Towards Automated Reasoning in Herbrand Structures

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Abstract

Herbrand structures have the advantage, computationally speaking, of being guided by the definability of all elements in them. A salient feature of the logics induced by them is that they internally exhibit the induction scheme, thus providing a congenial, computationally-oriented framework for formal inductive reasoning. Nonetheless, their enhanced expressivity renders any effective proof system for them incomplete. Furthermore, the fact that they are not compact poses yet another proof-theoretic challenge. This paper offers several layers for coping with the inherent incompleteness and non-compactness of these logics. First, two types of infinitary proof system are introduced—one of infinite width and one of infinite height—which manipulate infinite sequents and are sound and complete for the intended semantics. The restriction of these systems to finite sequents induces a completeness result for finite entailments. Then, in search of effectiveness, two finite approximations of these systems are presented and explored. Interestingly, the approximation of the infinite-width system via an explicit induction scheme turns out to be weaker than the effective cyclic fragment of the infinite-height system.

1 Introduction

Herbrand structures are a particularly useful subclass of first-order structures due to their strong definitional character. In these structures every element has a unique name, i.e. each semantical element is uniquely definable by some closed term of the language. This strong definability property of the logic induced by Herbrand structures offers both computational and pedagogical advantages over classical first-order logic. For instance, it enables a convenient way of performing symbolic computations, its models can be reduced to sets of atomic sentences, and it has a strong expressive power. It is for these reasons that Herbrand structures are widely used in various subfields of computer science. In logic they are often used as a key component in completeness proofs; in automated reasoning they are used for automated theorem proving [12, 7]; in deductive databases and logic programming they provide semantics for logic programs [28, 1]; and in logic education [23] they provide a simplified semantics for first-order languages.

This paper studies the logics induced by Herbrand structures on their own merit, not solely as an instrumental tool in subfields of artificial intelliegence and logic-related research.[1] Thus, we start by identifying the different components comprising Herbrand structures: (i) every element of the domain has a name in the formal language; and (ii) every such name is unique. The latter property is called "structure-definability". Structures that satisfy only (i) are called semi-Herbrand structures, and those who satisfy both (i) and (ii) are called Herbrand structures. We also consider the addition of equality to the

[1] Such line of research was suggested in [24], and the work presented here is an extension of [17].

language, and thus obtain four variants of the logic, which we call "Herbrand logics". Herbrand logics are "super-classical", in the sense that classical first-order logic is strictly contained in them.

There are some critical differences between Herbrand logics and first-order logic. First, the rich expressive power of Herbrand logics entails that they are inherently incomplete, in the sense that there cannot exist sound, effective proof systems for these logics that are also complete. Second, they are not even compact and thus, for instance, from the possibly infinite set of assumptions $\{\varphi(t) : t \in cl(\mathcal{L})\}$ (where $cl(\mathcal{L})$ is the set of closed terms of the language $\mathcal{L}$) one can infer $\forall x\varphi$, whereas in the general case it cannot be inferred from any finite subset of the assumptions. Finally, while both downward and upward Löwenheim-Skolem theorems hold for classical first-order logic, only the former holds for Herbrand logics.[2] The first two properties present major challenges in the development of a proof theory for Herbrand logics.

Accordingly, the main contribution of this paper is in the introduction of several sequent calculi [25] for (each of the variants of) Herbrand logics, as well as their comparison to one another. Sequent calculi are widely used in automated reasoning, especially when one is interested in the computational aspects of a logic (e.g., [33]). It has been employed in a variety of logical frameworks, e.g., many-valued and fuzzy logics [30, 11], modal logics [40, 31], paraconsistent logics [4], and also logical argumentation [2].

The standard way to recover completeness for logics that are not compact is to forgo effectiveness and develop an infinitary system (see, e.g., [37]). Thus, we present two approaches for developing *infinitary* sequent calculi which are both sound and complete for the various Herbrand logics. The first is an *infinite-width* system whose main component is a generalization of the ω-rule [37]. The fact that this well-known, natural derivation rule suffices for a deductive characterization of Herbrand structures is non-trivial, as evidenced by the construction employed in the completeness proofs. The second system is of *infinite height*, i.e. a non-well-founded proof system adhering to the principle of infinite descent [8, 9, 10, 16, 18, 35, 39]. The non-compactness of the logic entails that one must allow both systems to manipulate infinite sequents. Indeed, these systems are complete for their intended semantics. If the sequents permitted by the systems are only finite, completeness is retained with respect to finite inferences.

In order for proof systems for Herbrand logics to serve as a baseline for useful automation in any of the applications in computer science mentioned above one must consider effective approximations. We present two such approximations corresponding to the two infinite approaches. Finitary and sound approximations of the infinite-width systems are obtained by replacing the generalized ω-rule with a generalization of its finitary counterpart, namely the standard induction rule for Peano Arithmetics [25]. For the infinite-height systems, finitary and sound approximations are obtained by restricting our attention only to regular proofs, i.e. proofs that can be represented by finite, possibly cyclic, graphs. Interestingly, these two natural approximations are not equivalent as we show that the cyclic subsystem is stronger than the induction-based system.

The rest of this paper is organized as follows: In Section 2 variants of Herbrand structures are defined, and the semantics they induce are investigated. Sound and complete infinite-width proof systems for Herbrand logics are introduced in Section 3, followed by the equivalent infinite-height proof systems that are presented in Section 4. Section 5 presents finitary approximations of these systems: the induction-based system (Subsection 5.1) and the cyclic system (Subsection 5.2). Then, the latter is shown to be strictly stronger than the former (Subsection 5.3). We conclude with Section 6, where other directions for further research are suggested.

[2]Since Herbrand logics are super-classical, Lindström's theorem (see, e.g., [21]) tells us that they cannot be both compact and admit the downward Löwenheim-Skolem theorem. The fact that they do admit one of these properties makes Herbrand logics closer to first-order logics than other super-classical logics.

2 Herbrand Logics

This section is devoted to the various logics Herbrand structures induce. After the preliminary Section 2.1, in Section 2.2 we study structures that admit two well-known variants of the Herbrand property, namely Herbrand and semi-Herbrand structures. In Section 2.3 we add a distinguished equality symbol to the language, and explore its effect on the results of Section 2.2.

2.1 Preliminaries

A first-order language $\mathcal{L}$ consists of a set of variables, the standard logical connectives $\wedge, \vee, \supset$ and $\neg$, the logical quantifiers $\forall$ and $\exists$, and a signature that consists of a set $func\left(\mathcal{L}\right)$ of function symbols, and a set $pred\left(\mathcal{L}\right)$ of predicate symbols. We use x, y, z, w (possibly with subscripts) to denote variables of $\mathcal{L}$. We also denote by $consts\left(\mathcal{L}\right)$ the set of 0-ary function symbols (constants), and by $func^{+}\left(\mathcal{L}\right)$ the set $func\left(\mathcal{L}\right) \setminus consts\left(\mathcal{L}\right)$. The arity of a function symbol f is denoted by $ar(f)$. A term (formula) is called closed if no free variable occurs in it, and is called open otherwise. The set of closed terms of $\mathcal{L}$ is denoted by $cl\left(\mathcal{L}\right)$, and the set of its closed atomic formulas is denoted by $catoms\left(\mathcal{L}\right)$. We adopt standard conventions for eliminating parentheses (e.g., when writing $\varphi_1 \wedge \varphi_2 \supset \psi$ rather than $(\varphi_1 \wedge \varphi_2) \supset \psi$. We write $\varphi\left\{\frac{t}{x}\right\}$ to denote the formula obtained from φ by simultaneously substituting any free occurrence of x in φ with t.

A first-order structure for $\mathcal{L}$ is a pair $\langle D, I\rangle$, such that D is a non-empty set and I is an interpretation function, assigning an n-ary function (relation) over D to every n-ary function (predicate) symbol of $\mathcal{L}$. Given a first-order structure $M = \langle D, I\rangle$ for $\mathcal{L}$, an M-valuation is a function v from the set of variables to D. M-valuations are extended to be applied on arbitrary $\mathcal{L}$-terms in the usual manner. The satisfaction relation $M, v \models \varphi$ is standardly defined. We write $M \models \varphi$ if $M, v \models \varphi$ for every M-valuation v.

In what follows, we fix a first-order language $\mathcal{L}$ which has at least one constant symbol.[1] By a structure we mean a first-order structure for $\mathcal{L}$. We denote the classical satisfaction relation between structures and formulas by $\models$, and its induced consequence relation by $\vdash$.

2.2 Herbrand and Semi-Herbrand Structures

We start the investigation of Herbrand structures by disassembling their standard definition into its two components: the existence of a name for every domain element, and its uniqueness.[2]

Definition 2.1. A structure $M = \langle D, I\rangle$ is called *semi-Herbrand* if for every $a \in D$ there is some $t \in cl\left(\mathcal{L}\right)$ such that $I\left(t\right) = a$. If for each $a \in D$, there is a ***unique*** $t \in cl\left(\mathcal{L}\right)$ such that $I\left(t\right) = a$, M is called a *Herbrand structure*.

Since in Herbrand structures every element has a unique name, it is possible (and beneficial from a computational point of view) to fix the domain to consist solely of these names. Thus, in the reminder of this paper, we utilize the more common definition of a Herbrand structure, requiring that $D = cl\left(\mathcal{L}\right)$ and $I\left(t\right) = t$ for every $t \in D$. It is routine to show that every Herbrand structure is isomorphic to such a structure.

Herbrand structures allow for an even more computationally-oriented definition, as they admit a strong notion of definability, which we call "structure-definability". That is, every Herbrand structure is uniquely

[1] Dismissing this requirement yields a Herbrand counterpart of free-logic [27].

[2] These two components correspond to well-known concepts from inductive reasoning: a set of operators generating a set, and the question whether each element is generated in a unique way [22].

determined by a set of closed atoms, by setting $I(P) = \{\langle t_1, \ldots, t_n\rangle \in cl(\mathcal{L})^n : P(t_1, \ldots, t_n) \in A\}$ for every predicate symbol P.[3]

The fact that every element in the domain has a name allows for an equivalent, more convenient definition of quantifier semantics and of satisfaction of open formulas.

Lemma 2.2. *Let M be a semi-Herbrand structure.*

- *$M \models \exists x\psi$ iff $M \models \psi\left\{\frac{t}{x}\right\}$ for some $t \in cl(\mathcal{L})$.*
- *$M \models \forall x\psi$ iff $M \models \psi\left\{\frac{t}{x}\right\}$ for every $t \in cl(\mathcal{L})$.*

Definition 2.3. Let $\varphi(x_1, \ldots, x_n)$ be an open formula whose free variables are a subset of $\{x_1, \ldots, x_n\}$. An *$\mathcal{L}$-instance of* φ is a formula of the form $\varphi\left\{\frac{t_1}{x_1}, \ldots, \frac{t_n}{x_n}\right\}$, where $t_1, \ldots, t_n \in cl(\mathcal{L})$.

Lemma 2.4. *Let M be a semi-Herbrand structure. Then $M \models \varphi$ iff $M \models \varphi'$ for every $\mathcal{L}$-instance φ' of φ.*

The consequence relations induced by (semi-)Herbrand structures are defined as follows:

Definition 2.5. Let $T \cup \{\varphi\}$ be a set of $\mathcal{L}$-formulas. $T \vdash^{H} \varphi$ ($T \vdash^{SH} \varphi$) if for every (semi-) Herbrand structure M, $M \models T$ (i.e. $M \models \psi$ for every $\psi \in T$) implies $M \models \varphi$.

Notice that the relations $\vdash^{SH}$ and $\vdash^{H}$ are parametrized by the underlying language $\mathcal{L}$, and the answer to derivability questions highly depends on the identity of $\mathcal{L}$. For readability, we fix an arbitrary first-order language $\mathcal{L}$ as the underlying language for the rest of this paper, unless specifically stated otherwise.

The next proposition summarizes the relations between the logics that are induced by Herbrand, semi-Herbrand and arbitrary structures. Note that the proposition assumes $\mathcal{L}$ does not include a distinguished equality symbol (such languages are the subject of the next subsection).

Proposition 2.6. $\vdash \subsetneq \vdash^{SH} = \vdash^{H}$.

Proof. Since every Herbrand structure is also semi-Herbrand, and every semi-Herbrand structure is a classical structure, we have that $\vdash \subseteq \vdash^{SH} \subseteq \vdash^{H}$. To see that $\vdash \neq \vdash^{SH}$, consider e.g. the case where $\mathcal{L}$ consists of one constant symbol c, and a unary predicate P. Clearly, $P(c) \vdash^{SH} \forall x P(x)$, whereas $P(c) \nvdash \forall x P(x)$.[4] As for $\vdash^{SH}$ and $\vdash^{H}$, for every semi-Herbrand structure $M = \langle D, I\rangle$, an equivalent Herbrand structure $H_M = \langle cl(\mathcal{L}), I'\rangle$ can be constructed by taking $I'(t) = t$ for every $t \in cl(\mathcal{L})$, and $I'(P) = \{\langle t_1, \ldots, t_n\rangle : \langle I(t_1), \ldots, I(t_n)\rangle \in I(P)\}$ for each predicate symbol P. It can be shown using structural induction on the formulas of $\mathcal{L}$ that M and H_M satisfy the same formulas. □

2.3 Handling Equality

In this section we use the classical axiomatization of equality to study equality in Herbrand logics. Throughout, we assume $\mathcal{L}$ includes a binary predicate symbol $=$, and abbreviate $\neg(s = t)$ by $(s \neq t)$. A structure $M = \langle D, I\rangle$ is called *normal* if $I(=)$ is $\{\langle a, a\rangle \mid a \in D\}$.

Definition 2.7. $\vdash^{H_=}$ ($\vdash^{SH_=}$) is defined similarly to $\vdash^{H}$ ($\vdash^{SH}$), but is restricted to normal Herbrand (semi-Herbrand) structures.

The addition of equality to the signature separates the consequence relations induced by Herbrand and semi-Herbrand structures.

[3]This is how Herbrand structures were defined in [24].

[4]A more general argument was given in [24] for the proper inclusion of $\vdash$ in $\vdash^{H}$, which amounts to the failure of the compactness theorem for $\vdash^{H}$.

Proposition 2.8. *For languages with at least two distinct closed terms,* $\vdash^{SH_=} \subsetneq \vdash^{H_=}$.[5]

Proof. For t_1 and t_2 two distinct closed terms we have $\vdash^{H_=} t_1 \neq t_2$, while $\nvdash^{SH_=} t_1 \neq t_2$. $\square$

Semi-Herbrand structures allow for the same axiomatization of equality that is used in classical logic.

Definition 2.9. Let $Equiv = \{x = x, x = y \supset y = x, x = y \wedge y = z \supset x = z\}$, and let $E(\mathcal{L})$ be the set consisting of the following formulas:

- $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$ for every n-ary ($n > 0$) function symbol f.
- $x_1 = y_1 \wedge \ldots \wedge x_n = y_n \supset (P(x_1, \ldots, x_n) \supset P(y_1, \ldots, y_n))$ for every n-ary predicate symbol P.

