphi}$ for every context set C.

Definition 33. Let $\vdash$ be a consequence relation for L.

1. Rule N is *valid* in $\vdash$ if for every theory T, formula φ and context C, if $T \vdash \varphi$ then $T \vdash [C]\,\varphi$.

2. Rule N is *admissible* in $\vdash$ if the above holds for $T = \emptyset$.

Lemma 37. *Let M be a first order structure which doesn't interpret predicate symbols, let φ be an L-formula and let $t_1,...,t_n$ be terms. Assume that for every F such that (M,F) is an S-structure, $(M,F) \models_{GS} \varphi$. Then for every such F, $(M,F) \models_{GS} [t_1,...,t_n]\,\varphi$.*

Proof. Let $[s_1,...,s_n]\,\varphi'$ be an M-instance of $[t_1,...,t_n]\,\varphi$. Let $Y = \{I(s_1),...,I(s_n)\}$. Let F be a function as defined in Definition 9. Since $(M,F') \models_{GS} \varphi$ for every F', $(M,F') \models_{GS} \varphi'$ for every F'. In particular, $(M,F_Y) \models_{GS} \varphi'$, which means that $(M,F) \models_{GS} [s_1,...,s_n]\,\varphi'$.
□

Corollary 5. *Rule N is admissible in $\vdash^S_{GS}$: Let φ be a formula which is GS-valid in every S-structure and let $t_1, ..., t_n$ be terms. Then $[t_1, ..., t_n]\,\varphi$ is also GS-valid in every S-structure.*

Proof. Let (M, F) be an S-structure. Then for every F' such that (M, F') is an S-structure, $(M, F') \models_{GS} \varphi$. By Lemma 37, $(M, F') \models_{GS} [t_1, ..., t_n]\,\varphi$ for every such F' and in particular $(M, F) \models_{GS} [t_1, ..., t_n]\,\varphi$. □

Lemma 38. *Let φ be an axiom of HC. Then for every C, $\vdash_{HC} [C]\,\varphi$.*

Proof. It is enough to show that for every C, $\vdash_C [C]\,\varphi$. Let (M, F) be a C-structure and let C be a context set. Let $[C']\,\varphi'$ be an M-instance of $[C]\,\varphi$. By Lemma 28, φ' is an instance of the same axiom scheme as φ.

1. Assume $\varphi' = \psi'_1 \to (\psi'_2 \to \psi'_1)$. We need to show that $(M, F) \models_C [C']\,(\psi'_1 \to (\psi'_2 \to \psi'_1))$, i.e. that if $(M, F) \models_C [C']\,\psi'_1$ then $(M, F) \models_C [C']\,(\psi'_2 \to \psi'_1)$. Assume $(M, F) \models_C [C']\,\psi'_1$. In particular, it holds that either $(M, F) \models_C [C']\,\psi'_1$ or $(M, F) \not\models_C [C']\,\psi'_2$. Hence $(M, F) \models_C [C']\,(\psi'_2 \to \psi'_1)$.
2. If φ' is an instance of another propositional axiom, then this is shown similarly.
3. Assume $\varphi' = \forall x\psi' \to \psi'\,\{t'/x\}$ for some ground term t'. We need to show that $(M, F) \models_C [C']\,(\forall x\psi' \to \psi'\,\{t'/x\})$. Assume $(M, F) \models_C [C']\,\forall x\psi'$. Then for every $a \in |M|$, $(M, F) \models_C [C']\,\psi'\,\{\overline{a}/x\}$. It is easy to see that this means that for every ground term t, $(M, F) \models_C [C']\,\psi'\,\{t/x\}$, and in particular for t'.
4. If φ' is an instance of another first order axiom then this is shown similarly.
5. Assume $\varphi' = [A']\,[B']\,\psi' \leftrightarrow [A', B']\,\psi'$. For clarity, we explicitly write $[A]\,([B]\,\psi)$ instead of $[A]\,[B]\,\psi$. $(M, F) \models_C [C']\,([A']\,([B']\,\psi'))$ iff $(M, F) \models_C [C', A']\,([B']\,\psi')$ iff $(M, F) \models_C [C', A', B']\,\psi'$. $(M, F) \models_C [C']\,([A', B']\,\psi')$ iff $(M, F) \models_C [C', A', B']\,\psi'$.
6. Assume $\varphi' = [A']\,\neg\psi' \leftrightarrow \neg\,[A']\,\psi'$. We need to show that $(M, F) \models_C [C']\,[A']\,\neg\psi'$ iff $(M, F) \models_C [C']\,\neg\,[A']\,\psi'$. $(M, F) \models_C [C']\,[A']\,\neg\psi'$ iff $(M, F) \models_C [C', A']\,\neg\psi'$ iff $(M, F) \not\models_C [C', A']\,\psi'$. Now, $(M, F) \models_C [C']\,\neg\,[A']\,\psi'$ iff $(M, F) \not\models_C [C']\,[A']\,\psi'$ iff $(M, F) \not\models_C [C', A']\,\psi'$.
7. Assume $\varphi' = [A']\,(\psi'_1 \to \psi'_2) \leftrightarrow ([A']\,\psi'_1 \to [A']\,\psi'_2)$. We need to show that $(M, F) \models_C [C']\,[A']\,(\psi'_1 \to \psi'_2)$ iff $(M, F) \models_C [C']\,([A']\,\psi'_1 \to [A']\,\psi'_2)$.
$(M, F) \models_C [C']\,[A']\,(\psi'_1 \to \psi'_2)$ iff $(M, F) \models_C [C', A']\,(\psi'_1 \to \psi'_2)$ iff either $(M, F) \models_C [C', A']\,\psi'_2$ or $(M, F) \not\models_C [C', A']\,\psi'_1$. $(M, F) \models_C [C']\,([A']\,\psi'_1 \to [A']\,\psi'_2)$ iff either $(M, F) \models_C [C']\,[A']\,\psi'_2$ or $(M, F) \not\models_C [C']\,[A']\,\psi'_2$ iff either $(M, F) \models_C [C', A']\,\psi'_2$ or $(M, F) \not\models_C [C', A']\,\psi'_2$.
8. Assume $\varphi = [A]\,\forall x\psi \leftrightarrow \forall x\,[A]\,\psi$ where x doesn't occur in A. We need to show that $(M, F) \models_C [C']\,[A']\,\forall x\psi'$ iff $(M, F) \models_C [C']\,\forall x\,[A']\,\psi'$. $(M, F) \models_C [C']\,[A']\,\forall x\psi'$ iff $(M, F) \models_C [C', A']\,\forall x\psi'$ iff for every $a \in |M|$, $(M, F) \models_C [C', A']\,\psi'\{\overline{a}/x\}$. Now, $(M, F) \models_C [C']\,\forall x\,[A']\,\psi'$ iff for every $a \in |M|$, $(M, F) \models_C [C']\,[A']\,\psi'\,\{\overline{a}/x\}$ iff for every $a \in |M|$, $(M, F) \models_C [C', A']\,\psi'\,\{\overline{a}/x\}$.

□

We now turn to the actual results regarding rule N.

Lemma 39.

1. *Rule N is admissible in $\vdash_C$.*

2. *Rule N is admissible in* $\vdash_C^R$.

3. *Rule N isn't admissible in* $\vdash_C^T$.

Proof.

1. It is enough to show that Rule N is admissible in $\vdash_{HC}$. Let φ be a theorem of HC and let C be any context set. We prove that $[C]\,\varphi$ is also a theorem of HC by induction on the length of the proof of φ in HC.

   (a) If φ is an axiom of HC then by Lemma 38, $\vdash_{HC} [C]\,\varphi$.

   (b) If φ is proved from ψ and $\psi \to \varphi$ using $M.P$ then by the i.h., $\vdash_{HC} [C]\,\psi$ and $\vdash_{HC} [C]\,(\psi \to \varphi)$. By $C_\to$, $\vdash_{HC} [C]\,\psi \to [C]\,\varphi$. Using $M.P$ we get that $\vdash_{HC} [C]\,\varphi$.

   (c) If φ is proved from ψ using Gen then $\varphi = \forall x\psi$. By the i.h., $\vdash_{HC} [C]\,\psi$. First assume x doesn't appear in C. Using Gen, we get that $\vdash_{HC} \forall x\,[C]\,\psi$ and using $C_\forall$ we get that $\vdash_{HC} [C]\,\forall x\psi$. Now assume x appears in C. Assume w.l.g that $C = \{x_1, ..., x_n\}$ and that $x = x_1$. Let $y_1, ..., y_n$ be fresh variables. By the i.h. $\vdash_{HC} [y_1, ..., y_n]\,\psi$. Using Gen, we get that $\vdash_{HC} \forall x\,[y_1, ..., y_n]\,\psi$. Using $C_\forall$ we get that $\vdash_{HC} [y_1, ..., y_n]\,\forall x\psi$ (since $y_1, ..., y_n$ are fresh). Using Gen n more times, we get that $\vdash_{HC} \forall y_1 ... \forall y_n\,([y_1, ..., y_n]\,\forall x\psi)$. Now using the axiom $\forall xA \to A\,\{t/x\}$ of $HFOL$ n times we get that $\vdash_{HC} [C]\,\forall x\psi$.

2. By Corollary 5 and Theorem 4.

3. By Lemmas 11 and 12.

□

Lemma 40. *Rule N isn't valid in* $\vdash_C$ *and* $\vdash_C^R$.

Proof. For $\vdash_C^R$, since $\vdash_C^R = \vdash_{GS}^S$, it is enough to present an S-structure (M, F) and a formula φ such that $(M, F) \models_{GS} \varphi$ and $(M, F) \not\models_{GS} [C]\,\varphi$: $|M| = \{1, 2\}$, $I(c) = 1$, $I(d) = 2$. $f^P(\emptyset) = f^P(\{1\}) = f^P(\{2\}) = \emptyset$, $f^P(\{1, 2\}) = \{1\}$. $(M, F) \models_{GS} \neg P(c)$ since $I(c) = 1 \notin f^P(\{I(c)\}) = \emptyset$. Nevertheless, $(M, F) \not\models_{GS} [d]\,\neg P(c)$ since $(M, F_Y) \not\models_{GS} \neg P(c)$ with $Y = \{I(d)\}$ since $(M, F_Y) \models_{GS} P(c)$ since $I(c) = 1 \in f_Y^P(\{I(c)\}) = f(\{1, 2\}) \cap \{1\} = \{1\}$. For $\vdash_C$: We present a C-structure (M, F) and a formula φ such that $(M, F) \models_C \varphi$ and $(M, F) \not\models_C [C]\,\varphi$, namely the above S-structure. $(M, F) \models_C \neg P(c)$ since $I(c) = 1 \notin f^P(\emptyset) = \emptyset$. However, $(M, F) \not\models_C [c, d]\,\neg P(c)$ since $(M, F) \models [c, d]\,P(c)$ since $I(c) = 1 \in \{1\} = f^P(\{1, 2\}) = f^P(\{I(d), I(c)\})$. □

6.2 Axiom T

Recall that axiom T is $P(x_1, ..., x_n) \leftrightarrow [x_1, ..., x_n]\,P(x_1, ..., x_n)$. Another way of presenting this axiom is: $\varphi(x_1, ..., x_n) \leftrightarrow [x_1, ..., x_n]\,\varphi(x_1, ..., x_n)$ for every atomic φ. A natural question which arises is whether or not $\varphi(x_1, ..., x_n) \leftrightarrow [x_1, ..., x_n]\,\varphi(x_1, ..., x_n)$ is a theorem of the discussed deductive systems (or, equivalently - valid in the discussed semantic consequence relations).

Lemma 41. Let $\vdash \in \left\{\vdash_C, \vdash_C^T, \vdash_C^R\right\}$. Then it doesn't hold that for every $\varphi(x_1, ..., x_n)$ $\vdash \varphi(x_1, ..., x_n) \leftrightarrow [x_1, ..., x_n]\,\varphi(x_1, ..., x_n)$.