Finally, let $Eq(\mathcal{L})$ be the set $Equiv \cup E(\mathcal{L})$. For normal semi-Herbrand structures, we have:

Proposition 2.10. $T \vdash^{SH_=} \varphi$ *iff* $T \cup Eq(\mathcal{L}) \vdash^{SH} \varphi$.

$Eq(\mathcal{L})$, however, is insufficient for characterizing normal Herbrand structures. For example, $\vdash^{H_=} t_1 \neq t_2$, but $Eq(\mathcal{L}) \nvdash^{H} t_1 \neq t_2$, whenever t_1 and t_2 are distinct closed terms. Interestingly, in Herbrand structures, capturing equality requires also the *inequalities* of the language, which are succinctly described using the following inequality axioms. Together with the equality axioms above, we obtain a variant of Clark equality theory [13].

Definition 2.11. Let $inE(\mathcal{L})$ be the set consisting of the following formulas:

- $f(x_1, \ldots, x_n) \neq g(y_1, \ldots, y_m)$ for every distinct function symbols f and g with arities n and m, respectively.[6]
- $x_i \neq y_i \supset f(x_1, \ldots x_i, \ldots, x_n) \neq f(x_1, \ldots y_i, \ldots, x_n)$ for every n-ary ($n > 0$) function symbol f and $1 \leq i \leq n$.

Notice that $inE(\mathcal{L})$ is determined solely by $func(\mathcal{L})$. Therefore, whenever this set is finite, so is $inE(\mathcal{L})$. This is in contrast to $Eq(\mathcal{L})$, which also depends on $pred(\mathcal{L})$.

Example 2.12. [24] defines a theory (which is here denoted by NAT) in a language (which is here denoted by $\mathcal{L}_{\mathsf{NAT}}$) for arithmetics, that consists of a constant 0, a unary function s, two ternary predicate symbols $plus$ and $times$, and a binary predicate symbol $equal$. For this particular language, we have $inE(\mathcal{L}_{\mathsf{NAT}}) = \{0 \neq s(x), s(x) \neq 0, x \neq y \supset s(x) \neq s(y)\}$. These axioms, together with $x = x$, are essentially the axioms that were used for the predicate $equal$ in NAT.

Lemma 2.13. *The following hold:*

- $inE(\mathcal{L}) \vdash^{H} s \neq t$ *for every distinct closed terms* s *and* t*.*
- *Let* M *be a Herbrand structure.* $M \models inE(\mathcal{L}) \cup \{x = x\}$ *iff* M *is normal.*
- *Let* M *be a semi-Herbrand structure.* $M \models inE(\mathcal{L})$ *iff* M *is Herbrand.*

With $inE(\mathcal{L})$, we obtain the following counterpart of Proposition 2.10.

Proposition 2.14. $T \vdash^{H_=} \varphi$ *iff* $T \cup inE(\mathcal{L}) \cup \{x = x\} \vdash^{H} \varphi$.

[5] If the language has only one closed term the consequence relations are identical.

[6] Note that f and g may be constant symbols.

Figure 1: Rules of the Basic Calculus G

$$\text{(id)}\ \frac{}{\Gamma,\varphi\Rightarrow\varphi,\Delta}\qquad \text{(weak)}\ \frac{\Gamma\Rightarrow\Delta}{\Gamma,\Gamma'\Rightarrow\Delta',\Delta}\qquad \text{(cut)}\ \frac{\Gamma\Rightarrow\varphi,\Delta\quad \Gamma,\varphi\Rightarrow\Delta}{\Gamma\Rightarrow\Delta}$$

$$(\wedge\Rightarrow)\ \frac{\Gamma,\varphi,\psi\Rightarrow\Delta}{\Gamma,\varphi\wedge\psi\Rightarrow\Delta}\qquad (\Rightarrow\wedge)\ \frac{\Gamma\Rightarrow\varphi,\Delta\quad \Gamma\Rightarrow\psi,\Delta}{\Gamma\Rightarrow\varphi\wedge\psi,\Delta}$$

$$(\vee\Rightarrow)\ \frac{\Gamma,\varphi\Rightarrow\Delta\quad \Gamma,\psi\Rightarrow\Delta}{\Gamma,\varphi\vee\psi\Rightarrow\Delta}\qquad (\Rightarrow\vee)\ \frac{\Gamma\Rightarrow\varphi,\psi,\Delta}{\Gamma\Rightarrow\varphi\vee\psi,\Delta}$$

$$(\supset\Rightarrow)\ \frac{\Gamma\Rightarrow\varphi,\Delta\quad \Gamma,\psi\Rightarrow\Delta}{\Gamma,\varphi\supset\psi\Rightarrow\Delta}\qquad (\Rightarrow\supset)\ \frac{\Gamma,\varphi\Rightarrow\psi,\Delta}{\Gamma\Rightarrow\varphi\supset\psi,\Delta}$$

$$(\neg\Rightarrow)\ \frac{\Gamma\Rightarrow\varphi,\Delta}{\Gamma,\neg\varphi\Rightarrow\Delta}\qquad (\Rightarrow\neg)\ \frac{\Gamma,\varphi\Rightarrow\Delta}{\Gamma\Rightarrow\neg\varphi,\Delta}$$

$$(\forall\Rightarrow)\ \frac{\Gamma,\varphi\{\frac{t}{x}\}\Rightarrow\Delta}{\Gamma,\forall x\varphi\Rightarrow\Delta}\qquad (\Rightarrow\exists)\ \frac{\Gamma\Rightarrow\varphi\{\frac{t}{x}\},\Delta}{\Gamma\Rightarrow\exists x\varphi,\Delta}$$

Corollary 2.15. *$T\vdash^{H_=}\varphi$ iff $T\cup inE(\mathcal{L})\vdash^{SH_=}\varphi$.*

To conclude this section, we make a brief remark about the axiomatization of arithmetics within the framework of Herbrand structures. The theory NAT, mentioned in Example 2.12, provides an axiomatization that categorically characterizes the natural numbers under Herbrand structures. This axiomatization is actually (the relational version of) Peano Arithmetics (PA) without the induction scheme, called Π_2 (see, e.g., [29]). Robinson [34] added to this system the axiom $\forall x.x\neq 0\supset\exists y.x=s(y)$, thus forming his famous system Q. When only considering Herbrand structures for $\mathcal{L}_{\mathsf{NAT}}$, this additional axiom is valid, and thus the induction-free part of PA suffices. Moreover, the induction scheme itself is also valid in such structures. Therefore, Π_2, Q and PA are equivalent in Herbrand structures.

It was already shown in [24] that $\vdash^{H}$ is not recursively enumerable, and the same holds for the other variants of Herbrand logics (due to the above translations between them). Consequently, Herbrand logics are inherently incomplete. Nevertheless, as shown below, there are natural formal systems which allow for infinitary proofs, that are sound and complete for Herbrand logics.

3 Infinite-width Systems

In this section we provide derivation systems for Herbrand logics that do not make use of free variables. This provides a proof-theoretical counterpart of Lemma 2.2, in which an alternative semantics were given for quantified formulas without using free variables.

3.1 Manipulating Infinite Sequents

The proof-theoretic mechanism we use is that of sequent calculi [25]. To obtain completeness for our systems, however, the standard definition of a sequent must be relaxed to allow infinite sequents.

Definition 3.1. A *sequent* is an expression of the form $\Gamma\Rightarrow\Delta$, where Γ and Δ are (possibly infinite) sets of formulas. A sequent containing only closed formulas is called closed; otherwise, it is called open.

Figure 2: Derivation Rules for Herbrand and semi-Herbrand Logics

$$(\Rightarrow\forall)_H \quad \frac{\left\{\Gamma \Rightarrow \varphi\left\{\frac{t}{x}\right\}, \Delta : t \in cl\left(\mathcal{L}\right)\right\}}{\Gamma \Rightarrow \forall x\varphi, \Delta} \qquad (\exists\Rightarrow)_H \quad \frac{\left\{\Gamma, \varphi\left\{\frac{t}{x}\right\} \Rightarrow \Delta : t \in cl\left(\mathcal{L}\right)\right\}}{\Gamma, \exists x\varphi \Rightarrow \Delta}$$

$$(\mathrm{PM}_1) \quad \frac{\Gamma \Rightarrow \varphi\left\{\frac{s}{x}\right\}, \Delta}{\Gamma, s = t \Rightarrow \varphi\left\{\frac{t}{x}\right\}, \Delta} \qquad (\mathrm{PM}_2) \quad \frac{\Gamma \Rightarrow \varphi\left\{\frac{t}{x}\right\}, \Delta}{\Gamma, s = t \Rightarrow \varphi\left\{\frac{s}{x}\right\}, \Delta}$$

$$(\Rightarrow=) \quad \frac{}{\Gamma \Rightarrow t = t, \Delta} \quad t \in cl\left(\mathcal{L}\right) \qquad (=\Rightarrow) \quad \frac{}{\Gamma, s = t \Rightarrow \Delta} \quad s \neq t \in cl\left(\mathcal{L}\right)$$

Let $\mathcal{LK}$ denote a variant of Gentzen's calculus for classical logic [25], in which sequents are taken to be pairs of finite sets, rather than pairs of lists (in particular, contraction, expansion and permutation are not needed). Denote by G the calculus that consists of the rules in Figure 1. Derivations in G are restricted to closed (possibly infinite) sequents. Note that the rules of G are obtained from those of $\mathcal{LK}$ by dismissing $(\Rightarrow\forall)$ and $(\exists\Rightarrow)$ (which are the $\forall$-introduction rule and $\exists$-elimination rule, respectively), replacing the axiom $\varphi \Rightarrow \varphi$ by $\Gamma, \varphi \Rightarrow \varphi, \Delta$ for every closed sequent $\Gamma, \varphi \Rightarrow \varphi, \Delta$, and also replacing the two original weakening rules by the following single weakening rule.

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma, \Gamma' \Rightarrow \Delta', \Delta}$$

The key idea behind capturing the essence of Herbrand logics in a formal proof system is to formalize the syntactic restriction on the domain. This is obtained by replacing the standard $(\Rightarrow\forall)$ rule by a language-based introduction rule for $\forall$, which is a generalization of the ω-rule employed in some proof systems for arithmetic (see, e.g., [37]). Similar modification is required in the $(\exists\Rightarrow)$ rule. The adjustments in the axiom and weakening rules allow the introduction of infinite sequents.

Definition 3.2. We define the following proof systems using the derivation rules in Figs. 1 and 2.

1. $G_H = G + (\Rightarrow\forall)_H + (\exists\Rightarrow)_H$.
2. $G_{SH_=} = G_H + (\Rightarrow=) + (\mathrm{PM}_1) + (\mathrm{PM}_2)$.
3. $G_{H_=} = G_H + (\Rightarrow=) + (=\Rightarrow)$.

Note that (PM_1) and (PM_2) are not included in $G_{H_=}$ since they are derivable using $(\Rightarrow=)$ and $(=\Rightarrow)$.

Let $\mathcal{G}$ be one of the three calculi in the above definition. A derivation of a sequent s in $\mathcal{G}$ is a (possibly infinite) tree with a finite height, rooted with s, in which every node in the tree is the result of an application of some rule of X on the set of its predecessors. We write $\vdash_{\mathcal{G}} s$ if there is a derivation of s in $\mathcal{G}$.

It is important to note that every derivation in the above calculi does not contain occurrences of free variables. Also note that, in general, the above calculi are not effective as they allow for both infinite derivations (by using $(\Rightarrow\forall)_H$ or $(\exists\Rightarrow)_H$ when $cl\left(\mathcal{L}\right)$ is infinite) and infinite sequents (by using an instance of (id) with an infinite sequent, or introducing one with (weak)).

First we show that all three calculi extend $\mathcal{LK}$. For this we consider $\mathcal{L}$-instances of sequents.

Definition 3.3. A *substitution* is a function σ assigning an element of $cl\left(\mathcal{L}\right)$ to every variable. Given a formula φ, φ^σ is obtained from φ by substituting every free occurrence of a variable x with $\sigma\left(x\right)$. An $\mathcal{L}$-instance of a sequent $\Gamma \Rightarrow \Delta$ is any sequent of the form $\Gamma^\sigma \Rightarrow \Delta^\sigma$ for some σ, where $X^\sigma = \{\varphi^\sigma \mid \varphi \in X\}$.

Proposition 3.4. *Let s be a finite closed sequent derivable in $\mathcal{LK}$. Then s is derivable in G_H.*

Proof. Denote by $(\Gamma \Rightarrow \Delta)^{\sigma}$ the sequent $\Gamma^{\sigma} \Rightarrow \Delta^{\sigma}$ (see Definition 3.3). We prove by induction on the length of the proof of s in $\mathcal{LK}$ that $\vdash_{G_H} s^{\sigma}$ for every substitution σ and any finite (possibly open) sequent s. Proposition 3.4 then immediately follows.

1. The cases where s was derived using a structural rule (id, cut, weakening) or an introduction rule of a propositional connective are routine. We explicitly show the case where the last rule in the derivation of s was $(\supset\Rightarrow)$: in this case, s has the form $\Gamma, \varphi \supset \psi \Rightarrow \Delta$, and there are shorter derivations of $\Gamma \Rightarrow \varphi, \Delta$ and $\Gamma, \psi \Rightarrow \Delta$. Let σ be a substitution. By the induction hypothesis, $\vdash_{G_H} \Gamma^{\sigma} \Rightarrow \varphi^{\sigma}, \Delta^{\sigma}$ and $\vdash_{G_H} \Gamma^{\sigma}, \psi^{\sigma} \Rightarrow \Delta^{\sigma}$, and thus $\vdash_{G_H} s^{\sigma}$.