Proof. Since $\vdash_C \subsetneq \vdash_C^T \subsetneq \vdash_C^R$, It is enough to present a formula $\varphi(x_1, ..., x_n)$ such that $\nvdash_C^R \varphi(x_1, ..., x_n) \leftrightarrow [x_1, ..., x_n]\,\varphi(x_1, ..., x_n)$. Let $\varphi = P(x) \to P(y)$. We present an R-structure (M, F) such that $(M, F) \not\models_C (P(x) \to P(y)) \leftrightarrow [x, y]\,(P(x) \to P(y))$: $|M| =$

$\{1,2\}$, $f^P(\{1\}) = f^P(\{1,2\}) = \{1,2\}$, $f^P(\emptyset) = f^P(\{2\}) = \{1\}$. (M,F) is indeed an R-structure. Consider the following M-instance of $(P(x) \to P(y)) \leftrightarrow [x,y](P(x) \to P(y))$: $\left(P\left(\overline{1}\right) \to P\left(\overline{2}\right)\right) \leftrightarrow \left[\overline{1},\overline{2}\right]\left(P\left(\overline{1}\right) \to P\left(\overline{2}\right)\right)$. We show that (M,F) doesn't satisfy it. Since $1 \in f^P(\emptyset)$ and $2 \notin f^P(\emptyset)$, $(M,F) \not\models_C P\left(\overline{1}\right) \to P\left(\overline{2}\right)$. However, since $1,2 \in f^P(\{1,2\})$, $(M,F) \models_C \left[\overline{1},\overline{2}\right]\left(P\left(\overline{1}\right) \to P\left(\overline{2}\right)\right)$. All together,
$(M,F) \not\models_C (P(x) \to P(y)) \leftrightarrow [x,y](P(x) \to P(y))$. □

6.3 Replacement of Equivalents

Loosely speaking, a consequence relation admits replacement of equivalents if whenever two formulas are equivalent, it is possible to replace each occurrence of the first with the other. In this section we check which of our consequence relations admits this rule.

Definition 34. Let $\vdash$ be any consequence relation for L.

1. Let T be an L-theory and let A and A' be L-formulas. A and A' are $\vdash$-equivalent in T if $T \vdash A \leftrightarrow A'$.

2. $\vdash$ admits *strong* replacement of equivalents if for every L-theory T and L-formulas φ, φ', A, A' such that φ' is obtained from φ by replacing zero or more occurrences of A with A', if A and A' are $\vdash$-equivalent in T then so are φ and φ'. $\vdash$ admits *weak* replacement of equivalents if the above holds for $T = \emptyset$.

Lemma 42. *$\vdash_C$ and $\vdash_C^R$ admit weak replacement of equivalents.*

Proof. It is enough to show that $\vdash_{HC}$ and $\vdash_{HCR}$ admit weak replacement of equivalents. Let $\vdash \in \{\vdash_{HC}, \vdash_{HCR}\}$. Let $k \geq 0$ be the number of replacements of A by A' made in φ. If $k = 0$ then this is trivial. Let $k > 0$. If $A = \varphi$ then $\varphi' = A'$ and this is also trivial. Hence we may assume that $A \neq \varphi$ (i.e. that A is a distinct sub-formula of φ). We prove the claim by induction on the complexity of φ.

1. Assume φ is $P(t_1, ..., t_n)$. Then $A = P(t_1, ..., t_n)$ and this falls under one of the above cases.

2. If φ is $\neg\psi$, $\psi_1 \to \psi_2$ or $\forall x\psi$ then this is shown similarly to the proof of the theorem for classical first order logic.

3. Assume φ is $[C]\,\psi$ for some context set C. $\varphi' = [C]\,\psi'$, where ψ' is a formula obtained from ψ by replacing the same occurrences of A with A'. By the i.h., $\vdash \psi \leftrightarrow \psi'$. By Lemma 39 and Theorem 6, $\vdash [C](\psi \leftrightarrow \psi')$, and by Lemma 27, $\vdash [C]\,\psi \leftrightarrow [C]\,\psi'$.

□

Lemma 43. *$\vdash_C^T$ doesn't admit weak replacement of equivalents.*

Proof. We prove this claim for $\vdash_{HCT}$. Let $\varphi := [d]\,[c]\,P(c) \leftrightarrow [d,c]\,P(c)$, $A := [c]\,P(c)$ and $A' := P(c)$. Let $\varphi' := [d]\,P(c) \leftrightarrow [d,c]\,P(c)$, obtained from φ by replacing one occurrence of A with A'. By axiom T, $\vdash_{HCT} A \leftrightarrow A'$. However, $\nvdash_{HCT} \varphi \leftrightarrow \varphi'$: Assume for contradiction that $\vdash_{HCT} \varphi \leftrightarrow \varphi'$. Then since $\vdash_{HCT} \varphi$ (axiom CC), $\vdash_{HCT} \varphi'$, i.e. $\vdash_{HCT}$ $[d]\,P(c) \leftrightarrow [d,c]\,P(c)$. Since $\vdash_{HCT} [d,c]\,P(c) \leftrightarrow [d]\,[c]\,P(c)$, $\vdash_{HCT} [d]\,P(c) \leftrightarrow [d]\,[c]\,P(c)$ and hence $\vdash_{HCT} [d](P(c) \leftrightarrow [c]\,P(c))$. This means that $\vdash_C^T [d](P(c) \leftrightarrow [c]\,P(c))$, but this is not the case, as seen in Lemma 12. □

Lemma 44. *$\vdash_C$ and $\vdash_C^R$ don't admit strong replacement of equivalents.*

Proof. We start with $\vdash_C$. Let φ, φ', A, A' be the same as in Lemma 43. Define $T := \{A \leftrightarrow A'\} = \{[c]\,P(c) \leftrightarrow P(c)\}$. Obviously, $T \vdash_C A \leftrightarrow A'$. We prove that $T \nvdash_C \varphi \leftrightarrow \varphi'$. It is enough to find a C-structure (M,F) such that $(M,F) \models_C T$ and $(M,F) \not\models_C \varphi \leftrightarrow \varphi'$. Consider the following C-structure (M,F): $|M| = \{1,2\}$. $I(c) = 1$, $I(d) = 2$. $f^P(\emptyset) = f^P(\{1\}) = f^P(\{1,2\}) = \{1\}$. $f^P(\{2\}) = \emptyset$. Clearly, $(M,F) \models_C [c]\,P(c) \leftrightarrow P(c)$. In addition, $(M,F) \models_C [d]\,[c]\,P(c) \leftrightarrow [d,c]\,P(c)$ (by Theorem 5). However, $(M,F) \not\models_C [d]\,P(c) \leftrightarrow [d,c]\,P(c)$. Hence $(M,F) \not\models_C \varphi \leftrightarrow \varphi'$. Now we turn to $\vdash^R_C$. If $\vdash^R_C$ would admit strong replacement of equivalents, then we could infer that $P(c) \leftrightarrow P(d) \vdash^R_C [e]\,P(c) \leftrightarrow [e]\,P(d)$, since $\vdash^R_C [e]\,P(c) \leftrightarrow [e]\,P(c)$, and $[e]\,P(c) \leftrightarrow [e]\,P(d)$ is obtained from it by a replacement of one occurrence of $P(c)$ with $P(d)$. However, $P(c) \leftrightarrow P(d) \nvdash^R_C [e]\,P(c) \leftrightarrow [e]\,P(d)$: Consider the following R-structure: $|M| = \{1,2,3\}$. $I(c) = 1$, $I(d) = 2$, $I(e) = 3$. $f^P(\{3\}) = f^P(\{1,3\}) = \{1\}$. $f^P(X) = \emptyset$ for every other $X \in FIN(|M|)$. First, we show that (M,F) is indeed an R-structure, i.e. that f^P is reflexive: Let $x \in |M|$ and $X \in FIN(|M|)$. $x \in f^P(X)$ iff $x = 1$ and $X \in \{\{3\},\{1,3\}\}$ iff $x = 1$ and $X \cup \{x\} = \{1,3\}$ iff $x \in f^P(X \cup \{x\})$. Second, $(M,F) \models_C P(c) \leftrightarrow P(d)$ since $I(c) = 1 \notin f^P(\emptyset)$ and $I(d) = 2 \notin f^P(\emptyset)$. However, $(M,F) \not\models_C [e]\,P(c) \leftrightarrow [e]\,P(d)$ since $I(c) = 1 \in f^P(\{I(e)\})$ but $I(d) = 2 \notin f^P(\{I(e)\})$. $\square$

6.4 Replacement of Congruent Formulas

We have seen that only two of the studied consequence relations admit weak replacement of equivalents and that none of them admits strong replacement of equivalents.

We now define a stronger notion of equivalence, namely congruence, in order to obtain a more positive result.

Definition 35. Let $\vdash$ be any consequence relation for L.

1. Let T be an L-theory and let A and A' be L-formulas. A and A' are $\vdash$-congruent in T if for every context set C, $T \vdash [C]\,(A \leftrightarrow A')$.

2. $\vdash$ admits (strong) replacement of congruent formulas if for every L-theory T and L-formulas φ, φ', A, A' such that φ' is obtained from φ by replacing zero or more occurrences of A with A', if A and A' are $\vdash$-congruent in T then so are φ and φ'.

Lemma 45. *Let* $\vdash \in \left\{\vdash_C, \vdash^T_C, \vdash^R_C\right\}$. *Then* $\vdash$ *admits replacement of congruent formulas.*

Proof. We prove the claim for $\vdash \in \{\vdash_{HC}, \vdash_{HCT}, \vdash_{HCR}\}$. Let T be an L-theory and φ, φ', A, A' be L-formulas such that φ' is obtained from φ by replacing zero or more occurrences of A with A'. Assume A and A' are $\vdash$-congruent in T. We prove that $T \vdash [C]\,(\varphi \leftrightarrow \varphi')$ for every C. Let $k \geq 0$ be the number of replacements of A by A' made in φ. If $k = 0$ then we need to prove that for every C, $\vdash [C]\,(\varphi \leftrightarrow \varphi)$. This is equivalent to showing that $\vdash [C]\,\varphi \leftrightarrow [C]\,\varphi$, which of course holds. Let $k > 0$. If $A = \varphi$ then $\varphi' = A'$ and this is trivial. Hence we may assume that $A \neq \varphi$ (i.e. that A is a distinct sub-formula of φ). We prove the claim by induction on the complexity of φ.

1. Assume φ is $P(t_1, ..., t_n)$. Then $A = P(t_1, ..., t_n)$ and this falls under one of the above cases.

2. Assume φ is $\neg\psi$. Let ψ' be the formula which is obtained from ψ by the same k replacements of A by A' made in φ. Obviously, $\varphi' = \neg\psi'$. Let C be any context set. By the induction hypothesis, $T \vdash [C]\,(\psi \leftrightarrow \psi')$. Therefore, $T \vdash [C]\,\psi \leftrightarrow [C]\,\psi'$. Since $\vdash$ has all axioms and rules of $HFOL$, $T \vdash \neg\,[C]\,\psi \leftrightarrow \neg\,[C]\,\psi'$. By $C_\neg$, $T \vdash \neg\,[C]\,\psi \leftrightarrow [C]\,\neg\psi$ and $T \vdash \neg\,[C]\,\psi' \leftrightarrow [C]\,\neg\psi'$. Therefore, $T \vdash [C]\,\neg\psi \leftrightarrow [C]\,\neg\psi'$ and hence $T \vdash [C]\,(\neg\psi \leftrightarrow \neg\psi')$.

3. Assume φ is $\psi_1 \to \psi_2$. Then this shown similarly.

4. Assume φ is $\forall x\psi$. Let ψ' be the formula which is obtained from ψ by the same k replacements of A by A' made in φ. Obviously, $\varphi' = \forall x\psi'$. Let C be any context set. By the induction hypothesis, $T \vdash [C]\,\psi \leftrightarrow [C]\,\psi'$. By the $HFOL$ axiom $\forall xA \to A\{t/x\}$, $T \vdash \forall x\,[C]\,\psi \to [C]\,\psi$. Hence $T \vdash \forall x\,[C]\,\psi \to [C]\,\psi'$. By Gen, $T \vdash \forall x\,(\forall x\,[C]\,\psi \to [C]\,\psi')$. By the $HFOL$ axiom $\forall x\,(A \to B) \to (A \to \forall xB)$ and the fact that x isn't free in $\forall x\,[C]\,\psi$, $T \vdash \forall x\,[C]\,\psi \to \forall x\,[C]\,\psi'$. Similarly, $T \vdash \forall x\,[C]\,\psi' \to \forall x\,[C]\,\psi$. Hence $T \vdash \forall x\,[C]\,\psi \leftrightarrow \forall x\,[C]\,\psi'$. First assume x doesn't appear in C. Then $T \vdash \forall x\,[C]\,\psi \leftrightarrow [C]\,\forall x\psi$ and $T \vdash \forall x\,[C]\,\psi' \leftrightarrow [C]\,\forall x\psi'$. Then $T \vdash [C]\,\forall x\psi \leftrightarrow [C]\,\forall x\psi'$ and hence $T \vdash [C]\,(\forall x\psi \leftrightarrow \forall x\psi')$. Now assume x does appear in C. Assume w.l.g. that $C = \{x_1, ..., x_n\}$ and that $x = x_1$. Let $y_1, ..., y_n$ be fresh variables. By the induction hypothesis, $T \vdash [y_1, ..., y_n]\,\psi \leftrightarrow [y_1, ..., y_n]\,\psi'$. By Gen, $T \vdash \forall x\,([y_1, ..., y_n]\,\psi \leftrightarrow [y_1, ..., y_n]\,\psi')$. Since $\vdash$ contains all axioms and rules of $HFOL$, , $T \vdash \forall x\,[y_1, ..., y_n]\,\psi \leftrightarrow \forall x\,[y_1, ..., y_n]\,\psi'$. Now, by $C_\forall$, $T \vdash \forall x\,[y_1, ..., y_n]\,\psi \leftrightarrow [y_1, ..., y_n]\,\forall x\psi$ and $T \vdash \forall x\,[y_1, ..., y_n]\,\psi' \leftrightarrow [y_1, ..., y_n]\,\forall x\psi'$ (since $y_1, ..., y_n$ are fresh). All together, $T \vdash [y_1, ..., y_n]\,\forall x\psi \leftrightarrow [y_1, ..., y_n]\,\forall x\psi'$. Using Gen and the instantiation axiom we obtain $T \vdash [C]\,\forall x\psi \leftrightarrow [C]\,\forall x\psi'$ and hence $T \vdash [C]\,(\forall x\psi \leftrightarrow \forall x\psi')$.

5. Assume φ is $[D]\,\psi$. Let ψ' be the formula which is obtained from ψ by the same k replacements of A by A' made in φ. Obviously, $\varphi' = [D]\,\psi'$. Let C be any context set. By the induction hypothesis, $T \vdash [C, D]\,(\psi \leftrightarrow \psi')$ and hence $T \vdash [C, D]\,\psi \leftrightarrow [C, D]\,\psi'$. By CC, $T \vdash [C, D]\,\psi \leftrightarrow [C]\,[D]\,\psi$ and $T \vdash [C, D]\,\psi' \leftrightarrow [C]\,[D]\,\psi'$. This means that $T \vdash [C]\,[D]\,\psi \leftrightarrow [C]\,[D]\,\psi'$, and hence $T \vdash [C]\,([D]\,\psi \leftrightarrow [D]\,\psi')$.

□

6.5 Summary

The following table summarizes the results of this section. r.o.e. stands for replacement of equivalents and r.o.c stands for replacement of congruent formulas.

	$\vdash_C$	$\vdash_C^T$	$\vdash_C^R$
rule N	admissible	not admissible	admissible
axiom T	valid for atomic formulas	valid for atomic formulas	valid for atomic formulas
r.o.e	holds weakly	doesn't hold	holds weakly
r.o.c	holds	holds	holds

7 Reduction to Classical Logic

In this section we show that although in the frameworks of C-languages it is more convenient to express the role of context in propositions, its expressive power is not greater than the expressive power of first order classical logic.

Throughout this section we consider an arbitrary C-language L. For the purpose of reducing to classical logic, it is easier to use a slightly different version of C-languages than the one defined in Definition 6 and to make some assumptions:

1. Like in [Ga-2010], we use finite *sequences* of terms, rather than finite *sets* of terms. Proof theoretically, this requires the addition (or more precisely, the reinstatement) of the axiom scheme $[x_1, ..., x_n]\,\varphi \to [x'_1, ..., x'_m]\,\varphi$ where $\{x_1, ..., x_n\} = \{x'_1, ..., x'_m\}$. Semantically, this does not matter, since the cdfs are defined over finite sets and not over finite sequences.

2. We use $\models$ with no subscripts for the satisfaction relation for classical first order logic (Definition 2).

3. We assume that L has an infinite number of variable symbols, and that all the terms of L are ordered by some well ordering, which we denote by "$\leq$".

7.1 Translation of Formulas

Definition 36. The first order language $tr(L)$ is obtained from L by the following steps:

1. Omitting the context operator.
2. Replacing every predicate symbol P of arity n with $\aleph_0$ predicate symbols, namely $P_{m,n}$ for every $m \in \mathbb{N} \cup \{0\}$. The arity of $P_{m,n}$ is $m+n$.

Definition 37. Let φ be a formula (in L or $tr(L)$). $sfv(\varphi)$ [10] is the smallest variable which is bigger than all the variables that occur (free or bound) in φ (with respect to the well ordering of the terms of L).

Definition 38. Let φ be an L-formula. $tr(\varphi)$ is a $tr(L)$-formula, and is defined by induction on the complexity of φ. In what follows $m \geq 0$ and $n, k > 0$.

1. If φ is $[s_1, ..., s_m] P(t_1, ..., t_n)$ then $tr(\varphi)$ is $P_{m,n}(s_1, ..., s_m, t_1, ..., t_n)$.
2. If φ is $[s_1, ..., s_m](\neg\psi)$ then $tr(\varphi)$ is $\neg(tr([s_1, ..., s_m]\psi))$.
3. If φ is $[s_1, ..., s_m](\psi_1 \to \psi_2)$ then $tr(\varphi)$ is $(tr([s_1, ..., s_m]\psi_1)) \to (tr([s_1, ..., s_m]\psi_2))$.
4. If φ is $\forall x\psi$ then $tr(\varphi)$ is $\forall x(tr(\psi))$.
5. If φ is $[s_1, ..., s_k]\forall x\psi$ then $tr(\varphi)$ is $\forall y(tr([s_1, ..., s_m]\psi\{y/x\}))$ where $y = sfv(\varphi)$.
6. If φ is $[t_1, ..., t_k][s_1, ..., s_m]\psi$ then $tr(\varphi)$ is $tr([t_1, ..., t_k, s_1, ..., s_m]\psi)$.

Remark 8. Precedence of the tr operator in the meta language: $tr(\psi_1) \to \psi_2 = (tr(\psi_1)) \to \psi_2$, $\psi_1 \to tr(\psi_2) = \psi_1 \to (tr(\psi_2))$, $\neg tr(\psi) = \neg(tr(\psi))$, $\forall x tr(\psi) = \forall x(tr(\psi))$, $[C]\, tr(\psi) = [C](tr(\psi))$.

Lemma 46. *Let φ be an L-formula. Then $Fv(\varphi) = Fv(tr(\varphi))$.*

Proof. The only case of Definition 38 in which some variables are changed is case 5. In this case, the only variables that are changed are bound variables, and they are replaced with other (fresh) bound variables. □

Remark 9. Our translation reduces the number of operators in a given formula, since it eliminates all context operators and doesn't change any classical connective or quantifier. However, the translated formula is dramatically longer than the original formula.

7.2 Translation of Semantics

Definition 39. Let (M, F) be a C-structure for L. $tr(M) := \langle |M|, tr(I) \rangle$ such that:

1. For every constant symbol c and function symbol g $tr(I)(c) = I(c)$ and $tr(I)(g) = I(g)$.
2. For every $m \geq 0$, $n > 0$ and predicate symbol $P_{m,n}$ of arity $m+n$:

$$tr(I)(P_{m,n}) = \left\{ \langle b_1, ..., b_m, a_1, ..., a_n \rangle : \langle a_1, ..., a_n \rangle \in f^P(\{b_1, ..., b_m\}) \right\}$$

[10] Short for "smallest fresh variable"

Definition 40. Let $M = \langle |M|, I \rangle$ be a first order structure for $tr(L)$.

1. M is a CC*-structure* if $M \models P_{m,n}(y_1, ..., y_m, x_1, ..., x_n) \leftrightarrow P_{k,n}(z_1, ..., z_k, x_1, ..., x_n)$ for every $P \in PRED(L)$, $m, k \geq 0$, $n > 0$ and $x_1, ..., x_n$,$y_1, ..., y_m$,$z_1, ..., z_k$ such that $\{y_1, ..., y_m\} = \{z_1, ..., z_k\}$.

2. M is a CT-structure if it is a CC-structure and in addition, $M \models P_{0,n}(x_1, ..., x_n) \leftrightarrow P_{n,n}(x_1, ..., x_n, x_1, ..., x_n)$ for every $P \in PRED(L)$, $n > 0$ and $x_1, ..., x_n$.

3. M is a CR-structure if it is a CC-structure, and in addition, $M \models P_{m,n}(y_1, ..., y_m, x_1, ..., x_n) \leftrightarrow P_{m+n,n}(y_1, ..., y_m, x_1, ..., x_n, x_1, ..., x_n)$ for every $P \in PRED(L)$, $m \geq 0$, $n > 0$ and $y_1, ..., y_m, x_1, ..., x_n$.

Lemma 47.

1. *The mapping tr from C-structures for L to first-order structures for $tr(L)$ is onto the class of CC-structures for $tr(L)$.*

2. *The mapping tr from T-structures for L to first-order structures for $tr(L)$ is onto the class of CT-structures for $tr(L)$.*

3. *The mapping tr from R-structures for L to first-order structures for $tr(L)$ is onto the class of CR-structures for $tr(L)$.*

Proof. We start by constructing a structure which we shall use in all the parts of this proof. Let $M = \langle |M|, I \rangle$ be a CC-structure for $tr(L)$. Then (M', F') is defined as follows:

- $|M'| = |M|$.
- I''s interpretation of constant symbols and function symbols is the same as I's.
- F' is defined as follows:

$$\begin{aligned} F' &= \lambda P \in PRED(L).\lambda X \in FIN(|M'|). \\ &\quad \{\langle a_1, ..., a_n \rangle : \exists b_1, ..., b_m. \{b_1, ..., b_m\} = X \wedge \langle b_1, ..., b_m, a_1, ..., a_n \rangle \in I(P_{m,n})\} \end{aligned}$$

Now we turn to the actual proof.

1. Let M be a CC-structure. Obviously, (M', F') is a C-structure. We claim that $tr(M', F') = M$. By definition, $tr(I')$ and I are identical over constant and function symbols, and $|M'| = |M|$. Let $P \in PRED(L)$ and let $m \geq 0$ and $n > 0$. We need to show that $tr(I')(P_{m,n}) = I(P_{m,n})$. $\langle b_1, ..., b_m, a_1, ..., a_n \rangle \in tr(I')(P_{m,n})$ iff $a_1, ..., a_n \in F'(P)(\{b_1, ..., b_m\})$ iff $\langle c_1, ..., c_m, a_1, ..., a_n \rangle \in I(P_{m,n})$ for some $\{c_1, ..., c_m\} = \{b_1, ..., b_m\}$ (by F''s construction) iff $\langle b_1, ..., b_m, a_1, ..., a_n \rangle \in I(P_{m,n})$ (since M is a CC-structure).

2. Let M be a CT-structure for $tr(L)$. Consider the C-structure (M', F') which was defined above. It is now clear that $tr(M', F') = M$. We claim that (M', F') is a T-structure. $\langle a_1, ..., a_n \rangle \in F'(\emptyset)$ iff $\langle a_1, ..., a_n \rangle \in I(P_{0,n})$ (by F''s construction) iff $\langle a_1, ..., a_n, a_1, ..., a_n \rangle \in I(P_{n,n})$ (since M is a CT-structure) iff $\langle a_1, ..., a_n \rangle \in F'(P)(\{a_1, ..., a_n\})$ (by F''s construction).