2. If s was derived by introducing $\forall$ on the right side, then s has the form $\Gamma \Rightarrow \forall x\varphi, \Delta$, and there is a shorter derivation of $\Gamma \Rightarrow \varphi\left\{\frac{y}{x}\right\}, \Delta$ for some variable y that does not occur free in $\Gamma \cup \Delta$. Let σ be a substitution. We prove that $\vdash_{G_H} \Gamma^{\sigma} \Rightarrow (\forall x\varphi)^{\sigma}, \Delta^{\sigma}$. For every $t \in cl(\mathcal{L})$, denote by $\sigma_{y\mapsto t}$ the substitution defined by: $\sigma_{y\mapsto t}(z) = t$ if $z = y$, and $\sigma_{y\mapsto t}(z) = \sigma(z)$ otherwise. By the induction hypothesis, $\vdash_{G_H} \Gamma^{\sigma_t} \Rightarrow \left(\varphi\left\{\frac{y}{x}\right\}\right)^{\sigma_t}, \Delta^{\sigma_t}$ for every $t \in cl(\mathcal{L})$. Now, let $t \in cl(\mathcal{L})$. Since y does not occur free in $\Gamma \cup \Delta$, we must have $\Gamma^{\sigma_t} = \Gamma^{\sigma}$ and $\Delta^{\sigma_t} = \Delta^{\sigma}$. Moreover, $\left(\varphi\left\{\frac{y}{x}\right\}\right)^{\sigma_t} = \left(\varphi\left\{\frac{t}{x}\right\}\right)^{\sigma}$. Denote by φ' the formula obtained from φ by substituting all free occurrences of variables z in φ by $\sigma(z)$, except for x, which is left free in φ'. Since t is a closed term, $\left(\varphi\left\{\frac{t}{x}\right\}\right)^{\sigma} = \varphi'\left\{\frac{t}{x}\right\}$. Thus for every $t \in cl(\mathcal{L})$, we have $\vdash_{G_H} \Gamma^{\sigma} \Rightarrow \varphi'\left\{\frac{t}{x}\right\}, \Delta^{\sigma}$. Using $(\Rightarrow\forall)_H$, we obtain that $\vdash_{G_H} \Gamma^{\sigma} \Rightarrow \forall x\varphi', \Delta^{\sigma}$. Clearly, $\forall x\varphi' = (\forall x\varphi)^{\sigma}$, and so we have $\vdash_{G_H} \Gamma^{\sigma} \Rightarrow (\forall x\varphi)^{\sigma}, \Delta^{\sigma}$.

3. If s was derived by introducing $\forall$ on the left side, then s has the form $\Gamma, \forall x\varphi \Rightarrow \Delta$, and there is a shorter derivation of $\Gamma, \varphi\left\{\frac{\alpha}{x}\right\} \Rightarrow \Delta$ for some (possibly open) term α that can be substituted for x in φ. Let σ be a substitution. We prove that $\vdash_{G_H} \Gamma^{\sigma}, (\forall x\varphi)^{\sigma} \Rightarrow \Delta^{\sigma}$. By the induction hypothesis, $\vdash_{G_H} \Gamma^{\sigma}, \left(\varphi\left\{\frac{\alpha}{x}\right\}\right)^{\sigma} \Rightarrow \Delta^{\sigma}$. Let α^{σ} denote the closed term obtained from α by substituting every free occurrence of a variable z in α by $\sigma(z)$. Let φ' be defined as in the previous case. Then $\left(\varphi\left\{\frac{\alpha}{x}\right\}\right)^{\sigma} = \varphi'\left\{\frac{\alpha^{\sigma}}{x}\right\}$. Thus we have $\vdash_{G_H} \Gamma^{\sigma}, \varphi'\left\{\frac{\alpha^{\sigma}}{x}\right\} \Rightarrow \Delta^{\sigma}$. α^{σ} is a closed term, and thus can be substituted for x in φ'. Using $(\forall\Rightarrow)$, we obtain $\vdash_{G_H} \Gamma^{\sigma}, \forall x\varphi' \Rightarrow \Delta^{\sigma}$. Clearly, $\forall x\varphi' = (\forall x\varphi)^{\sigma}$, and so we have $\vdash_{G_H} \Gamma^{\sigma}, (\forall x\varphi)^{\sigma} \Rightarrow \Delta^{\sigma}$.

4. The cases where the last rule in the derivation introduces $\exists$ are symmetric. □

We now prove that all three calculi are sound and complete with respect to their corresponding Herbrand logics. We begin with the soundness results.

Definition 3.5. Let M be a structure for $\mathcal{L}$ and $s = \Gamma \Rightarrow \Delta$ a closed sequent. $M \models s$ if $M \not\models \varphi$ for some $\varphi \in \Gamma$ or $M \models \psi$ for some $\psi \in \Delta$. We write $\vdash_{G_H} s$ if $M \models s$ for every Herbrand structure M for $\mathcal{L}$, and use this notation similarly for $\vdash_{G_{SH_=}}$ and $\vdash_{G_{H_=}}$.

Theorem 3.6 (Soundness)**.** *Let s be a closed sequent. The following hold:*

1. $\vdash_{G_H} s$ *only if* $\vdash^{H} s$.
2. $\vdash_{G_{H_=}} s$ *only if* $\vdash^{H_=} s$.
3. $\vdash_{G_{SH_=}} s$ *only if* $\vdash^{SH_=} s$.

Proof.

1. Let s be a closed sequent, and assume $\vdash_{G_H} s$. We prove that every Herbrand structure is a model of s. The proof is carried out by induction on the proof tree of s in G_H. The base cases are trivial, and the rules in G are proven to be sound just as in the classical case. We here consider the $(\Rightarrow\forall)_H$ rule. The case for $(\exists\Rightarrow)_H$ is symmetric. Let $\Gamma \Rightarrow \forall x\varphi, \Delta$ and $\left\{\Gamma \Rightarrow \varphi\left\{\frac{t}{x}\right\}, \Delta : t \in cl\left(\mathcal{L}\right)\right\}$ be the conclusion and set of premises, respectively of an application of $(\Rightarrow\forall)_H$ and M a Herbrand structure. We prove that $M \models \Gamma \Rightarrow \forall x\varphi, \Delta$. By the induction hypothesis, $M \models \Gamma \Rightarrow \varphi\left\{\frac{t}{x}\right\}, \Delta$ for every $t \in cl\left(\mathcal{L}\right)$. If $M \models \Gamma \Rightarrow \Delta$, then we have that $M \models \Gamma \Rightarrow \forall x\varphi, \Delta$. Otherwise, using Lemma 2.2, we must have $M \models \varphi\left\{\frac{t}{x}\right\}$ for every $t \in cl\left(\mathcal{L}\right)$. In such a case, $M \models \forall x\varphi$, and in particular, $M \models \Gamma \Rightarrow \forall x\varphi, \Delta$.
2. For $G_{SH_=}$, the additional rules are proved to be sound similarly to the classical case.
3. For $G_{H_=}$, $(\Rightarrow=)$ is proved to be sound similarly to the classical case. As for $(=\Rightarrow)$, let $s = t \Rightarrow$ be an application of $(=\Rightarrow)$. Since s and t are distinct and M is a Herbrand structure, $M \not\models s = t$, and so $M \models s = t \Rightarrow$. □

We now turn to completeness. The main challenge in the completeness proofs is the unavailability of Lindenbaum's lemma, due to the infinitary nature of the systems. This, in turn, renders the standard construction of a maximal unprovable sequent inapplicable. To solve this problem, we employ a similar method to the one used in [20, 38], and generate a sequent that admits all the necessary properties for inducing a countermodel, without necessarily being maximal-unprovable. This is achieved by the addition of Henkin witnesses from $\mathcal{L}$ itself (unlike in the classical case, where the language is extended with new constant symbols that serve as Henkin witnesses). We first describe the constructions for the system G_H, and then consider their amendments for the $G_{H_=}$ and $G_{SH_=}$.

Lemma 3.7.

1. *If* $\not\vdash_{G_H} \Gamma \Rightarrow \Delta$ *and* $\forall x\varphi \in \Delta$*, then there exists* $t \in cl\left(\mathcal{L}\right)$ *such that* $\not\vdash_{G_H} \Gamma \Rightarrow \varphi\left\{\frac{t}{x}\right\}, \Delta$.
2. *If* $\not\vdash_{G_H} \Gamma \Rightarrow \Delta$ *and* $\exists x\varphi \in \Gamma$*, then there exists* $t \in cl\left(\mathcal{L}\right)$ *such that* $\not\vdash_{G_H} \Gamma, \varphi\left\{\frac{t}{x}\right\} \Rightarrow \Delta$.

Proof.

1. Assume otherwise. Then $\vdash_{G_H} \Gamma \Rightarrow \varphi\left\{\frac{t}{x}\right\}, \Delta$ for every $t \in cl\left(\mathcal{L}\right)$. Using $(\Rightarrow\forall)_H$, $\vdash_{G_H} \Gamma \Rightarrow \Delta$ which is a contradiction.
2. Assume otherwise. Then $\vdash_{G_H} \Gamma, \varphi\left\{\frac{t}{x}\right\} \Rightarrow \Delta$ for every $t \in cl\left(\mathcal{L}\right)$. Using $(\exists\Rightarrow)_H$, $\vdash_{G_H} \Gamma \Rightarrow \Delta$ which is a contradiction. □

For the following constructions and properties, we assume some given sequent $\Gamma \Rightarrow \Delta$ that is unprovable, and an enumeration $\psi_0, \psi_1, \ldots$ of the closed formulas of $\mathcal{L}$. We inductively define a sequence $s_0 = \Gamma_0 \Rightarrow \Delta_0, s_1 = \Gamma_1 \Rightarrow \Delta_1, \ldots$ of sequents as follows.

$$s_0 = \Gamma \Rightarrow \Delta$$

$$s_{2i+1} = \begin{cases} \Gamma_{2i}, \psi_i \Rightarrow \Delta_{2i} & \text{if } \not\vdash_{G_H} \Gamma_{2i}, \psi_i \Rightarrow \Delta_{2i} \\ \Gamma_{2i} \Rightarrow \psi_i, \Delta_{2i} & \text{otherwise} \end{cases}$$

$$s_{2i+2} = \begin{cases} \Gamma_{2i+1} \Rightarrow \varphi\left\{\frac{t}{x}\right\}, \Delta_{2i+1} & \text{if } \psi_i = \forall x\varphi \in \Delta_{2i+1} \\ \Gamma_{2i+1}, \varphi\left\{\frac{t}{x}\right\} \Rightarrow \Delta_{2i+1} & \text{if } \psi_i = \exists x\varphi \in \Gamma_{2i+1} \\ s_{i+1} & \text{otherwise} \end{cases}$$

where t is taken to be the minimal closed term given by Lemma 3.7 w.r.t. a chosen well-ordering on $cl(\mathcal{L})$. This construction follows the usual construction of a maximal unprovable sequent, with the crucial change that the so-called “Henkin-witnesses” (added to sequents with even indexes) consist of closed terms in the language, and not new constant symbols.

This sequence contains each of the closed formulas of $\mathcal{L}$.

Lemma 3.8. *$\psi_i \in \Gamma_{2i+1} \cup \Delta_{2i+1}$ for every $i \in \mathbb{N}$.*

Moreover, when the sequent $\Gamma \Rightarrow \Delta$ is unprovable, so are all the sequents in the sequence.

Lemma 3.9. *$\nvdash_{G_H} \Gamma_i \Rightarrow \Delta_i$ for every $i \in \mathbb{N}$.*

Proof. By induction on i. The base case follows from the assumption regarding $\Gamma \Rightarrow \Delta$. Let $i \geq 0$. Assume that $\nvdash_{G_H} s_{2i}$. We prove that $\nvdash_{G_H} s_{2i+1}$ and $\nvdash_{G_H} s_{2i+2}$. We start with $\nvdash_{G_H} s_{2i+1}$. If $\nvdash_{G_H} \Gamma_{2i}, \psi_i \Rightarrow \Delta_{2i}$, then $\nvdash_{G_H} s_{2i+1}$. Otherwise, $\vdash_{G_H} \Gamma_{2i}, \psi_i \Rightarrow \Delta_{2i}$, and then using cut, we must have that $\nvdash_{G_H} s_{2i+1}$, as otherwise the induction hypothesis would be contradicted. Now for s_{2i+2}. If $\psi_i = \forall x\varphi \in \Delta_{2i+1}$ then the fact that $\nvdash_{G_H} s_{i+2}$ follows from the choice of t according to Lemma 3.7, using the fact that $\nvdash_{G_H} s_{2i+1}$. If $\psi_i = \exists x\varphi \in \Gamma_{2i+1}$ then this is similar. Otherwise, $s_{i+2} = s_{i+1}$, and we have already shown that $\nvdash_{G_H} s_{i+1}$. □

Let $L = \bigcup_{i\in\mathbb{N}} \Gamma_i$ and $R = \bigcup_{i\in\mathbb{N}} \Delta_i$.

Lemma 3.10. *$\{L, R\}$ is a partition of the set of closed formulas.*

Proof. By Lemma 3.8, for every formula ψ_i, $\psi_i \in \Gamma_{2i+1} \cup \Delta_{2i+1} \subseteq L \cup R$. Now assume $L \cap R \neq \emptyset$. Then there are $i, j \in \mathbb{N}$ such that $\psi \in \Gamma_i \cap \Delta_j$. Let $k = max\{i, j\}$. Then $\psi \in \Gamma_k \cap \Delta_k$, contradicting Lemma 3.9. Finally, neither L nor R can be empty, as L contains all tautologies and R all contradictions. □

Given the sequence of sequents from $\Gamma \Rightarrow \Delta$ defined above, we then define the following Herbrand structure: $M = \langle D, I\rangle$, where $D = cl(\mathcal{L})$, $I(t) = t$ for every $t \in cl(\mathcal{L})$, and $I(P) = \{\langle t_1, \ldots, t_n\rangle \in D^n \mid P(t_1, \ldots, t_n) \in L\}$. This structure has the following property.

Lemma 3.11. *$\varphi \in L$ iff $M \models \varphi$.*

Proof. By induction on φ.

1. If $\varphi \in catoms(\mathcal{L})$ then this follows by the definition of M.
2. If $\varphi = \varphi_1 \supset \varphi_2$ then:
   (a) Suppose $\varphi \in L$, and assume for contradiction that $M \not\models \varphi$. Then $M \models \varphi_1$ and $M \not\models \varphi_2$. By the induction hypothesis, $\varphi_1 \in L$ and $\varphi_2 \notin L$, that is, $\varphi_2 \in R$. Thus there exists $k \in \mathbb{N}$ such that $\varphi_1, \varphi \in \Gamma_k$, while $\varphi_2 \in \Delta_k$, contradicting Lemma 3.9 using ($\supset\Rightarrow$).
   (b) Suppose $\varphi \notin L$. Then $\varphi \in R$. Assume for contradiction that $M \models \varphi$. Then either $M \not\models \varphi_1$ or $M \models \varphi_2$. By the induction hypothesis, either $\varphi_1 \notin L$ (that is, $\varphi_1 \in R$) or $\varphi_2 \in L$. Thus there exists $k \in \mathbb{N}$ such that $\varphi \in \Delta_k$, and either $\varphi_2 \in \Gamma_k$ or $\varphi_1 \in \Delta_k$. Either way, we obtain a contradiction to Lemma 3.9 using ($\Rightarrow\supset$).
3. If φ has one of the forms $\varphi_1 \wedge \varphi_2$, $\varphi_1 \supset \varphi_2$ or $\neg\psi$ then this is shown similarly.
4. If $\varphi = \forall x\psi$ then:

(a) Suppose $\varphi \in L$, and assume for contradiction that $M \not\models \varphi$. Then $M \not\models \psi\left\{\frac{t}{x}\right\}$ for some $t \in cl(\mathcal{L})$. By the induction hypothesis, $\psi\left\{\frac{t}{x}\right\} \notin L$, that is, $\psi\left\{\frac{t}{x}\right\} \in R$, which means that there exists $k \in \mathbb{N}$ such that $\varphi \in \Gamma_k$ and $\psi\left\{\frac{t}{x}\right\} \in \Delta_k$, which contradicts Lemma 3.9 using $(\forall\Rightarrow)$.

(b) Suppose $\varphi \notin L$. Then $\varphi \in R$. Assume for contradiction that $M \models \varphi$. Then $M \models \psi\left\{\frac{t}{x}\right\}$ for every $t \in cl(\mathcal{L})$. By the induction hypothesis, $\psi\left\{\frac{t}{x}\right\} \in L$ for every $t \in cl(\mathcal{L})$. Let $k \in \mathbb{N}$ such that $\varphi = \psi_k$. By Lemma 3.8, $\varphi \in \Gamma_{2k+1} \cup \Delta_{2k+1}$. $\varphi \notin L$ and thus $\varphi \in \Delta_{2k+1}$. By definition, there exists some $t \in cl(\mathcal{L})$ such that $\psi\left\{\frac{t}{x}\right\} \in \Delta_{2k+2}$. Thus there exists some $n \in \mathbb{N}$ such that $\psi\left\{\frac{t}{x}\right\} \in \Gamma_n \cap \Delta_n$, which contradicts Lemma 3.9.

5. If $\varphi = \exists x \psi$ then this is shown similarly. $\square$

We now describe the necessary modifications in the above arguments for $G_{SH_=}$ and $G_{H_=}$.

For $G_{SH_=}$, the definition of L and R, as well as the amended Lemmas 3.9 and 3.10, are analogues. The definition of M, however, is changed as follows. Define a binary relation $\sim$ over $cl(\mathcal{L})$ by: $s \sim t$ iff $s = t \in L$. The fact that this is an equivalence relation can be shown using $(\Rightarrow=)$, (PM_1), (PM_2), and some inessential cuts (cuts on equations). We define a countermodel $M = \langle D, I\rangle$ by setting $D = cl(\mathcal{L}) / \sim$ (that is, D is the quotient set of $cl(\mathcal{L})$ by $\sim$), $(I(f))([t_1]_\sim, \ldots, [t_n]_\sim) = [f(t_1, \ldots, t_n)]_\sim$ for every $f \in \mathit{func}(\mathcal{L})$, and $I(P)$ is given by: $\{\langle [t_1]_\sim, \ldots, [t_n]_\sim\rangle \in D^n \mid P(t_1, \ldots, t_n) \in L\}$ for every $P \in \mathit{pred}(\mathcal{L})$. First, let us show that the definitions of $I(f)$ and $I(P)$ do not depend on the chosen representatives. If $t_i \sim t'_i$ for every $1 \le i \le n$, then $t_i = t'_i \in L$ for every such i, and using $(\Rightarrow=)$ and repeated applications of (PM_1) and (PM_2), we obtain that $f(t_1, \ldots, t_n) = f(t'_1, \ldots, t'_n) \in L$, and thus $f(t_1, \ldots, t_n) \sim f(t'_1, \ldots, t'_n)$. It is similarly shown that $P(t_1, \ldots, t_n) \in L$ iff $P(t'_1, \ldots, t'_n) \in L$. Also, we have that $I(t) = [t]_\sim$ for every $t \in cl(\mathcal{L})$. This is shown by induction on t. If t is a constant then this is immediate from the definition of I. If $t = f(t_1, \ldots, t_n)$, then we have by the induction hypothesis that $I(t_i) = [t_i]_\sim$ for every $1 \le i \le n$. Since I does not depend on the representatives, we have that $I(f(t_1, \ldots, t_n)) = (I(f))([t_1]_\sim, \ldots, [t_n]_\sim) = [f(t_1, \ldots, t_n)]_\sim$. For the amended Lemma 3.11, we prove the two base cases, for equations and atomic formulas.

1. If φ is $s = t$ then we have that $\varphi \in L$ iff $s \sim t$, iff $I(s) = I(t)$, which holds iff $M \models \varphi$.

2. If $\varphi = P(t_1, \ldots, t_n)$ then we have that $\varphi \in L$ iff $\langle [t_1]_\sim, \ldots, [t_n]_\sim\rangle \in I(P)$ iff $M \models P(t_1, \ldots, t_n)$.

For $G_{H_=}$, we adjust M to be normal by setting $I(=)$ to be $\{\langle t, t\rangle \mid t \in cl(\mathcal{L})\}$. In Lemma 3.11, a second base case, in which φ is $s = t$, has to be considered:

1. If $s = t \in L$, then there exists some $k \in \mathbb{N}$ such that $s = t \in \Gamma_k$. Assume for contradiction that $M \not\models s = t$. Then s and t are distinct terms, as M is a Herbrand structure. This contradicts Lemma 3.9 using $(=\Rightarrow)$.

2. If $s = t \notin L$, then $s = t \in R$, and so there exists some $k \in \mathbb{N}$ such that $s = t \in \Delta_k$. Assume for contradiction that $M \models s = t$. Then since it is a Herbrand structure, s and t are the same term. Using $(\Rightarrow=)$, we get a contradiction to Lemma 3.9.

We now prove the completeness results.

Theorem 3.12 (Completeness)**.** *Let s be a closed sequent. The following hold:*

1. $\vdash_{G_H} s$ *if* $\vdash^H s$.

2. $\vdash_{G_{H_=}} s$ *if* $\vdash^{H_=} s$.

3. $\vdash_{G_{SH_=}} s$ *if* $\vdash^{SH_=} s$.

Proof. We show the contrapositive. Assume s is not provable. Then, by (the appropriate variant of) Lemma 3.11, for every $\varphi \in \Gamma \subseteq L$ we have $M \models \varphi$, and for every $\psi \in \Delta \subseteq R$ we have $\psi \notin L$ and so $M \not\models \varphi$. Therefore, $M \not\models s$. □

Note that Theorem 3.12 provides an alternative, less constructive proof of Proposition 3.4 (together with Proposition 2.6 and the classical soundness theorem).

Due to Proposition 2.6, G_H is also sound and complete for $\vdash^{SH}$.

3.2 Manipulating Finite Sequents

The proof systems introduced in this section are not effective, as they manipulate both infinite sequents and infinite proof trees. Are these two sources of non-effectiveness tightly coupled? Consider, for example, a finite sequent s. Although s is finite, it may still have infinite derivations, that involve applications of $(\Rightarrow\forall)_H$ with a finite context sequent $\Gamma \Rightarrow \Delta$. However, do we at least have that all sequents that occur in its (possibly infinite) proof are finite? The answer is positive, as shown in the following proposition:

Proposition 3.13. *Let s be a finite sequent and let $\mathcal{G}$ be one of the systems G_H, $G_{H_=}$, or $G_{SH_=}$. If s is provable in $\mathcal{G}$, then s has a derivation in $\mathcal{G}$ that consists solely of finite sequents.*

Proof. By induction on the height of the proof of s, using the fact that no rule in these systems concludes a finite sequent if an infinite sequent exists in the assumptions. □

4 Infinite-height Systems

In this section, we explore infinite-height (but finite-width) proof systems for Herbrand logics. The soundness of such infinitary proof theories is underpinned by the principle of *infinite descent* [8, 9, 10, 16, 18, 35, 39]: proofs are permitted to be infinite, non-well-founded trees, but subject to the restriction that every infinite path in the proof admits some infinite descent. In our case the descent is witnessed by tracing terms descending via the subterm ordering, which is sound precisely because elements in Herbrand structures are identified by some (finite) term. For this reason, the systems we describe below can be considered systems of implicit induction, as opposed to those which employ explicit rules for applying induction principles, like those presented in the previous section.

A basic feature of our infinite-height systems is that they replace the infinite width $(\Rightarrow\forall)_H$ and $(\exists\Rightarrow)_H$ rules with 'case-split' rules. In contrast to the systems of Section 3, the infinite-height systems crucially rely on the presence of free variables in sequents, in order to form traces from which to construct infinite descent arguments. Moreover, in order to obtain systems of finite width we assume $func(\mathcal{L})$ is finite. Note that, as in Definition 3.1 above, the systems we consider in this section contain infinite sequents but that, similarly to the infinite-width systems, proofs of finite sequents require finite sequents only.

Definition 4.1. The infinitary proof systems G_H^∞, $G_{H_=}^\infty$ and $G_{SH_=}^\infty$ are obtained from G_H, $G_{H_=}$ and $G_{SH_=}$, respectively, by removing $(\Rightarrow\forall)_H$ and $(\exists\Rightarrow)_H$, allowing open sequents, adding the following substitution rule:

$$\frac{\Gamma \Rightarrow \Delta}{\Gamma\left\{\frac{t}{x}\right\} \Rightarrow \Delta\left\{\frac{t}{x}\right\}} \quad \text{(subst)}$$

restoring the standard $(\Rightarrow\forall)$ and $(\exists\Rightarrow)$ rules of $\mathcal{LK}$:

$$(\Rightarrow\forall) \quad \frac{\Gamma \Rightarrow \varphi\{\frac{y}{x}\}, \Delta}{\Gamma \Rightarrow \forall x\varphi, \Delta} \qquad (\exists\Rightarrow) \quad \frac{\Gamma, \varphi\{\frac{y}{x}\} \Rightarrow \Delta}{\Gamma, \exists x\varphi \Rightarrow \Delta}$$

where in both rules y does not occur free in the lower sequent, and then adding the following case-split rules, (case $\mathcal{L}$) to G_H^∞, and (case $\mathcal{L}_=$) to both $G_{H=}$ and $G_{SH=}$:

$$\frac{\left\{\Gamma\left\{\frac{f(x_1,\ldots,x_n)}{y}\right\} \Rightarrow \Delta\left\{\frac{f(x_1,\ldots,x_n)}{y}\right\} : f \in \mathit{func}(\mathcal{L})\right\}}{\Gamma\left\{\frac{t}{y}\right\} \Rightarrow \Delta\left\{\frac{t}{y}\right\}} \quad \text{(case } \mathcal{L}\text{)}$$

$$\frac{\{\Gamma, t = f(x_1, \ldots, x_n) \Rightarrow \Delta : f \in \mathit{func}(\mathcal{L})\}}{\Gamma \Rightarrow \Delta} \quad \text{(case } \mathcal{L}_=\text{)}$$

where in both rules, each variable x_i is distinct and does not occur in t or any formula in $\Gamma \cup \Delta$.

Example 4.2. For $\mathcal{L}_{\mathsf{NAT}}$, with nullary and unary function symbols 0 and s, respectively, the case split rules are the following:

$$\frac{\Gamma\left\{\frac{0}{y}\right\} \Rightarrow \Delta\left\{\frac{0}{y}\right\} \quad \Gamma\left\{\frac{s(x)}{y}\right\} \Rightarrow \Delta\left\{\frac{s(x)}{y}\right\}}{\Gamma\left\{\frac{t}{y}\right\} \Rightarrow \Delta\left\{\frac{t}{y}\right\}} \quad \text{(case } \mathcal{L}_{\mathsf{NAT}}\text{)}$$

$$\frac{\Gamma, t = 0 \Rightarrow \Delta \quad \Gamma, t = s(x) \Rightarrow \Delta}{\Gamma \Rightarrow \Delta} \quad \text{(case } \mathcal{L}_{\mathsf{NAT}=}\text{)}$$

where x is a fresh variable not occurring in the conclusion.

We allow derivations in this system to be non-well-founded (i.e. have infinite height). However, not all such derivations yield valid conclusions, so we use the terminology 'pre-proof'.

Definition 4.3 (Pre-proofs). A *pre-proof* is a possibly infinite (i.e. non-well-founded) derivation tree formed using the inference rules. A *path* in a pre-proof is a possibly infinite sequence of sequents $s_0, s_1, \ldots (, s_n)$ such that s_0 is the root sequent of the proof, and s_{i+1} is a premise of s_i for each $i < n$.

To distinguish the pre-proofs that derive valid conclusions from those that don't, we use the following notion of a *trace* of terms through a pre-proof.

Definition 4.4 (Trace Pairs). Let t and t' be terms occurring (possibly as subterms) in the conclusion s and a premise s', respectively, of (an instance of) an inference rule; then (t, t') is said to be a *trace pair* for (s, s') when the following conditions hold:

- t and t' only contain free variables (i.e. they do not contain variables that are bound by any quantifiers under whose scope they occur);
- if the rule is an instance of the substitution rule, then t is the result of applying the substitution of the instance to t';
- if the rule is an instance of any other rule and t and t' are not identical, then either:
  - the rule is an instance of (case $\mathcal{L}_=$), with t the left-hand term and t' one of the variables x_i in the equality formula eliminated from the premise; or
  - the rule is an instance of (case $\mathcal{L}$), with t the term substituted in the conclusion and t' one of the variables x_i in the term $f(x_1, \ldots, x_n)$ substituted in the premise.

  In both these cases, we say that (t, t') is *progressing*.

We use trace pairs to construct traces along paths in a pre-proof.

Definition 4.5 (Traces). A *trace* is a (possibly infinite) sequence of terms. We say that a trace $t_1, t_2, \ldots (, t_n)$ *follows* a path $s_1, s_2, \ldots (, s_m)$ in a pre-proof if, for some $k \geq 0$, each consecutive pair of terms (t_i, t_{i+1}) is a trace pair for (s_{i+k}, s_{i+k+1}). If (t_i, t_{i+1}) is progressing then we say that the trace *progresses* at i, and we say that the trace is *infinitely progressing* if it progresses at infinitely many points.

Proofs, then, are pre-proofs which satisfy a global trace condition.

Definition 4.6 (Infinite Proofs). A *proof* is a pre-proof in which every infinite path is followed by some infinitely progressing trace.

The soundness of the non-well-founded systems is a consequence of a strong form of local soundness of the inference rules. This strength relates to the fact that we can endow the domains of the underlying (semi-)Herbrand structures with a well-ordering as follows. First, we take some well-founded path ordering on the set of (closed) terms, e.g. the lexicographic path ordering induced by some total ordering on the function symbols of the signature [19]. A crucial point is that this includes the subterm relation. Now, since every element of a semi-Herbrand structure has some representation as a closed term we can then lift this ordering to the domain of the structure by taking the smallest term representation corresponding to each element. That is, for a semi-Herbrand structure $\langle D, I \rangle$, $d \in D$ is smaller than $d' \in D$ iff the smallest closed term t such that $d = I(t)$ is smaller than the smallest closed term u such that $d' = I(u)$. Let us denote this ordering as $\sqsubset$ and its non-strict counterpart as $\sqsubseteq$.