3. Let M be a CR-structure for $tr(L)$. Consider the C-structure (M', F') which was defined above. It is now clear that $tr(M', F') = M$. We claim that (M', F') is an R-structure, i.e. that for every $P \in PRED(L)$, $F'(P)$ is reflexive. Let $P \in PRED(L)$ be an n-ary predicate and let $\{b_1, ..., b_m\} \in FIN(|M|)$. $\langle a_1, ..., a_n \rangle \in$

$F'(P)(\{b_1,...,b_m\})$ iff $\langle d_1,...,d_m,a_1,...,a_n\rangle \in I(P_{m,n})$ with $\{d_1,...,d_m\} = \{b_1,...,b_m\}$ (by F''s construction) iff $\langle d_1,...,d_m,a_1,...,a_n,a_1,...,a_n\rangle \in I(P_{m+n,n})$ (since M is a CR-structure) iff $\langle b_1,...,b_m,a_1,...,a_n,a_1,...,a_n\rangle \in I(P_{m+n,n})$ (since M is a CC-structure) iff $\langle a_1,...,a_n\rangle \in F'(P)(\{b_1,...,b_m,a_1,...,a_n\})$ (by F''s construction).

□

Lemma 48. *Let (M,F) be a C-structure.*

1. *$tr(M,F)$ is a CC-structure.*
2. *If (M,F) is a T-structure then $tr(M,F)$ is a CT-structure.*
3. *If (M,F) is an R-structure then $tr(M,F)$ is a CR-structure.*

Proof.

1. Assume $\{b_1,...,b_m\} = \{c_1,...,c_k\}$. $\langle b_1,...,b_m,a_1,...,a_n\rangle \in tr(I)(P_{m,n})$ iff $\langle a_1,...,a_n\rangle \in f^P(\{b_1,...,b_m\}) = f^P(\{c_1,...,c_k\})$ iff $\langle c_1,...,c_k,a_1,...,a_n\rangle \in tr(I)(P_{k,n})$.
2. $\langle a_1,...,a_n\rangle \in tr(I)(P_{0,n})$ iff $\langle a_1,...,a_n\rangle \in f^P(\emptyset)$ iff $\langle a_1,...,a_n\rangle \in f^P(\{a_1,...,a_n\})$ (since (M,F) is a T-structure) iff $\langle a_1,...,a_n,a_1,...,a_n\rangle \in tr(I)(P_{n,n})$.
3. $\langle b_1,...,b_m,a_1,...,a_n\rangle \in tr(I)(P_{m,n})$ iff $\langle a_1,...,a_n\rangle \in f^P(\{b_1,...,b_m\})$ iff $\langle a_1,...,a_n\rangle \in f^P(\{a_1,...,a_n,b_1,...,b_m\})$ (since f^R is reflexive) iff $\langle b_1,...,b_m,a_1,...,a_n,a_1,...,a_n\rangle \in tr(I)(P_{m+n,n})$.

□

Lemma 49. *Let (M,F) be a C-structure, let $a_1,...,a_n \in |M|$ and let φ be an L-formula such that $Fv(\varphi) \subseteq \{x_1,...,x_n\}$. Then $tr(M,F) \models (tr(\varphi))\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}$ iff $tr(M,F) \models tr(\varphi\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$.*

The proof is tedious and routine, and is detailed in the appendix (Section 10.2).

Lemma 50. *Let (M,F) be a C-structure for L and let φ be an L-sentence. Then $(M,F) \models_C \varphi$ iff $tr(M,F) \models tr(\varphi)$.*

Proof. By induction on the complexity of φ.

1. $(M,F) \models_C P(t_1,...,t_n)$ iff $\langle I(t_1),...,I(t_n)\rangle \in f^P(\emptyset)$ iff $\langle tr(I)(t_1),...,tr(I)(t_n)\rangle \in tr(I)(P_{0,n})$ iff $tr(M,F) \models P_{0,n}(t_1,...,t_n)$ iff $tr(M,F) \models tr(P(t_1,...,t_n))$.
2. $(M,F) \models_C \forall x\psi$ iff for every $a \in |M|$, $(M,F) \models_C \psi\{\overline{a}/x\}$ iff for every $a \in |M|$ $tr(M,F) \models tr(\psi\{\overline{a}/x\})$ (by the i.h.) iff for every $a \in |M|$ $tr(M,F) \models tr(\psi)\{\overline{a}/x\}$ (by Lemma 49) iff $tr(M,F) \models \forall x(tr(\psi))$ iff $tr(M,F) \models tr(\forall x\psi)$.
3. $(M,F) \models_C [s_1,...,s_m]P(t_1,...,t_n)$ (with $m > 0$) iff $\langle I(t_1),...,I(t_n)\rangle \in f^P(\{I(s_1),...,I(s_m)\})$ iff $\langle tr(I)(s_1),...,tr(I)(s_m),tr(I)(t_1),...,tr(I)(t_n)\rangle \in tr(I)(P_{m,n})$ iff $tr(M,F) \models P_{m,n}(s_1,...,s_m,t_1,...,t_n)$ iff $tr(M,F) \models tr([s_1,...,s_m]P(t_1,...,t_n))$.
4. $(M,F) \models_C [s_1,...,s_m]\neg\psi$ (with $m \geq 0$) iff $(M,F) \not\models_C [s_1,...,s_m]\psi$ iff $tr(M,F) \not\models tr([s_1,...,s_m]\psi)$ (by the i.h.) iff $tr(M,F) \models \neg tr([s_1,...,s_m]\psi)$ iff $tr(M,F) \models tr([s_1,...,s_m]\neg\psi)$.
5. For $[s_1,...,s_m](\psi_1 \to \psi_2)$ (with $m \geq 0$) the proof is similar.

6. For $\varphi = [s_1, ..., s_m] \forall x\psi$ (with $m > 0$): Let $y := sfv(\varphi)$. $(M, F) \models_C [s_1, .., s_m] \forall x\psi$ iff for every $a \in |M|$, $(M, F) \models_C [s_1, ..., s_m] \psi \{\overline{a}/x\}$ iff for every $a \in |M|$, $tr(M, F) \models tr([s_1, ..., s_m] \psi \{\overline{a}/x\})$ (by the i.h.) iff for every $a \in |M|$, $tr(M, F) \models tr([s_1, ..., s_m] \psi \{y/x\} \{\overline{a}/y\})$ iff for every $a \in |M|$, $tr(M, F) \models tr(([s_1, ..., s_m] \psi \{y/x\}) \{\overline{a}/y\})$ (since y doesn't occur in $s_1, ..., s_m$) iff for every $a \in |M|$, $tr(M, F) \models tr([s_1, ..., s_m] \psi \{y/x\}) \{\overline{a}/y\}$ (by Lemma 49) iff $tr(M, F) \models \forall y (tr([s_1, ..., s_m] \psi \{y/x\}))$ iff $tr(M, F) \models tr([s_1, ..., s_m] \forall x\psi)$.

7. $(M, F) \models_C [s_1, ..., s_m] [t_1, ..., t_n] \psi$ (with $m, n > 0$) iff $(M, F) \models_C [s_1, ..., s_m, t_1, ..., t_n] \psi$ iff $tr(M, F) \models tr([s_1, ..., s_m, t_1, ..., t_n] \psi)$ (by the i.h.) iff $tr(M, F) \models tr([s_1, ..., s_m] [t_1, ..., t_n] \psi)$.

□

Lemma 51. *Let (M, F) be a C-structure for L and let φ be an L-formula. Then $(M, F) \models_C \varphi$ iff $tr(M) \models tr(\varphi)$.*

Proof. Let $x_1, ..., x_n$ be the free variables of φ. $(M, F) \models_C \varphi$ iff for every $a_1, ..., a_n \in |M|$, $(M, F) \models_C \varphi \{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\}$ iff for every $a_1, ..., a_n \in |M|$, $tr(M, F) \models tr(\varphi \{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\})$ (by Lemma 50) iff for every $a_1, ..., a_n \in |M|$, $tr(M, F) \models tr(\varphi) \{\overline{a_1}/x_n, ..., \overline{a_n}/x_n\}$ (by Lemma 49) iff $tr(M, F) \models tr(\varphi)$ (by Lemma 46). □

Definition 41.

1. $A :=$

$$\{P_{m,n}(y_1, ..., y_m, x_1, ..., x_n) \leftrightarrow P_{k,n}(z_1, ..., z_k, x_1, ..., x_n) : \\ \{y_1, ..., y_m\} = \{z_1, ..., z_k\}, P \in PRED(L), m, k \geq 0, n > 0\}$$

2. $B :=$

$$\{P_{0,n}(x_1, ..., x_n) \leftrightarrow P_{n,n}(x_1, ..., x_n, x_1, ..., x_n) : P \in PRED(L), n > 0\}$$

3. $C :=$

$$\{P_{m,n}(y_1, ..., y_m, x_1, ..., x_n) \leftrightarrow P_{m+n,n}(y_1, ..., y_m, x_1, ..., x_n, x_1, ..., x_n) : \\ P \in PRED(L), m \geq 0, n > 0\}$$

4. Let T be an L-theory. $tr(T) := \{tr(\varphi) : \varphi \in T\}$.

Theorem 7. *Let T be an L-theory and let φ be an L-formula.*

1. *$T \vdash_C \varphi$ iff $tr(T) \cup A \vdash_{FOL} tr(\varphi)$.*
2. *$T \vdash_C^T \varphi$ iff $tr(T) \cup A \cup B \vdash_{FOL} tr(\varphi)$.*
3. *$T \vdash_C^R \varphi$ iff $tr(T) \cup A \cup B \cup C \vdash_{FOL} tr(\varphi)$.*

Proof. We prove the case of $\vdash_C$. Other cases are similar. Assume $T \vdash_C \varphi$. Let M be a first-order model of $tr(T) \cup A$. In particular, $M \models A$ and hence M is a CC-structure. By Lemma 47, there exists a C-structure (M', F') such that $M = tr(M', F')$. By Lemma 51, $(M', F') \models_C T$. By our assumption, $(M', F') \models_C \varphi$ and again by Lemma 51, $M \models tr(\varphi)$. Hence $tr(T) \cup A \vdash_{FOL} tr(\varphi)$. Now assume $T \nvdash_C \varphi$. Then there exists a C-structure (M, F) such that $(M, F) \models_C T$ and $(M, F) \not\models_C \varphi$. By Lemma 51 and the fact that $tr(M, F)$ is a CC-structure (by Lemma 48), $tr(M, F) \models tr(T) \cup A$ and $tr(M, F) \not\models tr(\varphi)$. Therefore, $tr(T) \cup A \nvdash_{FOL} tr(\varphi)$. □

8 The Sorites Paradox

As mentioned, one of the motivations for developing TCL has been the fact that in classical logic the use of tolerant predicates can lead to the Sorites paradox. TCL offers a formulation in which the paradox can be avoided. We start this section with a rigorous presentation of the Sorites paradox. Then, we present a solution of fuzzy logic to the paradox. After that, we formalize Gaifman's original solution to the paradox using TCL.[11] We end this section with a comparison between the different approaches.

8.1 A Rigorous Formulation of the Paradox

Let L be a first order language. Let n be any natural number and let $a_0, ..., a_n$ be constant symbols for L. Let P be a unary predicate and let N_P be a binary predicate. Let T be the theory which consists of the following axioms:

1. axiom A: $P(a_0)$.
2. axiom B: $\neg P(a_n)$.
3. for every $0 \leq i \leq n-1$ axiom C_i is: $N_P(a_i, a_{i+1})$.
4. axiom TC: $N_P(a_i, a_j) \rightarrow (P(a_i) \rightarrow P(a_j))$.[12]

Lemma 52. *T is unsatisfiable in classical logic.*

However, if we consider P as a vague predicate and N_P as its nearness relation (i.e. $N_P(x, y)$ should hold if x is close to y), then T seems true. For example:

Example 6. Assume $n = 200$. Interpret P as "short", N_P as the nearness relation of "short" (e.g. $N_P(x, y)$ means "x and y are at most 1 centimeters apart in their heights") and $a_0, ..., a_{200}$ as a series of people such that: For each $0 \leq i \leq 200$ a_i denotes a person whose height is i.

The fact that T seems true by some interpretations although it is not satisfiable is one way to present the famous Sorites paradox.

8.2 The Solution(s) of Fuzzy Logic

The framework of [Ga-2010] may be viewed as a response to a collection of solutions to the Sorites paradox, offered by Fuzzy Logic[13]. In order to have a better understanding of the motivation for the contextual framework, we present an example for a solution from fuzzy logic.

We use the solution of Łukasiewicz fuzzy logic as it is presented on pages 263-271 of [Be-2008] as a basis. We made some modifications in order for it to be as similar as possible to our formulation of the paradox.

This solution shows (after well-defining the relevant notions) that T is "almost" satisfiable, by presenting an L-structure (Definition 5) which assigns to each axiom of T a very high truth value (at least $1 - \frac{1}{n}$). This solves the paradox, by showing the acceptable fact that T is indeed not satisfiable, but all of its formulas have a high truth value.

We present the following L-structure:

1. $|M| = \{0, ..., n\}$.

[11] However, we feel that this solution is not yet satisfactory and that further research is in order here.

[12] TC stands for the tolerance conditional. A more exact notation would be *e.g.* $TC_{i,j}$, however we will use TC to denote every such instance.

[13] Gaifman uses the term "Degree Theory".

2. $I(a_i) = i$.

3. $I(P) = \lambda x \in |M| . \frac{n-x}{n}$.

4. $I(N_P) = \lambda \langle x, y \rangle \in |M|^2 . \begin{cases} 1 & |x-y| \leq 1 \\ 0 & otherwise \end{cases}$

Let us calculate the truth values M assigns to the axioms of T:

1. $M(P(a_0)) = I(P)(0) = \frac{n-0}{n} = 1$.

2. $M(\neg P(a_n)) = 1 - M(P(a_n)) = 1 - \frac{n-n}{n} = 1$.

3. $M(N_P(a_i, a_{i+1})) = 1$ since $|I(a_i) - I(a_{i+1})| = 1$.

4. $M(N_P(a_i, a_j) \to (P(a_i) \to P(a_j))) =$
   $min\{1, 1 - M(N_P(a_i, a_j)) + M(P(a_i) \to P(a_j))\}$. If $M(N_P(a_i, a_j)) = 0$, then we are looking for $min\{1, 1 - 0 + M(P(a_i) \to P(a_j))\} = 1$ (as $M(P(a_i) \to P(a_j))$ is non-negative). Let us now assume $M(N_P(a_i, a_j)) = 1$ (recall that 0 and 1 are the only two options for $N_P(a_i, a_j)$). In this case, we are looking for

$$\begin{aligned}
min\{1, 1 - 1 + M(P(a_i) \to P(a_j))\} &= \\
min\{1, M(P(a_i) \to P(a_j))\} &= \\
min\{1, min\{1, 1 - M(P(a_i)) + M(P(a_j))\}\} &= \\
min\{1, 1 - M(P(a_i)) + M(P(a_j))\} &= \\
min\left\{1, 1 - \frac{n-i}{n} + \frac{n-j}{n}\right\} &= min\left\{1, 1 - \frac{i-j}{n}\right\}
\end{aligned}$$

   Since we are under the assumption that $M(N_P(a_i, a_j)) = 1$, $|i-j| = 1$. Therefore, $min\left\{1, 1 - \frac{i-j}{n}\right\} \geq 1 - \frac{1}{n}$. All together, $M(N_P(a_i, a_j) \to (P(a_i) \to P(a_j))) \geq 1 - \frac{1}{n}$.

Remark 10. Note that in the above example, N_P's interpretation is "crisp". However, it is possible that the notion of nearness is vague itself. There is a similar solution to the paradox which uses a fuzzy interpretation of N_P.

8.3 Gaifman's Solution

While the solution of fuzzy logic conforms with the fact that T is not satisfiable and solves the paradox by presenting an interpretation which doesn't fully satisfy T but almost does, Gaifman claims that T should be re-formulated in a new kind of language (C-language), resulting in a new theory T^*. T^* is (GS-) satisfiable, and hence the paradox is avoided.

For convenience, we assume the existence of a single context-dependent predicate P, which is unary. In addition, we use N_P to denote the binary nearness relation for P.

Definition 42. (From pages 8-9 of [Ga-2010]) A *Sorites chain* (for P) is a finite sequence of objects such that:

1. The first falls under P.

2. The last does not.

3. The difference between any two successive objects is so small that if one falls under P so does the other.

Remark 11. Here are two of the several equivalent definitions of a *feasible context* (inside square brackets are our comments):

1. "A context is feasible if its members can be partitioned into P's and non-P's without violating any instance of the tolerance conditional $[TC]$ or any of the semantic axioms governing P" (page 21 of [Ga-2010]).

2. "In all the standard examples of tolerant predicates, this [being a feasible context] is equivalent to the requirement that the context does not contain a Sorites chain" (page 17 of [Ga-2010]).

Proposition 1. *"For any given C we can express in the formal language the condition that C is feasible" (page 21 of [Ga-2010]).*

Remark 12. We denote the expression which states that C is a feasible context by $feasible\,(C)$.[14]

8.3.1 The Definition of $feasible\,(C)$

In this section we provide our[15] definition of the formula $feasible\,(C)$, which is meant to state that a certain context is a feasible one.

A context is feasible if it does not contain a Sorites chain. This means that if the context has an element which is considered P in that context and another element which is considered not P in that context, then there must be a gap between these two elements. This is formulated as follows:

1. For every $q \geq 2$: $feasible_q\,(\{t_1, ..., t_q\}) =$
$\bigwedge_{\pi \in S_q} \left(\left([t_1, ..., t_q]\, P\left(t_{\pi(1)}\right) \wedge [t_1, ..., t_q]\, \neg P\left(t_{\pi(q)}\right) \right) \to \bigvee_{i=1}^{q-1} \left(\neg N_P\left(t_{\pi(i)}, t_{\pi(i+1)}\right) \right) \right)$

2. For singletons: $feasible_1\,(\{t\})$ may be any $\vdash^S_{GS}$ valid formula (i.e. for every t, $\vdash^S_{GS} feasible_1\,(\{t\})$).

3. $feasible\,(C) = feasible_{|C|}\,(C)$.

8.3.2 The Solution

The main idea of the solution is the division of contexts into feasible ones and unfeasible ones. The ordinary use of tolerant predicates is restricted to feasible contexts. This is justified by the claim that "unfeasible contexts do not arise in practice" (page 17 of [Ga-2010]).

We define a new theory T^* in the C-language, rather than in the language of classical logic. T^* will keep the intuitiveness of all claims in T, but will restrict them to certain contexts. This way, the natural acceptance of T's axioms will be legitimate, only with minor acceptable changes. These changes will in fact turn our theory to a satisfiable one.

T^* is:

1. axiom A^*: $[C]\, P\,(a_0)$ for every context set C (including $\emptyset$).

2. axiom B^*: $[C]\, \neg P\,(a_n)$ for every context set C (including $\emptyset$).

3. for every $0 \leq i \leq n-1$ axiom C_i is: $N_P\,(a_i, a_{i+1})$.

4. The axiom IND_{N_P} (Definition 16).

[14] In fact, for every $n \in N$, there exists a formula $feasible_n\,(t_1, ..., t_n)$, which expresses the fact that $\{t_1, ..., t_n\}$ is feasible. Moreover, $feasible_n\,(t_1, ..., t_n)$ depends also on P and N_P (P's nearness relation). We ignore these dependencies since here we use a language in which the only context dependent predicate is P, and it has one associated nearness relation N_P.

[15] There was a flaw in the original construction of this formula in [Ga-2010] (confirmed by Gaifman in a personal communication).

5. axiom scheme TC^*: $((feasible\,(C) \wedge N_P\,(a_i, a_j)) \rightarrow [C]\,(P\,(a_i) \rightarrow P\,(a_j)))$ for every context set C which includes a_i and a_j.

Remark 13.

1. The formulas of the form $feasible\,(C)$ are only used in axiom scheme TC^*. The reason for that is the strong connection between feasible contexts and tolerance. According to [Ga-2010], "tolerance is, by definition, tolerance in all feasible contexts" (page 17 of [Ga-2010]). This means that unfeasible contexts are allowed, as long as tolerance isn't needed. If we introduce unfeasible contexts, we lose tolerance. This is why $feasible\,(C)$ is only used in the tolerance conditional TC^*.

2. A more general formulation of TC^* would be:

$$\forall x \forall y\,((x \in C \wedge y \in C \wedge feasible\,(C) \wedge N_P\,(x, y)) \rightarrow [C]\,(P\,(x) \rightarrow P\,(y)))$$

Using this scheme requires a language with the identity sign, in order to express "$\in$". This is not essential to the paradox nor its solutions. For technical simplicity, we use instead the scheme that was formalized above.

3. Note that in this solution (similarly to the solution of fuzzy logic that we presented above), N_P is context-independent. However, there can be situations in which N_P is context-dependent, but we shall not pursuit this here.

Lemma 53. *T^* is satisfiable in TCL (i.e. GS-satisfiable by an S-structure).*

Proof. We present an S structure which GS-satisfies T^*. [16]

We define (M, F) as follows:

1. $|M| = \{0, ..., n\}$.

2. $I\,(a_i) = i$ for every $0 \leq i \leq n$.

3. $f^{N_P}\,(X) = \left\{\langle i, j\rangle \in X^2 : |i - j| \leq 1\right\}$ for every $X \in FIN\,(|M|)$.

4. The definition of $f^P\,(X)$ is a bit more complicated:

   (a) For every $0 \leq i \leq n - 1$ $f^P\,(\{i\}) = \{i\}$.

   (b) $f^P\,(\{n\}) = \emptyset$.

   (c) For every $X \in FIN\,(|M|)$ which is not a singleton, if there exists $x, y \in X$ such that $x < y - 1$, and there isn't any $z \in X$ such that $x < z < y$, then $f^P\,(X) = \{w \in X : w \leq x\}$ for the minimal x for which there exists such y. [17]

   (d) For every other $X \in FIN\,(|M|)$ which is not a singleton, if $n \in X$, $f^P\,(X) = \{0\} \cap X$. Otherwise, $f^P\,(X) = X$.

Let us now show that (M, F) indeed models T^*. Let φ be an axiom of T^*. Let φ' be an M-instance of it. φ' may have 5 forms. We verify that in each possible form, $(M, F) \models_{GS} \varphi'$:

[16] Despite the fact that it is more convenient to work with R-structures *once they are defined*, it is easier to actually *define* S-structures, since for every finite subset X of $|M|$, one only needs to assign a subset of X. By Theorem 4, there also exists an R-structure which C-satisfies T^*, but it seems to have a more cumbersome definition.

[17] For clarity: If X isn't a singleton and it contains a gap (with respect to $\leq$ of the natural numbers), then all elements "left" to the gap are considered P in X.

1. $\varphi' = [C]\, P(a_0)$: If $C = \emptyset$ then $(M, F) \models_{GS} P(a_0)$ since $0 \in f^P(\{0\})$. Otherwise, it is easy to see that for every X such that $0 \in X$, $0 \in f^P(X)$. Hence $0 \in f^P(I(C) \cup \{0\}) \cap \{0\}$ for every C. This means that for every C, $\left(M, F_{I(C)}\right) \models_{GS} P(a_0)$ and hence $(M, F) \models_{GS} [C]\, P(a_0)$.

2. $\varphi' = [C] \neg P(a_n)$: If $C = \emptyset$ then $(M, F) \models_{GS} \neg P(a_n)$ since $n \notin f^P(\{n\})$. Otherwise, n in never an element of $f(X)$ for any X. Therefore, for every C, $n \notin f^P(I(C) \cup \{n\}) \cap \{n\}$, which means that for every C, $\left(M, F_{I(C)}\right) \models_{GS} \neg P(a_n)$ and hence $(M, F) \models_{GS} [C] \neg P(a_n)$.

3. $\varphi' = N_P(a_i, a_{i+1})$: Let $1 \leq i \leq n-1$. $|i - (i+1)| = 1$. Hence $\langle i, i+1 \rangle \in f^{N_P}(\{i, i+1\})$ and therefore $(M, F) \models_{GS} N_P(a_i, a_{i+1})$.