Lemma 4.7 (Trace Pair Local Soundness)**.** *Let $\mathcal{G}$ be one of the systems G^{∞}_{H}, $G^{\infty}_{H=}$, or $G^{\infty}_{SH=}$, $M = \langle D, I \rangle$ a (semi-)Herbrand structure appropriate for $\mathcal{G}$, and σ a substitution such that $M \not\models s^{\sigma}$ for the conclusion s of an instance of an inference rule in $\mathcal{G}$; then there is a substitution σ' such that $M \not\models {s_i}^{\sigma'}$ for some premise s_i of the rule and, moreover, for all trace pairs (t, t_i) for (s, s_i), $I({t_i}^{\sigma'}) \sqsubseteq I(t^{\sigma})$ and $I({t_i}^{\sigma'}) \sqsubset I(t^{\sigma})$ when (t, t') is progressing.*

Proof. The cases for the standard logical inference rules, (PM$_1$), (PM$_2$), ($\Rightarrow=$), and ($=\Rightarrow$) are straightforward. We give the proof for the case-split rules.

(**case** $\mathcal{L}_{=}$)**:** Suppose $M \not\models (\Gamma \Rightarrow \Delta)^{\sigma}$; that is, $M \models \varphi^{\sigma}$ for every $\varphi \in \Gamma$ but $M \not\models \psi^{\sigma}$ for each $\psi \in \Delta$. Now, $I(t^{\sigma})$ has some term representation and, in particular, it has a smallest one w.r.t. $\sqsubset$. That is $I(t^{\sigma}) = I(f(t_1, \ldots, t_n))$ for some $f \in \mathit{func}(\mathcal{L})$ and closed terms $t_1, \ldots, t_n$ with $I(f(t_1, \ldots, t_n)) \sqsubseteq I(u)$ for all closed terms u such that $I(t^{\sigma}) = I(u)$. So let $\sigma' = \sigma[x_1 \mapsto t_1, \ldots, x_n \mapsto t_n]$. Then, since each x_i does not occur in t, we have $I(t^{\sigma'}) = I(t^{\sigma}) = I(f(t_1, \ldots, t_n)) = I(f({x_1}^{\sigma'}, \ldots, {x_n}^{\sigma'})) = I(f(x_1, \ldots, x_n)^{\sigma'})$. That is, $M \models (t = f(x_1, \ldots, x_n))^{\sigma'}$. Moreover, since each x_i does not occur in Γ or Δ, we have that $I(\varphi^{\sigma'}) = I(\varphi^{\sigma})$ for each $\varphi \in \Gamma \cup \Delta$. Therefore, $M \models \varphi^{\sigma'}$ for each $\varphi \in \Gamma$ and $M \not\models \psi^{\sigma'}$ for each $\psi \in \Delta$. Hence $M \not\models (\Gamma, t = f(x_1, \ldots, x_n) \Rightarrow \Delta)^{\sigma'}$. Now, each (t, x_i) is a progressing trace pair, so we must show that $I({x_i}^{\sigma'}) \sqsubset I(t^{\sigma})$. Notice that $I({x_i}^{\sigma'}) = I(t_i)$ and so, by definition, the smallest closed term u such that $I({x_i}^{\sigma'}) = I(u)$ satisfies $u \sqsubseteq t_i$. Furthermore, $t_i \sqsubset f(t_1, \ldots, t_n)$, by the subterm ordering. Therefore $u \sqsubset f(t_1, \ldots, t_n)$ and since the latter term is the smallest closed representation of $I(t^{\sigma})$ we have that $I({x_i}^{\sigma'}) \sqsubset I(t^{\sigma})$, as required. Finally, $I(t^{\sigma}) = I(u^{\sigma'})$ for every other trace pair (t, u) since each x_i does not occur in t or u, thus trivially $I(t^{\sigma}) \sqsubseteq I(u^{\sigma'})$.

(**case** $\mathcal{L}$)**:** This case is similar to the above, except that we reason that $M \models \varphi\{\frac{f(x_1,\ldots,x_n)}{y}\}^{\sigma'}$ for each $\varphi \in \Gamma$ and $M \not\models \psi\{\frac{f(x_1,\ldots,x_n)}{y}\}^{\sigma'}$ from the assumption that $M \models \varphi\{\frac{t}{y}\}^{\sigma}$ for each $\varphi \in \Gamma$ and $M \not\models \psi\{\frac{t}{y}\}^{\sigma}$. This of course follows from that fact that $I(t^{\sigma}) = I(f(x_1, \ldots, x_n)^{\sigma'})$ as shown above. ☐

Given a non-well-founded proof (i.e. a pre-proof satisfying the global trace condition), this property allows us to construct an infinite descent-style argument that the conclusion is valid.

Theorem 4.8 (Soundness)**.** *Let s be a sequent. The following hold:*

1. $\vdash_{G^\infty_H} s$ *only if* $\vdash^H s$.
2. $\vdash_{G^\infty_{H_=}} s$ *only if* $\vdash^{H_=} s$.
3. $\vdash_{G^\infty_{SH_=}} s$ *only if* $\vdash^{SH_=} s$.

Proof. The structure of the proof is the same in each case. Suppose we have a proof of s and suppose, for contradiction, that s is not valid (w.r.t. the appropriate class of structures). Thus, by Lemma 2.4, there is a structure $M = \langle D, I \rangle$ (of the appropriate class) and substitution σ (cf. Definition 3.3) such that $M \not\models s^\sigma$. By Lemma 4.7, we can then construct an infinite path $s_1, s_2, \ldots$ through the proof along with an infinite sequence of substitutions $\sigma_1, \sigma_2, \ldots$ such that $M \not\models {s_i}^{\sigma_i}$ for each $i > 0$. Since we have a proof, there is an infinitely progressing trace $t_1, t_2, \ldots$ following this path; that is, there is some $k \geq 0$ such that (t_i, t_{i+1}) is a trace pair for (s_{i+k}, s_{i+k+1}) for each $i > 0$. Thus, again from Lemma 4.7, it follows that we have a descending chain $I({t_1}^{\sigma_{k+1}}) \sqsupseteq I({t_2}^{\sigma_{k+1}}) \sqsupseteq \ldots$ which, moreover, is infinitely descending since the trace is infinitely progressing. However, we have that $\sqsubset$ is well-founded, and thus we have derived a contradiction. We therefore conclude that $\Gamma \Rightarrow \Delta$ is valid (w.r.t. the appropriate class of structures) after all. □

Moreover, we can show for closed sequents that the non-well-founded proof systems are complete for the Herbrand semantics. This is due to the fact that we can simulate the $(\Rightarrow\forall)_H$ and $(\exists\Rightarrow)_H$ rules in the infinite height systems as the limit of an iteratively constructed sequence of open derivations.

Definition 4.9 (Simulation of $(\Rightarrow\forall)_H$ and $(\exists\Rightarrow)_H$). Take an instance of $(\Rightarrow\forall)_H$ or $(\exists\Rightarrow)_H$ with conclusion $\Gamma \Rightarrow \forall x\varphi, \Delta$ or $\Gamma, \exists x\varphi \Rightarrow \Delta$, respectively. Notice we may assume w.l.o.g. that x does not appear free in Γ or Δ. Fix an enumeration $x, x_1, x_2, \ldots$ of the variables not occurring free in the conclusion, starting with x, and let $<$ denote the corresponding induced ordering on variables. Let the function symbols of $\mathcal{L}$ be $f_1, \ldots, f_n$, of arities $k_1, \ldots, k_n$, respectively. We define infinite sequences $D_0, D_1, \ldots$ and $D'_0, D'_1, \ldots$ of open derivations inductively. We start with an application of the $(\Rightarrow\forall)$ or $(\exists\Rightarrow)$ rule.

$$D_0 = \frac{\Gamma \Rightarrow \varphi, \Delta}{\Gamma \Rightarrow \forall x\varphi, \Delta}\,(\Rightarrow\forall) \qquad\qquad D'_0 = \frac{\Gamma, \varphi \Rightarrow \Delta}{\Gamma, \exists x\varphi \Rightarrow \Delta}\,(\exists\Rightarrow)$$

The derivations D_{i+1} and D'_{i+1} are defined by applying inference rules to the open leaves of the derivations D_i and D'_i, respectively. The construction maintains the invariant that the sequents of the open leaves of each derivation D_{i+1} and D'_{i+1} are of the form $\Gamma \Rightarrow \psi, \Delta$ and $\Gamma, \psi \Rightarrow \Delta$, respectively, with the formula ψ being $\varphi\left\{\frac{t}{x}\right\}$ for some (possibly open) term t. The derivation D_{i+1} is defined by applying the following inference rules to each open leaf $\Gamma \Rightarrow \psi, \Delta$ in D_i containing free variables $\leq$-related to x, where x_j and x_q denote, respectively, the $<$-smallest and $<$-largest free variables in ψ.

$$\dfrac{\dfrac{\Gamma \Rightarrow \psi\left\{\frac{f_1(x_{q+1},\ldots,x_{q+k_1})}{x_j}\right\}, \Delta}{\Gamma, x_j = f_1(x_{q+1}, \ldots, x_{q+k_1}) \Rightarrow \psi, \Delta}\,(\mathrm{PM}_2) \quad \ldots \quad \dfrac{\Gamma \Rightarrow \psi\left\{\frac{f_n(x_{q+1},\ldots,x_{q+k_n})}{x_j}\right\}, \Delta}{\Gamma, x_j = f_n(x_{q+1}, \ldots, x_{q+k_n}) \Rightarrow \psi, \Delta}\,(\mathrm{PM}_2)}{\Gamma \Rightarrow \psi(x_j, \ldots, x_q), \Delta}\,(\text{case } \mathcal{L}_=)$$

$$\begin{array}{c} \Gamma \Rightarrow \psi(x_j, \ldots, x_q), \Delta \\ \vdots\; D_i \\ \Gamma \Rightarrow \forall x\varphi\Delta \end{array}$$

Similarly, each D'_{i+1} is obtained from D'_i be applying the following inference rules to each open leaf.

$$\dfrac{\dfrac{\Gamma, \psi\Big\{\frac{f_1(x_{q+1},\ldots,x_{q+k_1})}{x_j}\Big\} \Rightarrow \Delta}{\Gamma, x_j = f_1(x_{q+1},\ldots,x_{q+k_1}), \psi \Rightarrow \Delta}\,(\ddagger_{f_1}) \quad \ldots \quad \dfrac{\Gamma, \psi\Big\{\frac{f_n(x_{q+1},\ldots,x_{q+k_n})}{x_j}\Big\} \Rightarrow \Delta}{\Gamma, x_j = f_n(x_{q+1},\ldots,x_{q+k_n}), \psi \Rightarrow \Delta}\,(\ddagger_{f_n})}{\dfrac{\Gamma, \psi(x_j,\ldots,x_q) \Rightarrow \Delta}{\begin{array}{c}\vdots\, D'_i \\ \Gamma, \exists x \varphi \Rightarrow \Delta\end{array}}}\,(\text{case } \mathcal{L}_=)$$

The inferences marked $(\ddagger_{f_i})$ abbreviate the following schema.

$$\dfrac{\dfrac{\dfrac{}{\psi \Rightarrow \psi\Big\{\frac{x_j}{x_j}\Big\}}\,(\text{axiom})}{x_j = f_i(x_{q+1},\ldots,x_{q+k_i}), \psi \Rightarrow \psi\Big\{\frac{f_i(x_{q+1},\ldots,x_{q+k_i})}{x_j}\Big\}}\,(\text{PM}_1) \quad \Gamma, \psi\Big\{\frac{f_i(x_{q+1},\ldots,x_{q+k_i})}{x_j}\Big\} \Rightarrow \Delta}{\Gamma, x_j = f_i(x_{q+1},\ldots,x_{q+k_i}), \psi \Rightarrow \Delta}\,(\text{cut})$$

Thus each derivation D_i (resp. D'_i) is a prefix of D_{i+1} (resp. D'_{i+1}). The limits of the sequences $D_0, D_1, \ldots$ and $D'_0, D'_1, \ldots$ are the smallest derivations D_∞ and D'_∞ such that each D_i and D'_i are prefixes of D_∞ and D'_∞, respectively, for every $i \geq 0$. The derivations D_∞ and D'_∞ simulate, in both $G^\infty_{H_=}$ and $G^\infty_{SH_=}$, the $(\Rightarrow\forall)_H$ and $(\exists\Rightarrow)_H$ rules, respectively. To simulate the rules in G^∞_H, we adapt this construction by using the (case $\mathcal{L}$) rule instead of (case $\mathcal{L}_=$) and eliding the paramodulation steps, the effects of which are instead carried out directly by the substitutions in the (case $\mathcal{L}$) rule.

Example 4.10. Figures 3a and 3b show the derivations simulating the $(\Rightarrow\forall)_H$ and $(\exists\Rightarrow)_H$ rules for the language $\mathcal{L}_{\mathsf{NAT}}$, with nullary and unary function symbols 0 and s, respectively.

Theorem 4.11 (Completeness)**.** *Let s be a closed sequent. The following hold:*

1. $\vdash_{G^\infty_H} s$ *if* $\vdash^H s$.
2. $\vdash_{G^\infty_{H_=}} s$ *if* $\vdash^{H_=} s$.
3. $\vdash_{G^\infty_{SH_=}} s$ *if* $\vdash^{SH_=} s$.