4. $\varphi' = IND_{N_P}$: By Lemma 6, it is enough to show that N_P is context independent in (M, F). Indeed, let $X \in FIN(|M|)$. $\langle x, y \rangle \in f^{N_P}(X \cup \{x, y\})$ iff both $x, y \in X \cup \{x, y\}$ and $|x - y| \leq 1$, iff $|x - y| \leq 1$ iff $\langle x, y \rangle \in f^{N_P}(\{x, y\})$.

5. $\varphi' = (\mathit{feasible}(C) \wedge N_P(a_i, a_j)) \rightarrow [C]\,(P(a_i) \rightarrow P(a_j))$: Assume $(M, F) \models_{GS} \mathit{feasible}(C) \wedge N_P(a_i, a_j)$. Then $(M, F) \models_{GS} \mathit{feasible}(C)$ and $|i - j| \leq 1$. Let $Y = I(C)$. We need to prove that $(M, F_Y) \models_{GS} P(a_i) \rightarrow P(a_j)$. We therefore assume $(M, F_Y) \models_{GS} P(a_i)$ and prove $(M, F_Y) \models_{GS} P(a_j)$. Note that since $(M, F_Y) \models_{GS} P(a_i)$, $i \in f^P(I(C) \cup \{i\}) \cap \{i\}$. In particular, $i \in f^P(I(C) \cup \{i\})$. In addition, the examined axiom should hold only when a_i and a_j are elements of C. Therefore, $I(C) \cup \{i, j\} = I(C)$, and hence $i \in f^P(I(C))$. Similarly, in order to prove that $(M, F_Y) \models_{GS} P(a_j)$, it is enough to show that $j \in f^P(I(C))$. Now we check every possible case:

   (a) $i < j$ $(j = i + 1)$:

      i. $C = \{a_i, a_j\}$: Assume $j = n$. In this case, $f^P(\{i, j\}) = \{i, j\} \cap \{0\}$. If $i = 0$ then $(M, F) \models_{GS} [C]\, P(a_0)$, $(M, F) \models_{GS} [C] \neg P(a_1)$ and $(M, F) \not\models_{GS} \neg N_P(a_0, a_1)$. Therefore, $(M, F) \not\models_{GS} \mathit{feasible}(C)$ which is a contradiction to our assumption. Hence $i \neq 0$, which means that $f^P(\{i, j\}) = \emptyset$. Therefore, $i \notin f^P(I(C))$, which is also a contradiction to our assumptions. Now assume $j \neq n$. Then $f^P(\{i, j\}) = \{i, j\}$ and hence $j \in f^P(I(C))$.

      ii. $|C| > 2$: First assume there exists $k \neq i$ such that $k \in f^P(I(C))$. Then since $i, k \in f^P(I(C))$, $f^P(I(C)) \neq \{0\}$ and $f^P(I(C)) \neq \emptyset$. This means that one of the two following options must hold: The first: there exists a minimal $x \in I(C)$ for which there exists $y \in I(C)$ such that $x < y - 1$, and there isn't any $z \in I(C)$ such that $x < z < y$. The second: $n \notin I(C)$. Assume that the first option holds. Since $i \in f^P(I(C))$, $i \leq x$. In addition, $x \neq i$, since $i + 1 = j \in I(C)$ (and there are no elements between x and y in $I(C)$). Therefore $i < x$ and hence $j = i + 1 \leq x$. This means that $j \in f^P(I(C))$. If the second option holds (and the first one doesn't), $f^P(I(C)) = I(C)$. As said, $j \in I(C)$ and hence $j \in f^P(I(C))$. Now assume $\{i\} = f^P(I(C))$. Then there are two possibilities: The first: i is the smallest $x \in I(C)$ for which there exists $y \in I(C)$ such that $x < y - 1$ and there is no $z \in I(C)$ such that $x < z < y$. This case is impossible, as $i + 1 = j \in I(C)$ and $j < y$. The second: $i = 0$ and $n \in I(C)$. In this case, $I(C) = |M|$ and hence $(M, F) \not\models_{GS} \mathit{feasible}(C)$, which contradicts our assumption.

   (b) $j < i$ $(i = j + 1)$:

      i. $C = \{a_i, a_j\}$: $i \neq n$ since $i \in f^P(I(C))$ and $n \notin f^P(X)$ for every $X \in FIN(|M|)$. Hence $n \notin I(C)$ and hence $f^P(I(C)) = I(C)$. In particular, $j \in f^P(I(C))$.

ii. $|C| > 2$: Similarly to item a.ii, first assume that there exists $k \neq i$ such that $k \in f^P(I(C))$. Then $f^P(I(C)) \neq \{0\}$ and $f^P(I(C)) \neq \emptyset$. This means that either there exists a minimal $x \in I(C)$ for which there exists $y \in I(C)$ such that $x < y - 1$, and there isn't any $z \in I(C)$ such that $x < z < y$, or $n \notin I(C)$. Assume that the first option holds. Since $i \in f^P(I(C))$, $i \leq x$. Since $j < i$, $j < x$ and hence $j \in f^P(I(C))$. If the second option holds (and the first one doesn't), $f^P(I(C)) = I(C)$ and hence $j \in f^P(I(C))$. Now assume $\{i\} = f^P(I(C))$. Since $i = j + 1$, $i \neq 0$. Therefore, i is the smallest $x \in I(C)$ for which there exists $y \in I(C)$ such that $x < y - 1$ and there is no $z \in I(C)$ such that $x < z < y$. In this case, $j < i = x$ and hence $j \in f^P(I(C))$. This is a contradiction since $j \neq i$.

(c) $i = j$: This case is trivial, as we are under the assumption that $i \in f^P(I(C))$ and need to prove that $j \in f^P(I(C))$.

□

A Remark Regarding the Solution

It would be useful to emphasize a special characteristic of the above solution. Notice that axiom scheme TC^* requires that $a_i, a_j \in C$. At first, this seems redundant, as the use of axiom R of HCR is supposed to guarantee that the presence of the considered elements in the context is not important. However, axiom R only refers to atomic formulas, and not to complex ones. Now we show that omitting this requirement (of TC^*) results in an unsatisfiable theory. In the foregoing, we denote by T^{**} the theory obtained from T^* by omitting that requirement from axiom TC^*.

Lemma 54. $\vdash_{HCR} ([C]\,P(t) \to [C]\,P(s)) \to ([C,t]\,P(t) \to [C,s]\,P(s))$.

Proof. By Lemma 35, $\vdash_{HCR} [C]\,P(t) \leftrightarrow [C,t]\,P(t)$ and $\vdash_{HCR} [C]\,P(s) \leftrightarrow [C,s]\,P(s)$. Since $\vdash_{HCR}$ admits weak replacement of equivalents (Lemma 42 and the soundness and completeness theorems for $\vdash_{HCR}$), $\vdash_{HCR} ([C]\,P(t) \to [C]\,P(s)) \leftrightarrow ([C,t]\,P(t) \to [C]\,P(s))$ and $\vdash_{HCR} ([C,t]\,P(t) \to [C]\,P(s)) \leftrightarrow ([C,t]\,P(t) \to [C,s]\,P(s))$.
All together, $\vdash_{HCR} ([C]\,P(t) \to [C]\,P(s)) \leftrightarrow ([C,t]\,P(t) \to [C,s]\,P(s))$. □

Lemma 55. *T^{**} is not satisfiable.*

Proof. It is enough to show that T^{**} is inconsistent. Let C be $\{a_0\}$. Clearly, $\vdash_{HCR}$ $feasible(C)$. In addition, for every $1 \leq i \leq n$, $T^{**} \vdash_{HCR} feasible(C, a_i)$. For every $0 \leq i \leq n-1$, denote by $Deduction_i$ the following proof in HCR:

Line #	Formula	Justification
$7i+1$	$[C,a_i]\,P(a_i)$	If $i=0$: axiom A Otherwise: Repetition of line $7(i-1)+7$
$7i+2$	$feasible(C) \wedge N_P(a_i, a_{i+1})$	$C = \{a_0\}$ and axiom C_i
$7i+3$	$(feasible(C) \wedge N_P(a_i, a_{i+1})) \to [C](P(a_i) \to P(a_{i+1}))$	axiom TC^*
$7i+4$	$[C](P(a_i) \to P(a_{i+1}))$	M.P lines $7i+2$ and $7i+3$
$7i+5$	$[C]\,P(a_i) \to [C]\,P(a_{i+1})$	M.P of axiom $C_\to$ of HCR and line $7i+4$
$7i+6$	$[C,a_i]\,P(a_i) \to [C,a_{i+1}]\,P(a_{i+1})$	M.P. of Lemma 54 and line $7i+5$
$7i+7$	$[C,a_{i+1}]\,P(a_{i+1})$	M.P lines $7i+1$, $7i+6$

$Deduction_0$, $Deduction_1$,...,$Deduction_{n-1}$ is a proof $[C,a_n]\,P(a_n)$ from T^{**} in HCR, which means that $T^{**} \vdash_{HCR} [C,a_n]\,P(a_n)$. In addition, the single-line proof "$\neg [C,a_n]\,P(a_n)$" shows that $T^{**} \vdash_{HCR} \neg [C,a_n]\,P(a_n)$. □

T^{**} is not satisfiable, but seems true, just like the original T. Therefore, the requirement that $a_i, a_j \in C$ in axiom TC is crucial.

8.3.3 Other Contextual Solutions

Obviously, the proposed construction of $feasible\,(C)$ is not intended for an actual application - it isn't feasible to check whether a context is feasible using this construction. This fact shouldn't be too disturbing, because of the following two reasons:

1. The contextual framework can be used in situations where unfeasible contexts do not arise. This means that TCL should be used to describe tolerant predicates, in most of their ordinary usage (which is usually in feasible contexts). Moreover, as quoted above, unfeasible contexts do not arise in practice. This suggests the solution to the Sorites, by which the paradoxical inference does not arise in practice in its contextual form.

2. There exists a solution to the Sorites which uses the contextual approach without using the concept of feasible contexts at all. This solution appears in [Th] in a general explanation. We shall now present (our rigorous formulation of) this solution.

Thomason's Solution

The following description of Thomason's approach is based on pages 5-6 of [Th] and reflects our understanding of these pages.

For technical simplification, we make the intuitive assumption that $n > 1$. We define a new theory T' which is obtained from T^* by changing axiom TC^* to: $N_P\,(a_i, a_j) \to [a_i, a_j]\,(P\,(a_i) \to P\,(a_j))$.

Now, the following deduction in HCR fails to derive the paradoxical conclusion:

Line #	Formula	Justification
1	$[a_0, a_1]\,P\,(a_0)$	axiom A^*
2	$N_P\,(a_0, a_1)$	axiom C_0
3	$N_P\,(a_0, a_1) \to [a_0, a_1]\,(P\,(a_0) \to P\,(a_1))$	axiom TC^*
4	$[a_0, a_1]\,(P\,(a_0) \to P\,(a_1))$	M.P lines 2, 3
5	$[a_0, a_1]\,(P\,(a_0) \to P\,(a_1)) \leftrightarrow ([a_0, a_1]\,P\,(a_0) \to [a_0, a_1]\,P\,(a_1))$	$C_\to$
6	$[a_0, a_1]\,P\,(a_0) \to [a_0, a_1]\,P\,(a_1)$	M.P. lines 4,5
7	$[a_0, a_1]\,P\,(a_1)$	M.P lines 1, 6
8	$N_P\,(a_1, a_2)$	axiom C_1
9	$N_P\,(a_1, a_2) \to [a_1, a_2]\,(P\,(a_1) \to P\,(a_2))$	axiom TC^*
10	$[a_1, a_2]\,(P\,(a_1) \to P\,(a_2))$	M.P lines 8, 9
11	$[a_1, a_2]\,(P\,(a_1) \to P\,(a_2)) \leftrightarrow ([a_1, a_2]\,P\,(a_1) \to [a_1, a_2]\,P\,(a_2))$	$C_\to$
12	$[a_1, a_2]\,P\,(a_1) \to [a_1, a_2]\,P\,(a_2)$	M.P. lines 10,11

Obviously, it isn't possible to apply M.P. on lines 7 and 12. Hence, the paradoxical deduction is avoided. Indeed,

Lemma 56. *T' is satisfiable in TCL (i.e. GS-satisfiable by an S-structure).*

Proof. Consider the following structure (M, F): $|M| = \{0, ..., n\}$. For every $1 \le i \le n$, $I\,(a_i) = i$. $f^{N_P}\,(X) = \left\{\langle i, j\rangle \in X^2 : |i - j| \le 1\right\}$ for every $X \in FIN\,(|M|)$.

$$f^P = \lambda X \in FIN\,(|M|)\,.\begin{cases} \{0,1\} & 0,1 \in X \\ \{0\} & 0 \in X, 1 \notin X \\ \emptyset & 0 \notin X \end{cases}$$

We prove that $(M, F) \models_{GS} T'$. Let $\varphi \in T'$ and let φ' be an M-instance of φ. We check every case:

1. $(M,F) \models_{GS} [C]\, P(a_0)$: If $C = \emptyset$ then this holds since $0 \in f^P(\{0\})$. Assume $C \neq \emptyset$. $(M,F) \models_{GS} [C]\, P(a_0)$ iff $(M,F_Y) \models_{GS} P(a_0)$ with $Y = I(C)$ iff $0 \in f^P(\{0\} \cup I(C)) \cap \{0\}$, which is true by f^P's construction.