Proof. By completeness of the infinite-width proof systems (Theorem 3.12), it suffices to show that the $(\Rightarrow\forall)_H$ and $(\exists\Rightarrow)_H$ rules are derivable in each of the infinite-height proof systems, and that any pre-proof corresponding to an infinite-width proof via these derived rules satisfies the global trace condition. Derivability of the $(\Rightarrow\forall)_H$ and $(\exists\Rightarrow)_H$ rules in the infinite-height systems is shown in Definition 4.9. We now show that any pre-proofs using these derived rules satisfy the global trace condition. For languages with at least one constant the simulations of these rules have a possibly infinite (and non-zero) number of branches, but each path from the conclusion to an open leaf is of finite (although unbounded) height. Thus pre-proofs simulating $G_{H_=}$ and $G_{SH_=}$ proofs using these derivation schemes will not contain any infinite paths, and so trivially satisfy the global trace condition. Languages without constants actually have no closed terms, and thus the $(\Rightarrow\forall)_H$ and $(\exists\Rightarrow)_H$ rules are 0-premise rules. For these languages, the derivation schemes described in Definition 4.9 will also not contain any open leaves, but instead admit an infinite number of infinite paths. Notice, however, that along each of these infinite paths we can construct a trace following the variables $x, x_1, x_2, \ldots$ that progresses at each case-split rule, of which there are infinitely many. ☐

$$
\dfrac{
  \dfrac{\dfrac{\Gamma \Rightarrow \varphi\left\{\frac{0}{x}\right\}, \Delta}{\Gamma, x = 0 \Rightarrow \varphi(x), \Delta}\,(\mathrm{PM}_2)
  \qquad
  \dfrac{\dfrac{
    \dfrac{\Gamma \Rightarrow \varphi\left\{\frac{s(0)}{x}\right\}, \Delta}{\Gamma, x_1 = 0 \Rightarrow \varphi\left\{\frac{s(x_1)}{x}\right\}, \Delta}\,(\mathrm{PM}_2)
    \qquad
    \dfrac{\begin{matrix}\Gamma \Rightarrow \varphi\left\{\frac{s(s(0))}{x}\right\}, \Delta \quad \ldots \quad \Gamma \Rightarrow \varphi\left\{\frac{s^n(0)}{x}\right\}, \Delta \quad \ldots \\ \vdots \\ \Gamma \Rightarrow \varphi\left\{\frac{s(s(x_2))}{x}\right\}, \Delta\end{matrix}}{\Gamma, x_1 = s(x_2) \Rightarrow \varphi\left\{\frac{s(x_1)}{x}\right\}, \Delta}\,(\mathrm{PM}_2)
  }{\Gamma \Rightarrow \varphi\left\{\frac{s(x_1)}{x}\right\}, \Delta}\,(\text{case } \mathcal{L}_{\mathsf{NAT}=})}{\Gamma, x = s(x_1) \Rightarrow \varphi(x), \Delta}\,(\mathrm{PM}_2)
  }{\Gamma \Rightarrow \varphi(x), \Delta}\,(\text{case } \mathcal{L}_{\mathsf{NAT}=})
}{\Gamma \Rightarrow \forall x \varphi, \Delta}\,({\Rightarrow}\forall)
$$

(a) Simulation of the $({\Rightarrow}\forall)_H$ rule.

$$
\dfrac{
  \dfrac{\dfrac{\Gamma, \varphi\left\{\frac{0}{x}\right\} \Rightarrow \Delta}{\Gamma, x = 0, \varphi(x) \Rightarrow \Delta}\,(\ddagger_0)
  \qquad
  \dfrac{\dfrac{
    \dfrac{\Gamma, \varphi\left\{\frac{s(0)}{x}\right\} \Rightarrow \Delta}{\Gamma, x_1 = 0, \varphi\left\{\frac{s(x_1)}{x}\right\} \Rightarrow \Delta}\,(\ddagger_0)
    \qquad
    \dfrac{\begin{matrix}\Gamma, \varphi\left\{\frac{s(s(0))}{x}\right\} \Rightarrow \Delta \quad \ldots \quad \Gamma, \varphi\left\{\frac{s^n(0)}{x}\right\} \Rightarrow \Delta \quad \ldots \\ \vdots \\ \Gamma, \varphi\left\{\frac{s(s(x_2))}{x}\right\} \Rightarrow \Delta\end{matrix}}{\Gamma, x_1 = s(x_2), \varphi\left\{\frac{s(x_1)}{x}\right\} \Rightarrow \Delta}\,(\ddagger_s)
  }{\Gamma, \varphi\left\{\frac{s(x_1)}{x}\right\} \Rightarrow \Delta}\,(\text{case } \mathcal{L}_{\mathsf{NAT}=})}{\Gamma, x = s(x_1), \varphi(x) \Rightarrow \Delta}\,(\ddagger_s)
  }{\Gamma, \varphi(x) \Rightarrow \Delta}\,(\text{case } \mathcal{L}_{\mathsf{NAT}=})
}{\Gamma, \exists x \varphi \Rightarrow \Delta}\,(\exists{\Rightarrow})
$$

(b) Simulation of the $(\exists{\Rightarrow})_H$ rule.

Figure 3: Simulations of the infinite width rules in $G^{\infty}_{H=}/G^{\infty}_{SH=}$ for the language $\mathcal{L}_{\mathsf{NAT}}$.

Lastly, a counterpart of Proposition 3.13 can be obtained for the proof systems of this section.

Proposition 4.12. *Let s be a finite sequent and let $\mathcal{G}$ be one of the systems G^{∞}_{H}, $G^{\infty}_{H=}$, or $G^{\infty}_{SH=}$. If s is provable in $\mathcal{G}$, then s has a derivation in $\mathcal{G}$ that consists solely of finite sequents.*

Proof. By Theorem 4.8 and Proposition 3.13, there is an infinite-width proof using only finite sequents. The constructions of Definition 4.9 that simulate the infinite width rules do not introduce infinite sequents if the root sequent is finite. Therefore, we can construct an infinite-height proof containing only finite sequents. □

The purpose of studying the infinitary systems in the previous two sections, in particular the completeness results, is two-fold. Firstly, it provides a solid foundation for the effective approximations that we present in the next section: the induction-based system of Section 5.1, which approximates the infinite-width system, and the cyclic system of Section 5.2, which approximates the infinite-height system. We would like our approximations to capture as much as possible; there is therefore little point in starting with an already incomplete system. Secondly, it provides a context for the comparison of the effective systems. In Section 5.3 we show that, for some languages, the cyclic approximation is more expressive than the induction-based approximation. One might suppose that this is due to some intrinsic difference in expressiveness between infinite width and infinite height. Our soundness and completeness results show

Figure 4: Effective Derivation Rules for Herbrand and semi-Herbrand Logics

$$(\Rightarrow\forall)_{IND} \quad \frac{\left\{\Gamma, \varphi\left\{\frac{x_1}{x}\right\}, \ldots, \varphi\left\{\frac{x_n}{x}\right\} \Rightarrow \varphi\left\{\frac{f(x_1,\ldots,x_n)}{x}\right\}, \Delta : f \in func\left(\mathcal{L}\right)\right\}}{\Gamma \Rightarrow \forall x\varphi, \Delta}$$

$$(\exists\Rightarrow)_{IND} \quad \frac{\left\{\Gamma, \varphi\left\{\frac{f(x_1,\ldots,x_n)}{x}\right\} \Rightarrow \varphi\left\{\frac{x_1}{x}\right\}, \ldots, \varphi\left\{\frac{x_n}{x}\right\}, \Delta : f \in func\left(\mathcal{L}\right)\right\}}{\Gamma, \exists x\varphi \Rightarrow \Delta}$$

In $(\Rightarrow\forall)_{IND}$ and $(\exists\Rightarrow)_{IND}$, $x_1, \ldots, x_n$ do not occur free in $\Gamma \cup \Delta$.

$$(\Rightarrow=)_{IND} \quad \frac{}{\Gamma \Rightarrow x = x, \Delta}$$

$$(=\Rightarrow)_1 \quad \frac{}{\Gamma, f\left(x_1, \ldots, x_n\right) = g\left(y_1, \ldots, y_m\right) \Rightarrow \Delta} \qquad f \neq g \in func\left(\mathcal{L}\right)$$

$$(=\Rightarrow)_2 \quad \frac{\Gamma, x_i = y_i \Rightarrow \Delta}{\Gamma, f\left(x_1 \ldots, x_i, \ldots, x_n\right) = f\left(x_1, \ldots, y_i, \ldots, x_n\right) \Rightarrow \Delta} \qquad f \in func^{+}\left(\mathcal{L}\right)$$

that this is not the case: both infinitary systems capture the full expressivity of Herbrand logics. Thus we can pin-point the difference in expressiveness between the approximations as lying in the nature of the respective gaps between effectiveness and non-effectiveness in each case.

5 Effective Approximations

While the proof systems presented in Sections 3 and 4 are sound and complete with respect to Herbrand logics, they are not effective, in the sense that proofs cannot be verified. What is more, the set of proofs cannot even be enumerated. Thus, for systems that are more suitable for automated reasoning, one needs to sacrifice completeness for the sake of effectiveness. Accordingly, in this section, we provide finitary counterparts of the infinitary systems presented in Sections 3 and 4.

5.1 Induction-Based Systems

The finitary inductive systems that we introduce in this section can be seen as natural restrictions of the infinite-width systems of Section 3 above. However to recover some of the strength of those systems, they must incorporate free variables in sequents. Accordingly, let G^{IND} be the sequent calculus obtained from G (cf. Page 7) by allowing only finite sequents in applications of rules, and permitting open sequents. To obtain finitary systems, in this section we assume $func\left(\mathcal{L}\right)$ is finite.

Definition 5.1. Figure 4 includes the additional derivation rules employed.

1. $G_H^{IND} = G^{IND} + (\Rightarrow\forall)_{IND} + (\exists\Rightarrow)_{IND} + (\text{subst})$.
2. $G_{SH_=}^{IND} = G_H^{IND} + (\Rightarrow=)_{IND} + (\text{PM}_1) + (\text{PM}_2)$.
3. $G_{H_=}^{IND} = G_{SH_=}^{IND} + (=\Rightarrow)_1 + (=\Rightarrow)_2$

The key idea in constructing the above systems is to replace the infinitary rules $(\Rightarrow\forall)_H$ and $(\exists\Rightarrow)_H$ by finitary approximations. This is achieved by taking PA's induction scheme as a finite approximation of the ω-rule. Accordingly, we replace $(\Rightarrow\forall)_H$, which is a language-based version of the ω-rule, with a new rule, $(\Rightarrow\forall)_{IND}$, which is a language-based version of Gentzen's induction rule for PA. $(\exists\Rightarrow)_{IND}$ is treated symmetrically.

Following Example 2.12, we note that for $\mathcal{L}_{\mathsf{NAT}}$, applications of $(\Rightarrow\forall)_{IND}$ have the form

$$\frac{\Gamma\Rightarrow\varphi\left\{\frac{0}{x}\right\},\Delta\quad \Gamma,\varphi\left\{\frac{x_1}{x}\right\}\Rightarrow\varphi\left\{\frac{s(x_1)}{x}\right\},\Delta}{\Gamma\Rightarrow\forall x\varphi,\Delta}\tag{1}$$

Actually, Gentzen's original induction scheme for PA is easily derivable in the above systems, using (1) and $(\forall\Rightarrow)$. What enables the effective formulation of PA's induction rule is the fact that the language of PA is finite. The above systems can manipulate languages which are more expressive than that of PA, since they allow multiple constant symbols, as well as function symbols with arbitrary arities. However, we still assume that there are finitely many basic constructs for terms. This explains the requirement employed in this section that $func(\mathcal{L})$ is finite. The fact that $func(\mathcal{L})$ is finite entails that G_H^{IND}, $G_{SH_=}^{IND}$ and $G_{H_=}^{IND}$ are finitary, in the sense that their proof trees are always finite.

In search of effectiveness, we replace $(=\Rightarrow)$ in the system for Herbrand logics with equality by two rules $(=\Rightarrow)_1$ and $(=\Rightarrow)_2$, that correspond to sequent rules for disequalities in [36]. Those achieve precisely the same power, while being actual finite sets of rules, rather than schemes. In order to be able to derive in $G_{H_=}^{IND}$ all instances of the rule $(=\Rightarrow)$ of $G_{H_=}$ we include (PM_1) and (PM_2) in $G_{H_=}^{IND}$ (unlike in the original system $G_{H_=}$).

Lemma 5.2. *If t_1 and t_2 are two distinct closed terms of $\mathcal{L}$, then $\vdash_{G_{H_=}^{IND}} t_1=t_2\Rightarrow$.*

Proof. This is shown using induction on the sums of complexities of t_1 and t_2. The only interesting case is when $t_1=f(s_1,\ldots,s_n)$ and $t_2=f(r_1,\ldots,r_n)$ for some n-ary function symbol f. Since t_1 and t_2 are distinct, we must have some $1\leq i\leq n$ such that $s_i\neq r_i$. By the induction hypothesis, $\vdash_{G_{H_=}^{IND}} s_i=r_i\Rightarrow$.

It is easy to verify that the sequent $x=s_i, y=r_i, x_i=y_i\Rightarrow s_i=r_i$ is derivable in $G_{H_=}$. This, in turn, enables the proof of $t_1=t_2\Rightarrow$ in $G_{H_=}^{IND}$, using cuts on the conclusion of the following derivation:

$$\cfrac{\cfrac{\overline{x_i=s_i,y_i=r_i,x_i=y_i\Rightarrow s_i=r_i}}{x_i=s_i,y_i=r_i,f(x_1,\ldots,x_i,\ldots x_n)=f(x_1,\ldots,y_i,\ldots,x_n)\Rightarrow s_i=r_i}\,(=\Rightarrow)_2}{s_i=s_i,r_i=r_i,f(x_1,\ldots,s_i,\ldots,x_n)=f(x_1,\ldots,r_i,\ldots,x_n)\Rightarrow s_i=r_i}\,(\text{subst})\qquad\square$$

All the effective systems are complete for classical logic (as the rules $(\Rightarrow\forall)$ and $(\exists\Rightarrow)$ of $\mathcal{LK}$ are derivable in them). They are, however, not sound for it (since they include consecutions that are only valid in Herbrand logics). As a consequence of the finitary nature of the systems, one cannot expect that they are complete with respect to Herbrand logics. For example, in the case of the language of the natural numbers we obtain exactly the difference between the provable theorems of PA and the true statements of arithmetics. Notwithstanding, the effective systems are both sound and expressive enough to capture meaningful Herbrand-valid statements.

Theorem 5.3 (Soundness)**.** *Let s be a sequent. The following hold:*

1. *$\vdash_{G_H^{IND}} s$ only if $\vdash^H s$.*

2. *$\vdash_{G_{SH_=}^{IND}} s$ only if $\vdash^{SH_=} s$.*

3. $\vdash_{G^{IND}_{H_=}} s$ *only if* $\vdash^{H_=} s$.

Proof. We extend Definition 3.5 by setting $M \models s$ iff M satisfies every $\mathcal{L}$-instance of s. For G^{IND}_{H}, the only non-trivial cases are $(\Rightarrow\forall)_{IND}$ and $(\exists\Rightarrow)_{IND}$. We here provide the proof for the first, as the second is symmetrical.

Let M be a Herbrand model of the premises of $(\Rightarrow\forall)_{IND}$. We prove $M \models \Gamma \Rightarrow \forall x\varphi, \Delta$. If $M \models \Gamma \Rightarrow \Delta$, this clearly holds. Otherwise, $M \models \varphi\left\{\frac{x_1}{x}\right\}, \ldots, \varphi\left\{\frac{x_n}{x}\right\} \Rightarrow \varphi\left\{\frac{f(x_1,\ldots,x_n)}{x}\right\}$ for every $f \in func(\mathcal{L})$. Thus, M satisfies every $\mathcal{L}$-instance of such sequents. To show that $M \models \forall x\varphi$, we consider an arbitrary $\mathcal{L}$-instance $\forall x\varphi'$ of $\forall x\varphi$, and prove that $M \models \varphi'\left\{\frac{t}{x}\right\}$ for every $t \in cl(\mathcal{L})$, by inner induction on t. If $t \in consts(\mathcal{L})$, then this follows from the induction hypothesis (in this case, $n = 0$ and hence the premise has the form $\Gamma \Rightarrow \varphi\left\{\frac{t}{x}\right\}, \Delta$). Otherwise, $t = f(t_1, \ldots, t_n)$ for some $f \in func^+(\mathcal{L})$ and $t_1, \ldots, t_n \in cl(\mathcal{L})$ such that $t_1, \ldots, t_n$ all have lower complexities than t. By the induction hypothesis, $M \models \varphi'\left\{\frac{t_i}{x}\right\}$ for every $1 \leq i \leq n$. Since $M \models \varphi'\left\{\frac{x_1}{x}\right\}, \ldots, \varphi'\left\{\frac{x_n}{x}\right\} \Rightarrow \varphi'\left\{\frac{f(x_1,\ldots,x_n)}{x}\right\}$, we also have that $M \models \varphi'\left\{\frac{t}{x}\right\}$.