2. $(M,F) \models_{GS} \neg [C]\, P(a_n)$: If $C = \emptyset$ then this holds since $n \notin f^P(\{n\})$. Now assume $C \neq \emptyset$. $(M,F) \models_{GS} \neg [C]\, P(a_n)$ iff $(M,F) \not\models_{GS} [C]\, P(a_n)$ iff $(M,F_Y) \not\models_{GS} P(a_n)$ with $Y = I(C)$ iff $n \notin f^P(I(C) \cup \{n\}) \cap \{n\}$, which is true by f^P's construction and by the fact that $n > 1$.

3. $(M,F) \models_{GS} N_P(a_i, a_{i+1})$ and $(M,F) \models_{GS} IND_{N_P}$: Since f^{N_P} is defined exactly like in Lemma 53, the proof is the same.

4. $(M,F) \models_{GS} N_P(a_i, a_j) \to [a_i, a_j]\,(P(a_i) \to P(a_j))$: Assume $(M,F) \models_{GS} N_P(a_i, a_j)$. Let $Y = \{i, j\}$. We prove that $(M,F_Y) \models_{GS} P(a_i) \to P(a_j)$. If $i = j$ then this is obvious, and hence we assume $i \neq j$. Assume $(M,F_Y) \models_{GS} P(a_i)$. Then either $i = 0$ or $i = 1$. If $i = 0$ then since $(M,F) \models_{GS} N_P(a_i, a_j)$, $j = 1$ and $1 \in f^P(\{0,1\} \cup \{1\}) \cap \{1\}$. This means that $(M,F_Y) \models_{GS} P(a_j)$. Now assume $i = 1$. Then either $j = 0$ or $j = 2$. If $j = 0$, then since $0 \in f^P(\{0,1\} \cup \{0\}) \cap \{0\}$, $(M,F_Y) \models_{GS} P(a_j)$. If $j = 2$ then $1 \notin f^P(\{i,j\} \cup \{i\}) \cap \{i\}$ and hence $(M,F_Y) \not\models_{GS} P(a_i)$, which is a contradiction to our assumption.

□

8.4 Comparison between The Approaches

We have discussed above two approaches to the Sorites: one of fuzzy logic and one of TCL ($\vdash^S_{GS}$). Each approach suggests more than one solution to the paradox, out of which we have seen one for fuzzy logic and two for TCL. There are three main differences between these approaches. The first is concerned with the notion of truth degrees. In the semantics of fuzzy logic, each formula is assigned a truth value from $[0,1]$, while the contextual approach sticks to the classical use of only two truth values. The contextual approach is an extension of classical logic in another sense too: Classical logic is contained in contextual logic (in fact, both logics are identical when restricted to formulas without the context operator). Moreover, contextual logic can be reduced to classical logic (as was shown in Section 7). The second main difference is that in TCL the tolerance conditional, which states that if x and y are close to each other then if x is P then so is y, is acceptable (in principle). Indeed, this conditional expresses a "part of the semantic norms governing the use of tolerant predicates" [Ga-2010]. One should however be careful with the exact formalization of the conditional in order to take into account the effect of context on the meaning of tolerant predicates. Hence the contextual approach uses a richer language. Using the enriched language we obtain a new version of the tolerance conditional, which is satisfiable (along with the other axioms of the Sorites). In contrast, according to the solution of fuzzy logic, the tolerance conditional is not acceptable, i.e. not all of its instances are assigned the maximum truth value. By this solution, the tolerance conditional isn't true, and the paradox is avoided by lowering its truth value. To conclude this difference: The solution of TCL is based on a reformulation of the assumptions, while the solution of fuzzy logic is based on the denial of the absolute truth of the tolerance conditional. Another important difference between the approaches of fuzzy logic and TCL is that the solution of TCL strongly relies on the notion of feasibility. When formalizing the tolerance conditional in TCL, it is meant to hold only in feasible contexts. This is justified by the claim that unfeasible contexts do not arise in practice. On the other hand, in the solution of fuzzy logic, the claims of the Sorites are formalized in general and are not restricted to particular cases. Note however that TCL is useful even when neglecting the notion of feasible contexts, as can be seen by the solution of Thomason.

9 Conclusion and Further Research

In this work we have reconstructed Gaifman's logic for context dependency in a modular way.

Below is a graph which summarizes the explored deductive systems and their relations to the explored semantics. A one directional arrow from A to B ($A \to B$) means $A \subsetneq B$. A bidirectional arrow between A to B ($A \leftrightarrow B$) means $A = B$. If $A \subseteq B$ and $B \subseteq C$ we omit the arrow from A to C.

$$\begin{tikzcd}
 & \vdash_C \arrow[rr, leftrightarrow, dotted] \arrow[d] & & \vdash_{HC} \arrow[d] \\
 & \vdash_C^T \arrow[rr, leftrightarrow, dotted] \arrow[d] & & \vdash_{HCT} \arrow[d] \\
\vdash_{GS}^S \arrow[r, leftrightarrow] & \vdash_C^R \arrow[rr, leftrightarrow, dotted] & & \vdash_{HCR}
\end{tikzcd}$$

Recall that HCT is the deductive system from [Ga-2010] and that $\vdash_C^R$ and $\vdash_{GS}^S$ are equivalent to the semantical consequence relation from [Ga-2010]. We have seen that indeed, HCT is not complete for $\vdash_C^R$, but rather for $\vdash_C^T$. We have also seen that the new system HCR is complete for $\vdash_C^R$.

In addition, we have studied several properties of the consequence relations, including a reduction to classical logic.

In Section 8, two main approaches for the Sorites paradox were presented: fuzzy logic and TCL. The first is based on an infinite set of truth values, where the second is based on an extended language. A combination between the approaches can be achieved by extending fuzzy logic semantics to handle contexts and context dependent predicates. [18]

10 Appendix

10.1 Using The Substitutional Approach

In this small section we show that the objectual approach is equivalent to the substitutional approach when using TCL (see end of Section 3).

Let L be a C-language.

Definition 43. Let (M, F) be an S-structure. Let $n > 0$, let $\varphi(x_1, ..., x_n)$ be any formula and let $\{a_i\}_{i=1}^n \subseteq |M|$. $v_{\{a_i\}_{i=1}^n} : VAR(L) \to |M|$ is defined as follows: for every $1 \le i \le n$ $v'(x_i) = a_i$ and for every other $x \in VAR(L)$, $v_{\{a_i\}_{i=1}^n}(x) = a_1$.

Notations:

1. $v' := v_{\{a_i\}_{i=1}^n}$
2. for any term t, $t' := t\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\}$.

Lemma 57. *Let $t(x_1, ..., x_n)$ be any term in L and let $a_1, ..., a_n \in |M|$. Then $I(t') = v'(t)$.*

[18] In his paper, Gaifman initiates a similar project, by presenting a combination between TCL and another logic which is meant to model borderline vagueness by modal logic. In this logic, the truth degrees of fuzzy logic are replaced by iterated modalities, retaining two truth values.

Proof. By induction on the complexity of t:

1. If $t = x_1$ then $I\left(t\left\{\overline{a_1}/x_1\right\}\right) = I\left(\overline{a_1}\right) = a_1 = v'\left(x_1\right)$.

2. If $t = c$ then this is obvious.

3. If $t = f\left(t_1, ..., t_m\right)$ then $I\left(f\left(t_1, ..., t_m\right)\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}\right) = I\left(f\left(t_1', ..., t_m'\right)\right)$ $= I\left(f\right)\left(I\left(t_1'\right), ..., I\left(t_m'\right)\right) = I\left(f\right)\left(v'\left(t_1\right), ..., v'\left(t_m\right)\right)$ (by the induction hypothesis), which equals to $v'\left(f\left(t_1, ..., t_m\right)\right)$.

□

Definition 44. Let (M, F) be an S-structure and let φ be a formula. $(M, F) \models^S_{GS} \varphi$ if (M, F) models φ in the substitutional approach and $(M, F) \models^O_{GS} \varphi$ if (M, F) models φ in the objectual approach. $\vdash^{SO}_{GS}$ is the consequence relation induced by $\models^O_{GS}$ and $\vdash^S_{GS}$ is the consequence relation induced by $\models^S_{GS}$.

Lemma 58. *Let $\varphi\left(x_1, ..., x_n\right)$ be any formula. Then for every S-structure (M, F) and $a_1, ..., a_n \in |M|$, $(M, F) \models^S_{GS} \varphi\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}$ iff $(M, F, v') \models^O_{GS} \varphi$.*

Proof. By induction on the complexity of φ.

1. $(M, F) \models^S_{GS} P\left(t_1, ..., t_m\right)\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}$ iff $\left\langle I\left(t_1'\right), ..., I\left(t_m'\right)\right\rangle \in f^P\left(\left\{I\left(t_1'\right), ..., I\left(t_m'\right)\right\}\right)$ iff $\left\langle v'\left(t_1\right), ..., v'\left(t_m\right)\right\rangle \in f^P\left(\left\{v'\left(t_1\right), ..., v'\left(t_m\right)\right\}\right)$ (by Lemma 57) iff $(M, F, v') \models^O_{GS} P\left(t_1, ..., t_m\right)$.

2. If φ has the form $\psi_1 \to \psi_2$, $\neg\psi$ or $\forall x \psi$, then this is routine.

3. If φ has the form $\left[t_1, ..., t_m\right]\psi$, then: $(M, F) \models^S_{GS} \left(\left[t_1, ..., t_m\right]\psi\right)\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}$ iff $(M, F_Y) \models^S_{GS} \psi'$ with $Y = \left\{I\left(t_1'\right), ..., I\left(t_m'\right)\right\} = \left\{v'\left(t_1\right), ..., v'\left(t_m\right)\right\}$ (Lemma 57) iff $(M, F_Y, v') \models^O_{GS} \psi$ (by the induction hypothesis) iff $(M, F, v') \models^O_{GS} \left[t_1, ..., t_m\right]\psi$.

□

Lemma 59. *Let $\varphi\left(x_1, ..., x_n\right)$ be any formula. Then for every S-structure (M, F). $(M, F) \models^S_{GS} \varphi$ iff $(M, F) \models^O_{GS} \varphi$.*

Proof. $(M, F) \models^S_{GS} \varphi$ iff for every $a_1, ..., a_n \in |M|$ $(M, F) \models^S_{GS} \varphi\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}$ iff for every $a_1, ..., a_n \in |M|$ $(M, F, v') \models^O_{GS} \varphi$ (by Lemma 58) iff for every v $(M, F, v) \models^O_{GS} \varphi$ (since v''s definition on variables other than $x_1, ..., x_n$ is irrelevant) iff $(M, F) \models^O_{GS} \varphi$. □

Corollary 6. $\vdash^{SO}_{GS} = \vdash^S_{GS}$.