Soundness for $G^{IND}_{SH_=}$ is obtained similarly, with the addition of the same arguments for equality as in the classical case. For $G^{IND}_{H_=}$, the fact that only Herbrand structures are considered is used to show the validity of the rules $(=\Rightarrow)_1$ and $(=\Rightarrow)_2$. $\square$

Example 5.4. Following Example 2.12, consider $G_{H_=}$ in the language $\mathcal{L}_{\mathsf{NAT}}$. This system is also sound w.r.t. the $\mathcal{L}_{\mathsf{NAT}}$ version of PA (with multiplication and addition as predicates). This can be shown by induction on derivations in $G_{H_=}$, where the main case is the straightforward simulation of the induction rule of PA with $(\Rightarrow\forall)_{IND}$.

Generalizing Example 5.4, consider the following generalized induction scheme for languages $\mathcal{L}$ with finitely many function symbols, where each variable x_i is distinct and fresh for φ:

$$\left(\left(\bigwedge_{c\in consts(\mathcal{L})} \varphi\left\{\frac{c}{x}\right\}\right) \wedge \bigwedge_{f\in func^+(\mathcal{L})} \forall x_1 \ldots \forall x_{ar(f)} \left(\bigwedge_{1\leq i\leq ar(f)} \varphi\left\{\frac{x_i}{x}\right\}\right) \supset \varphi\left\{\frac{f(x_1,\ldots,x_n)}{x}\right\}\right) \supset \forall x\varphi \qquad (IND_{\mathcal{L}})$$

Lemma 5.5. *The following hold:*

1. G^{IND}_{H} *is sound w.r.t. first-order structures for* $\mathcal{L}$ *that satisfy* $(IND_{\mathcal{L}})$*.*

2. $G^{IND}_{SH_=}$ *is sound w.r.t. normal first-order structures for* $\mathcal{L}$ *that satisfy* $(IND_{\mathcal{L}})$*.*

3. $G^{IND}_{H_=}$ *is sound w.r.t. normal first-order structures* $\langle D, I\rangle$ *for* $\mathcal{L}$ *that satisfy* $(IND_{\mathcal{L}})$ *and, furthermore, in which: (a) the images of* $I(f)$ *and* $I(g)$ *in* D *are disjoint for each pair of distinct function symbols* $f, g \in func(\mathcal{L})$*; and (b) the interpretation* $I(f)$ *of each function symbol* $f \in func(\mathcal{L})$ *is injective.*

5.2 Cyclic Systems

The infinite-height proof systems presented in Section 4 can also be restricted to retrieve effectiveness by considering a subset of finitely representable proofs, namely the *regular* infinite proofs. These are (possibly) infinite derivations that have only a finite number of *distinct* subtrees. They can be represented by finite, possibly cyclic, graphs.

Definition 5.6 (Cyclic Pre-Proofs). A *cyclic* pre-proof is a pair (P, f) consisting of:

- a finite derivation tree P possibly containing open leaves (i.e. which are not conclusions of an axiom schema) called *buds*; and

$$\cfrac{\cfrac{(\dagger)\ \Gamma \Rightarrow \varphi(x), \Delta}{\Gamma \Rightarrow \varphi\Big\{\frac{x_n}{x}\Big\}, \Delta}\ \text{(subst)} \qquad \Gamma, \varphi\Big\{\frac{x_1}{x}\Big\}, \dots, \varphi\Big\{\frac{x_1}{x_n}\Big\} \Rightarrow \varphi\Big\{\frac{f(x_1,\dots,x_n)}{x}\Big\}, \Delta}{\Gamma, \varphi\Big\{\frac{x_1}{x}\Big\}, \dots, \varphi\Big\{\frac{x_{n-1}}{x}\Big\} \Rightarrow \varphi\Big\{\frac{f(x_1,\dots,x_n)}{x}\Big\}, \Delta}\ \text{(cut)}$$

$$\vdots$$

$$\cfrac{\cfrac{\cfrac{\cfrac{(\dagger)\ \Gamma \Rightarrow \varphi(x), \Delta}{\Gamma \Rightarrow \varphi\Big\{\frac{x_1}{x}\Big\}, \Delta}\ \text{(subst)} \qquad \Gamma, \varphi\Big\{\frac{x_1}{x}\Big\} \Rightarrow \varphi\Big\{\frac{f(x_1,\dots,x_n)}{x}\Big\}, \Delta}{\Gamma \Rightarrow \varphi\Big\{\frac{f(x_1,\dots,x_n)}{x}\Big\}, \Delta}\ \text{(cut)}}{\Gamma, x = f(x_1, \dots, x_n) \Rightarrow \varphi(x), \Delta}\ (\text{PM}_2) \qquad : f \in \mathit{func}(\mathcal{L})}{\cfrac{(\dagger)\ \Gamma \Rightarrow \varphi(x), \Delta}{\Gamma \Rightarrow \forall x\, \varphi(x), \Delta}\ (\Rightarrow\forall)}\ (\text{case } \mathcal{L}_=)$$

Figure 5: A derivation of the $(\Rightarrow\forall)_{IND}$ rule in $G^{\omega}_{H_=}$/$G^{\omega}_{SH_=}$.

- a function f assigning to each bud an internal node of the tree, called its *companion*, with a syntactically identical sequent.

The graph associated with a pre-proof is the one induced by identifying each bud with its companion.

For effectiveness, we here again consider only finite sequents.

Definition 5.7. The cyclic subsystems G^{ω}_{H}, $G^{\omega}_{H_=}$, and $G^{\omega}_{SH_=}$ are obtained by restricting to only the regular pre-proofs of G^{∞}_{H}, $G^{\infty}_{H_=}$ and $G^{\infty}_{SH_=}$, respectively, containing finite sequents and satisfying the global trace condition.

For cyclic pre-proofs the global trace condition, being an ω-regular property, is decidable. Since they are a special case of the non-well-founded proofs, they are also sound for Herbrand semantics.

Corollary 5.8 (Soundness)**.** *The cyclic systems G^{ω}_{H}, $G^{\omega}_{H_=}$ and $G^{\omega}_{SH_=}$ are sound with respect to their corresponding Herbrand semantics.*

5.3 Comparison Between the Approximations

This section establishes the connections between the two approaches described above. As in other systems for inductive reasoning, we show that the cyclic systems are strictly stronger than the induction-based ones. We start by showing that the cyclic proof systems subsume the explicit induction systems containing rules $(\Rightarrow\forall)_{IND}$ and $(\exists\Rightarrow)_{IND}$. This is a result of the following lemma.

Lemma 5.9. *The rules $(\Rightarrow\forall)_{IND}$ and $(\exists\Rightarrow)_{IND}$ are derivable in each of G^{ω}_{H}, $G^{\omega}_{H_=}$ and $G^{\omega}_{SH_=}$.*

Proof. The derivations in Figs. 5 and 6 simulate rules $(\Rightarrow\forall)_{IND}$ and $(\exists\Rightarrow)_{IND}$ in each of the cyclic systems. In these derivations we assume (w.l.o.g.) that x does not occur free in $\Gamma \cup \Delta$. We show a generic subderivation for the case $x = f(x_1, \dots, x_n)$ of $f \in \mathit{func}(\mathcal{L})$. Note that in the case $f \in \mathit{consts}(\mathcal{L})$, for the simulation of $(\Rightarrow\forall)_{IND}$ the premise of the (PM_2) rule is already the corresponding premise of the $(\Rightarrow\forall)_{IND}$ rule and so the remainder of the subderivation is not required. Similarly, in the simulation of $(\exists\Rightarrow)_{IND}$ instead of the subderivation concluding with (Cut), we may simply use the following:

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(id)}
\UnaryInfC{$\varphi(x) \Rightarrow \varphi(x)$}
\RightLabel{(PM$_1$)}
\UnaryInfC{$\varphi(x), x = f(x_1, \ldots, x_n) \Rightarrow \varphi\left\{\frac{f(x_1,\ldots,x_n)}{x}\right\}$}
\AxiomC{$\Gamma, \varphi\left\{\frac{f(x_1,\ldots,x_n)}{x}\right\} \Rightarrow \varphi\left\{\frac{x_1}{x}\right\}, \ldots, \varphi\left\{\frac{x_n}{x}\right\}, \Delta$}
\noLine
\UnaryInfC{$\vdots \; (\Rightarrow\vee) \times n$}
\noLine
\UnaryInfC{$\Gamma, \varphi\left\{\frac{f(x_1,\ldots,x_n)}{x}\right\} \Rightarrow \varphi\left\{\frac{x_1}{x}\right\} \vee \ldots \vee \varphi\left\{\frac{x_n}{x}\right\}, \Delta$}
\noLine
\UnaryInfC{$\vdots$}
\RightLabel{(cut)}
\BinaryInfC{$\Gamma, \varphi(x), x = f(x_1, \ldots, x_n) \Rightarrow \varphi\left\{\frac{x_1}{x}\right\} \vee \ldots \vee \varphi\left\{\frac{x_n}{x}\right\}, \Delta$}
\AxiomC{$(\dagger)\; \Gamma, \varphi(x) \Rightarrow \Delta$}
\RightLabel{(subst)}
\UnaryInfC{$\Gamma, \varphi\left\{\frac{x_1}{x}\right\} \Rightarrow \Delta$}
\AxiomC{$(\dagger)\; \Gamma, \varphi(x) \Rightarrow \Delta$}
\RightLabel{(subst)}
\UnaryInfC{$\Gamma, \varphi\left\{\frac{x_n}{x}\right\} \Rightarrow \Delta$}
\noLine
\UnaryInfC{$\vdots$}
\noLine
\UnaryInfC{$\Gamma, \varphi\left\{\frac{x_2}{x}\right\} \vee \ldots \vee \varphi\left\{\frac{x_n}{x}\right\} \Rightarrow \Delta$}
\RightLabel{$(\vee\Rightarrow)$}
\BinaryInfC{$\Gamma, \varphi\left\{\frac{x_1}{x}\right\} \vee \ldots \vee \varphi\left\{\frac{x_n}{x}\right\} \Rightarrow \Delta$}
\RightLabel{(cut)}
\BinaryInfC{$\Gamma, \varphi(x), x = f(x_1, \ldots, x_n) \Rightarrow \Delta \qquad : f \in \mathit{func}(\mathcal{L})$}
\RightLabel{(case $\mathcal{L}_=$)}
\UnaryInfC{$(\dagger)\; \Gamma, \varphi(x) \Rightarrow \Delta$}
\RightLabel{$(\Rightarrow\exists)$}
\UnaryInfC{$\Gamma, \exists x\, \varphi(x) \Rightarrow \Delta$}
\end{prooftree}
$$

Figure 6: A derivation of the $(\exists\Rightarrow)_{IND}$ rule in $G^{\omega}_{H_=}/G^{\omega}_{SH_=}$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{(id)}
\UnaryInfC{$\varphi(x) \Rightarrow \varphi(x)$}
\RightLabel{(PM$_1$)}
\UnaryInfC{$\varphi(x), x = f(x_1, \ldots, x_n) \Rightarrow \varphi\left\{\frac{f(x_1,\ldots,x_n)}{x}\right\}$}
\AxiomC{$\Gamma, \varphi\left\{\frac{f(x_1,\ldots,x_n)}{x}\right\} \Rightarrow \Delta$}
\RightLabel{(cut)}
\BinaryInfC{$\Gamma, \varphi(x), x = f(x_1, \ldots, x_n) \Rightarrow \Delta$}
\end{prooftree}
$$

In the case that $f \in \mathit{func}^+(\mathcal{L})$, the subderivation will have a non-zero number of buds, corresponding to each subterm x_i and marked by $(\dagger)$ in Figs. 5 and 6, which all link back to the conclusion of the (case $\mathcal{L}_=$) instance.

As in the proof of Theorem 4.11, we can adapt the derivation schemes in Figs. 5 and 6 for the system G^{ω}_{H} without equality by simply eliding the paramodulation rules, whose effect is achieved via substitutions in the (case $\mathcal{L}$) rule. □

Theorem 5.10. *The following inclusions hold:*

1. $G^{IND}_{H} \subseteq G^{\omega}_{H}$.
2. $G^{IND}_{H_=} \subseteq G^{\omega}_{H_=}$.
3. $G^{IND}_{SH_=} \subseteq G^{\omega}_{SH_=}$.

Proof. Consider the cyclic pre-proof obtained from an explicit induction proof by replacing each instance of $(\Rightarrow\forall)_{IND}$ and $(\exists\Rightarrow)_{IND}$ by the derivations given in Lemma 5.9. Any infinite path in this proof must eventually remain in a strongly connected component. However, notice that each such strongly connected component is contained in an instance of these derived rules. It is easy to see that we can construct a trace following this path. Notice that we can split the tail of this path into an infinite sequence of segments, each corresponding to a path from the companion $(\dagger)$ up to one of the buds. Along the segment we construct a trace as follows: start with x in the conclusion of the (case $\mathcal{L}$) or (case $\mathcal{L}_=$) rule; in the premise of the rule we then choose the x_i corresponding to the branch taken by the segment, which is a progressing trace pair; we may then trace x_i all the way to the bud, which then becomes x again via the substitution rule. Each segment contains a progression point, thus the trace is infinitely progressing. □

Next, we show that there are languages for which the converse inclusion does not hold. For this we can adapt the 'Hydra' counter-example given in [5] to establish the corresponding result for the explicit and cyclic systems for first-order logic with inductive definitions. Note that we do not here derive any

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{H \Rightarrow P(0,0)}\,(\text{id}) \qquad \dfrac{\dfrac{}{H \Rightarrow P(s(0),0)}\,(\text{id}) \quad \dfrac{\dfrac{(\dagger)\ H \Rightarrow P(x,y)}{H \Rightarrow P(s(x''),x'')}\,(\text{subst})}{H \Rightarrow P(s(s(x'')),0)}\,(H_6)}{H \Rightarrow P(s(x'),0)}\,(\text{case }\mathcal{L})}{H \Rightarrow P(x,0)}\,(\text{case }\mathcal{L}) \qquad \dfrac{\dfrac{\dfrac{}{H, P(x,s(0)) \Rightarrow P(x,s(0))}\,(\text{id})}{H \Rightarrow P(x,s(0))}\,(\forall\Rightarrow) \quad \dfrac{\dfrac{\dfrac{(\dagger)\ H \Rightarrow P(x,y)}{H \Rightarrow P(s(y''),y'')}\,(\text{subst})}{H \Rightarrow P(0,s(s(y'')))}\,(H_5) \quad \dfrac{\dfrac{(\dagger)\ H \Rightarrow P(x,y)}{H \Rightarrow P(x',y'')}\,(\text{subst})}{H \Rightarrow P(s(x'),s(s(y'')))}\,(H_4)}{H \Rightarrow P(x,s(s(y'')))}\,(\text{case }\mathcal{L})}{H \Rightarrow P(x,s(y'))}\,(\text{case }\mathcal{L})}{(\dagger)\ H \Rightarrow P(x,y)}\,(\text{case }\mathcal{L})}{H_1 \wedge \ldots \wedge H_6 \Rightarrow \forall x \forall y P(x,y)}\,(\wedge\Rightarrow),(\Rightarrow\forall)}{\Rightarrow (H_1 \wedge \ldots \wedge H_6) \supset \forall x \forall y P(x,y)}\,(\Rightarrow\supset)
$$

Figure 7: A G^{ω}_{H} proof of (2-Hydra).

essentially new results: we simply demonstrate that the constructions and results of [5] apply in the Herbrand setting.