10.2 Proof of Lemma 49

Recall Lemma 49:

Lemma. *Let (M, F) be a C-structure, let $a_1, ..., a_n \in |M|$ and let φ be an L-formula such that $Fv\left(\varphi\right) \subseteq \left\{x_1, ..., x_n\right\}$. Then $tr\left(M, F\right) \models \left(tr\left(\varphi\right)\right)\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}$ iff $tr\left(M, F\right) \models tr\left(\varphi\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}\right)$.*

Proof. By induction on the complexity of φ. Note that $\varphi\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}$ is a sentence, and by Lemma 46, so are $tr\left(\varphi\right)\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}$ and $tr\left(\varphi\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}\right)$.

1. If φ is $P\left(s_1, ..., s_m\right)$ then: $tr\left(M, F\right) \models \left(tr\left(\varphi\right)\right)\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}$
 iff $tr\left(M, F\right) \models P_{0,m}\left(s_1, ..., s_m\right)\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}$
 iff $tr\left(M, F\right) \models P_{0,m}\left(s_1\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}, ..., s_m\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}\right)$
 iff $tr\left(M, F\right) \models tr\left(P\left(s_1\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}, ..., s_m\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}\right)\right)$
 iff $tr\left(M, F\right) \models tr\left(P\left(s_1, ..., s_m\right)\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}\right)$
 iff $tr\left(M, F\right) \models tr\left(\varphi\left\{\overline{a_1}/x_1, ..., \overline{a_n}/x_n\right\}\right)$.

2. If φ is $\forall y\psi$ then: First assume $x_i \neq y$ for every $1 \leq i \leq n$:
$tr(M,F) \models (tr(\varphi))\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}$ iff
$tr(M,F) \models (\forall y(tr(\psi)))\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}$ iff
$tr(M,F) \models \forall y(tr(\psi)\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$ iff for every $a \in |M|$,
$tr(M,F) \models (tr(\psi)\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})\{\overline{a}/y\}$ iff for every $a \in |M|$,
$tr(M,F) \models tr(\psi)\{\overline{a_1}/x_1,...,\overline{a_n}/x_n,\overline{a}/y\}$ iff for every $a \in |M|$,
$tr(M,F) \models tr(\psi\{\overline{a_1}/x_1,...,\overline{a_n}/x_n,\overline{a}/y\})$ (by the i.h.) iff for every $a \in |M|$,
$tr(M,F) \models tr(\psi\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}\{\overline{a}/y\})$ iff for every $a \in |M|$,
$tr(M,F) \models tr(\psi\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})\{\overline{a}/y\}$ (by the i.h.) iff
$tr(M,F) \models \forall y(tr(\psi\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}))$ iff
$tr(M,F) \models tr(\forall y(\psi\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}))$ iff
$tr(M,F) \models tr((\forall y\psi)\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$ (since $x_i \neq y$ for every i) iff
$tr(M,F) \models tr(\varphi\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$. Now assume $x_i = y$ for some i. Without loss of generality, assume $i = 1$. $tr(M,F) \models (tr(\varphi))\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}$ iff
$tr(M,F) \models (\forall x_1(tr(\psi)))\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}$ iff
$tr(M,F) \models \forall x_1(tr(\psi)\{\overline{a_2}/x_2,...,\overline{a_n}/x_n\})$
(since x_1 doesn't occur free in $\forall x_1(tr(\psi))$) iff for every $a \in |M|$,
$tr(M,F) \models (tr(\psi)\{\overline{a_2}/x_2,...,\overline{a_n}/x_n\})\{\overline{a}/x_1\}$. From here, the proof continues similarly to the previous case.

3. If φ is $[l_1,...,l_k]P(s_1,...,s_m)$ with $k > 0$ then:
$tr(M,F) \models (tr(\varphi))\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}$
iff $tr(M,F) \models P_{k,m}(l_1,...,l_k,s_1,...,s_m)\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}$
iff

$$\begin{aligned} tr(M,F) \quad \models \quad & P_{k,m}(l_1\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\},...,l_k\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}, \\ & s_1\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\},...,s_m\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}) \end{aligned}$$

iff

$$\begin{aligned} tr(M,F) \models tr([& l_1\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\},...,l_k\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}] \\ & P(s_1\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\},...,s_m\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})) \end{aligned}$$

iff $tr(M,F) \models tr(([l_1,...,l_k]P(s_1,...,s_m))\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$ iff
$tr(M,F) \models tr(\varphi\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$.

4. If φ is $[s_1,...,s_m]\neg\psi$ with $m \geq 0$ then: $tr(M,F) \models (tr(\varphi))\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}$ iff
$tr(M,F) \models (\neg(tr([s_1,...,s_m]\psi)))\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}$ iff
$tr(M,F) \models \neg((tr([s_1,...,s_m]\psi))\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$ iff
$tr(M,F) \not\models (tr([s_1,...,s_m]\psi))\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}$ (since it is a sentence) iff
$tr(M,F) \not\models tr(([s_1,...,s_m]\psi)\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$ (by the i.h.)
iff $tr(M,F) \models \neg(tr(([s_1,...,s_m]\psi)\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}))$ (since it is a sentence)
iff $tr(M,F) \models tr(\neg(([s_1,...,s_m]\psi)\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}))$
iff $tr(M,F) \models tr((\neg[s_1,...,s_m]\psi)\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$
iff $tr(M,F) \models tr(\varphi\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$.

5. If φ is $[s_1,...,s_m](\psi_1 \to \psi_2)$ with $m \geq 0$ then the proof is similar.

6. If φ is $[s_1,...,s_m]\forall y\psi$ with $m > 0$ then: Let $z := sfv(\varphi)$,
$z' := sfv([s_1\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\},...,s_m\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}]\forall y\psi\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$.
Now, If for every $1 \leq i \leq n$ $x_i \neq y$ and $x_i \neq z$ then the latter holds
iff $tr(M,F) \models \forall z((tr([s_1,...,s_m]\psi\{z/y\}))\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$ iff for every $a \in |M|$,

$tr\,(M,F) \models ((tr\,([s_1,...,s_m]\,\psi\,\{z/y\}))\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})\,\{\overline{a}/z\}$ iff for every $a \in |M|$,
$tr\,(M,F) \models (tr\,([s_1,...,s_m]\,\psi\,\{z/y\}))\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}\,\{\overline{a}/z\}$ iff for every $a \in |M|$,
$tr\,(M,F) \models (tr\,([s_1,...,s_m]\,\psi\,\{z/y\}))\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n,\overline{a}/z\}$ iff for every $a \in |M|$,
$tr\,(M,F) \models tr\,(([s_1,...,s_m]\,\psi\,\{z/y\})\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n,\overline{a}/z\})$ (by the i.h. and the fact that $x_i \neq z$ for every i) iff for every $a \in |M|$,
$tr\,(M,F) \models tr\,(([s_1,...,s_m]\,\psi\,\{z/y\})\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}\,\{\overline{a}/z\})$ iff for every $a \in |M|$,
$tr\,(M,F) \models (tr\,(([s_1,...,s_m]\,\psi\,\{z/y\})\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}))\,\{\overline{a}/z\}$ (by the i.h. and the fact that z is the only free variable in $tr\,(([s_1,...,s_m]\,\psi\,\{z/y\})\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}))$
iff $tr\,(M,F) \models \forall z\,(tr\,(([s_1,...,s_m]\,\psi\,\{z/y\})\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}))$
iff

$$\begin{aligned} tr\,(M,F) \quad \models \quad & \forall z(tr(([s_1\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}\,,...,s_m\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}] \\ & \psi\,\{z/y\}\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}))) \end{aligned}$$

iff

$$\begin{aligned} tr\,(M,F) \quad \models \quad & \forall z(tr([s_1\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}\,,...,s_m\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}] \\ & \psi\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}\,\{z/y\})) \end{aligned}$$

iff

$$\begin{aligned} tr\,(M,F) \quad \models \quad & \forall z'(tr([s_1\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}\,,...,s_m\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}] \\ & \psi\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}\,\{z'/y\})) \end{aligned}$$

(by rule α) iff

$$\begin{aligned} tr\,(M,F) \quad \models \quad & tr([s_1\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}\,,...,s_m\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}] \\ & \forall y\,(\psi\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})) \end{aligned}$$

iff $tr\,(M,F) \models tr\,(([s_1,...,s_m]\,\forall y\psi)\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$ (since $x_i \neq y$ for every i)
iff $tr\,(M,F) \models tr\,(\varphi\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$.
Similar arguments can be given to each of the 3 other cases (e.g. there exists i such that $x_i = y$ and for every $1 \leq j \leq n$ $x_j \neq z$).

7. If φ is $[s_1,...,s_m]\,[l_1,...,l_k]\,\psi$ with $m,k>0$ then:
$tr\,(M,F) \models (tr\,(\varphi))\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}$
iff $tr\,(M,F) \models (tr\,([s_1,...,s_m,l_1,...,l_k]\,\psi))\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}$
iff $tr\,(M,F) \models tr\,(([s_1,...,s_m,l_1,...,l_k]\,\psi)\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$ (by the i.h.)
iff

$$\begin{aligned} tr\,(M,F) \quad \models \quad & tr([s_1\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}\,,...,s_m\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}\,, \\ & l_1\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}\,,...,l_k\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}]\psi\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}) \end{aligned}$$

iff

$$\begin{aligned} tr\,(M,F) \quad \models \quad & tr([s_1\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}\,,...,s_m\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}] \\ & [l_1\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}\,,...,l_k\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}]\,\psi\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\}) \end{aligned}$$

iff $tr\,(M,F) \models tr\,(([s_1,...,s_m]\,[l_1,...,l_k]\,\psi)\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$
iff $tr\,(M,F) \models tr\,(\varphi\,\{\overline{a_1}/x_1,...,\overline{a_n}/x_n\})$.

$\square$

References

[Ga-2010] Gaifman, H. (2010). Vagueness, Tolerance and Contextual Logic. Synthese, 174, 5-46.

[Ha-1998] Hajek P., Metamathematics of Fuzzy Logic, Kluwer Academic Publishers, 1998.

[Sh-1967] Shoenfield, J.R., Mathematical Logic, Association for Symbolic Logic, 1967.

[Be-2008] Bergmann, M., An Introduction to Many-Valued and Fuzzy Logic - Semantics, Algebras, and Derivation Systems, Cambridge University Press, 2008.

[Barr-1981] Barr, A & Feigenbaum, E.A, The Handbook of Artificial Intelligence, Addison-Wesley Publishing Company, 1981.

[Th] Thomason, R.H. (2005). Contextual Effects on Vagueness and the Sorites Paradox: A Preliminary Study. Proceedings of the CRR'05 Workshop on Context Representation and Reasoning.

תקציר

לוגיקה הקשרית של סובלנות (TCL) מיועדת להתמודד עם פרדיקטים "סובלניים" (כגון "גבוה"). הדבר נעשה על ידי הוספת אופרטור הקשר חדש לשפה של הלוגיקה הקלאסית, שנועד לבטא את ההקשר שבו טענה מסוימת נאמרת. כך בפתרון של פרדוקס הערימה על ידי TCL פרדיקט כמו "גבוה" נחשב תלוי־הקשר וסובלני (האחרון במובן שערך האמת המתאים לו אינו מושפע משינויים קטנים). TCL הוצגה ב־[Ga-2010], שם היא מאופיינת הן באופן סמנטי והן על ידי מערכת הוכחה. אולם כאן אנו מראים שמערכת ההוכחה והסמנטיקה הללו אינן תואמות זו את זו.

עם TCL כנקודת מוצא, מוצעות כאן שלוש סמנטיקות לטיפול בתלות בהקשר, ומסופקות מערכות הוכחה שלמות ונאותות עבורן. סמנטיקה אחת שקולה לסמנטיקה המקורית (מ־[Ga-2010]), שנייה שקולה למערכת ההוכחה המקורית, ואילו השלישית היא כללית יותר, ומוכלת (כיחס נביעה) בשתיים האחרות. לאחר הצגת המערכות הלוגיות הללו נחקרים מספר מאפיינים שלהן, קשרים בינן ובין עצמן וכן בינן ובין לוגיקה קלאסית מסדר ראשון. לסיום, מוצג פתרון של פרדוקס הערימה שמבוסס על סובלנות ותלות בהקשר.

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חקירה לוגית של תלות הקשרית

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