Consider a language that consists of a constant symbol 0, a unary function s, and a binary predicate P. The Hydra statement is formalized as follows:

$$
\begin{pmatrix} P(0,0) \wedge P(s(0),0) \wedge \forall x P(x,s(0)) \\ \quad \wedge \forall x \forall y (P(x,y) \supset P(s(x),s(s(y)))) \\ \quad \wedge \forall x \forall y (P(s(y),y) \supset P(0,s(s(y)))) \\ \quad \wedge \forall x \forall y (P(s(x),x) \supset P(s(s(y)),0)) \end{pmatrix} \supset \forall x \forall y P(x,y) \tag{2-Hydra}
$$

The antecedent of this implication characterises how the predicate P must be interpreted. It is essentially an inductive definition. We denote by H the sequence $H_1, \ldots, H_6$ of the conjuncts of the antecedent. Note that in [5], to reason over the domain an inductive unary predicate N was used, thus requiring quantifications in the Hydra statement to be bounded over N. However, since induction over the domain is a built-in feature in the Herbrand setting we do not need to bound the quantifications here.

Lemma 5.11. (2-Hydra) *is provable in the cyclic systems* G^{ω}_{H}, $G^{\omega}_{H_=}$, *and* $G^{\omega}_{SH_=}$.

Proof. Figure 7 shows a cyclic proof of (2-Hydra) in G^{ω}_{H}, adapted from [5]. The buds, labelled with $(\dagger)$, all have the same companion, which is the internal node of the proof labelled $(\dagger)$. The rule applications labelled H_4, H_5, H_6 abbreviate applications of (cut) that 'unfold' the right-hand side of the sequent according to the named component of H. For example, the derivation abbreviated by H_4 is the following:

$$
\dfrac{H \Rightarrow P(x',y'') \qquad \dfrac{\dfrac{\dfrac{}{H, P(x',y''), P(s(x'),s(s(y''))) \Rightarrow P(s(x'),s(s(y'')))}\,(\text{id}) \quad \dfrac{}{H, P(x',y'') \Rightarrow P(x',y''), P(s(x'),s(s(y'')))}\,(\text{id})}{H, P(x',y''), P(x',y'') \supset P(s(x'),s(s(y''))) \Rightarrow P(s(x'),s(s(y'')))}\,(\supset\Rightarrow)}{H, P(x',y'') \Rightarrow P(s(x'),s(s(y'')))}\,(\forall\Rightarrow)}{H \Rightarrow P(s(x'),s(s(y'')))}\,(\text{cut})
$$

The traces along each segment of the proof consist of variables: $x, \ldots, x', \ldots, x''$ and $y, \ldots, y', \ldots, y''$. For any infinite path through the proof, the instances of substitution in the proof (whose premises comprise the buds) allow these segments to be ‘glued together’ so as to form an infinitely progressing trace. More details may be found in [5]. Note that we straightforwardly obtain a proof in $G^{\omega}_{H_=}/G^{\omega}_{SH_=}$ by simulating the instances of (case $\mathcal{L}$) with an instance of (case $\mathcal{L}_=$) followed by applying paramodulation to each of the premises. $\square$

To show that (2-Hydra) is not provable in the induction-based systems, we demonstrate appropriate models for each system that satisfy $(IND_{\mathcal{L}})$ but falsify the Hydra statement. Then, by the soundness in Lemma 5.5 we obtain that the Hydra statement is not provable in the induction-based systems.

The structure underlying the models is that constructed in [5, §4] consisting of a disjoint sum of two domains, namely the natural numbers and the integers. The constant 0 is interpreted as the zero contained in the natural numbers, and the interpretation of s is the standard successor function over these domains. Notice that this structure is *not* a Herbrand structure for the language over the signature $\{0, s, P\}$ (nor is it even a semi-Herbrand structure): it contains a non-standard part consisting of elements that do not correspond to any closed term of the language (i.e. the integers).

The interpretation of the predicate P in this structure includes all pairs of elements in the domain, apart from a particular subset of pairs of integers carefully chosen so as to ensure that P still satisfies the properties defining the induction scheme H. Thus, this interpretation of P does *not* satisfy (2-Hydra), since it does not contain all pairs of elements. The pairs of elements ‘missing’ from P all contain at least one integer (i.e. lie outside the fully Herbrand part of the structure). Indeed, this must be the case since provability of (2-Hydra) in the cyclic system implies that any interpretation of P satisfying H must contain all pairs in which both elements are Herbrand for $\{0, s, P\}$ (i.e. are named by a closed term).

Finally, the results in [5] entail that this model satisfies $(IND_{\mathcal{L}})$ for the signature $\{0, s, P\}$. They also further entail that the extended (normal) model obtained by adding equality also satisfies $(IND_{\mathcal{L}})$ for the signature $\{0, s, P, =\}$. Furthermore, the underlying structure has the properties that the successor function is injective, and (natural number) zero is not in the image of the successor function. That is, the normal model also satisfies the conditions in Lemma 5.5(3). Thus, we have that (2-Hydra) is not provable in any of the systems G^{IND}_{H}, $G^{IND}_{SH_=}$, or $G^{IND}_{H_=}$.

In the language $\mathcal{L}_{\mathsf{NAT}}$, where addition, multiplication and equality are taken as relations, and with the appropriate additional axioms of PA, the cyclic and induction-based systems are equivalent to the implicit and explicit systems for PA (respectively). Thus, in this case the two systems become equivalent as shown in [6].[7]

6 Conclusion and Further Research

In this paper we provided a modular proof-theoretic study of Herbrand logics by considering Herbrand and semi-Herbrand structures, with and without equality. Sound and complete infinitary sequent-based proof systems were introduced for the various logics corresponding to two infinitary proof theoretic approaches. The first adopts an infinite rule and thus results in infinite-width proofs, and the second allows for non-well-founded proofs that admit the principle of infinite-descent. Sound finitary approximations of those infinitary systems were given as well, and the connections between them have been investigated. The adequacy of Herbrand logics as a convenient logical framework is supported by the naturality of the rules employed in all of these systems. The correspondence between Herbrand structures and the suggested systems is however non-trivial, as evident by the methods employed in order to achieve completeness.

We have shown that both approaches (infinite-width and infinite-height) are equivalent (for closed sequents) over languages with a finite number of function symbols. However this equivalence was shown

[7]This is also true for extensions of the language by new predicates, but not by new function symbols.

using the completeness theorem, going through the semantics. A first avenue for further research is to obtain a constructive translation between these two systems that will be purely syntactic. Such a translation would potentially reveal deeper connections between the approaches.

Further study is required to find appropriate counterparts (whenever such exist) to central proof-theoretical and meta-logical properties in the context of Herbrand logics. For example, it seems non-trivial to find a suitable notion of an interpolant that would entail some meaningful version of Craig interpolation. A possible direction for finding such a notion could potentially be obtained by translation to related systems [32]. Other standard properties that could be explored in this context are function elimination, extension by definitions, etc.

The work presented here offers a proof-theoretical interpretation of Herbrand structures that we believe will lay the foundations for further developments and applications of Herbrand logics that are more naturally handled proof-theoretically. One such application is in the realm of logic programming (see, e.g., [28]). The expressiveness of Herbrand logics allows for a finite axiomatization of the minimal model of any safe stratified logic program [24]. Reasoning about such programs can thus gain from the proof systems proposed here, by replacing validity checking in the minimal model with efficient proof search, having the aforementioned axiomatization as the set of premises. Other potential applications of Herbrand logics are described in [26]. Such applications could also gain from the availability of the corresponding proof systems presented here.

Lastly, we note that Herbrand structures are a robust concept, not limited only to the usual first-order structures of classical logic. The structure-definability property embedded in Herbrand logics can be studied in every logic that employs structures, including first-order modal and temporal logics, intuitionistic logic, second order logic, many-valued logics and more. Due to the computational aspect of Herbrand logics, as well as their expressiveness, it seems that the study of Herbrand structures in the context of computationally useful logics is of special interest for automated reasoning and AI. Two such prominent logics seem to be intuitionistic first-order logic which by itself already carries constructive computational content, and ancestral logic (or transitive closure logic) which is an extension of first-order logic that has been established as a congenial framework for inductive reasoning (see, e.g., [3, 15, 14]).

References

[1] Serge Abiteboul, Richard Hull, and Victor Vianu, editors. *Foundations of Databases: The Logical Level*. Addison-Wesley Longman Publishing Co., Inc., Boston, MA, USA, 1995.

[2] O. Arieli, AM. Borg, and C. Straßer. Argumentative approaches to reasoning with consistent subsets of premises. In *International Conference on Industrial, Engineering and Other Applications of Applied Intelligent Systems*, pages 455–465. Springer, 2017.

[3] A. Avron. Transitive closure and the mechanization of mathematics. In *Thirty Five Years of Automating Mathematics*, pages 149–171. Springer, 2003.

[4] A. Avron, B. Konikowska, and A. Zamansky. Modular construction of cut-free sequent calculi for paraconsistent logics. In *Logic in Computer Science (LICS), 2012 27th Annual IEEE Symposium on*, pages 85–94, 2012.

[5] Stefano Berardi and Makoto Tatsuta. Classical System of Martin-Löf's Inductive Definitions Is Not Equivalent to Cyclic Proof System. In *Proceedings of the 20th International Conference on Foundations of Software Science and Computation Structures, FOSSACS 2017, Uppsala, Sweden,*

April 22–29, 2017, pages 301–317, Berlin, Heidelberg, 2017. Springer Berlin Heidelberg. doi:10.1007/978-3-662-54458-7_18.

[6] Stefano Berardi and Makoto Tatsuta. Equivalence of Inductive Definitions and Cyclic Proofs Under Arithmetic. In *Proceedings of the 32nd Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2017, Reykjavik, Iceland, June 20–23, 2017*, pages 1–12, 2017. doi:10.1109/LICS.2017.8005114.

[7] E. Börger, E. Grädel, and Y. Gurevich. *The classical decision problem*. Springer Science & Business Media, 2001.

[8] James Brotherston. Formalised Inductive Reasoning in the Logic of Bunched Implications. In *Proceedings of Static Analysis, 14th International Symposium, SAS 2007, Kongens Lyngby, Denmark, August 22–24, 2007*, pages 87–103, 2007. doi:10.1007/978-3-540-74061-2_6.

[9] James Brotherston, Richard Bornat, and Cristiano Calcagno. Cyclic Proofs of Program Termination in Separation Logic. In *Proceedings of the 35th ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages, POPL 2008, San Francisco, California, USA, January 7–12, 2008*, pages 101–112, 2008. doi:10.1145/1328438.1328453.

[10] James Brotherston and Alex Simpson. Sequent Calculi for Induction and Infinite Descent. *Journal of Logic and Computation*, 21(6):1177–1216, 2010. doi:10.1093/logcom/exq052.

[11] W. A. Carnielli. On sequents and tableaux for many-valued logics. *Journal of Non-Classical Logic*, 8(1):59–76, 1991.

[12] CL. Chang and R. CT. Lee. *Symbolic logic and mechanical theorem proving*. Academic press, 2014.

[13] Keith L. Clark. Negation as failure. In Hervé Gallaire and Jack Minker, editors, *Logic and Data Bases*, pages 293–322, Boston, MA, 1978. Springer US.

[14] Liron Cohen. Completeness for ancestral logic via a computationally-meaningful semantics. In Renate A. Schmidt and Cláudia Nalon, editors, *Automated Reasoning with Analytic Tableaux and Related Methods*, pages 247–260, Cham, 2017. Springer International Publishing.

[15] Liron Cohen and Arnon Avron. The Middle Ground–Ancestral Logic. *Synthese*, pages 1–23, 2015. doi:10.1007/s11229-015-0784-3.

[16] Liron Cohen and Reuben N. S. Rowe. Uniform Inductive Reasoning in Transitive Closure Logic via Infinite Descent. In *Proceedings of the 27th EACSL Annual Conference on Computer Science Logic, CSL 2018, September 4–7, 2018, Birmingham, UK*, pages 16:1–16:17, 2018. doi:10.4230/LIPIcs.CSL.2018.16.

[17] Liron Cohen and Yoni Zohar. Reasoning inside the box: Deduction in herbrand logics. In Christoph Benzmüller, Christine Lisetti, and Martin Theobald, editors, *GCAI 2017. 3rd Global Conference on Artificial Intelligence*, volume 50 of *EPiC Series in Computing*, pages 107–120. EasyChair, 2017.

[18] Anupam Das and Damien Pous. A Cut-Free Cyclic Proof System for Kleene Algebra. In *Proceedings of the 26th International Conference Automated Reasoning with Analytic Tableaux and Related Methods, TABLEAUX 2017, Brasília, Brazil, September 25–28, 2017*, pages 261–277, 2017. doi:10.1007/978-3-319-66902-1_16.

[19] Nachum Dershowitz and Jean-Pierre Jouannaud. Rewrite Systems. In J. van Leeuwen, editor, *Handbook of Theoretical Computer Science, Volume B: Formal Models and Sematics (B)*, pages 243–320. MIT Press, 1990.

[20] S. Feferman. Lectures on proof theory. In M. H. Löb, editor, *Proceedings of the Summer School in Logic*, pages 1–107. Springer Berlin Heidelberg, 1968.

[21] J Flum. First-order logic and its extensions. In $\models$*ISILC Logic Conference*, pages 248–310. Springer, 1975.

[22] J. H. Gallier. *Logic for computer science: foundations of automatic theorem proving*. Courier Dover Publications, 2015.

[23] M. Genesereth and E. Kao. *Introduction to Logic*. Synthesis Lectures on Computer Science. Morgan & Claypool Publishers, 2012.

[24] M. Genesereth and E. Kao. The Herbrand manifesto - thinking inside the box. In N. Bassiliades, G. Gottlob, F. Sadri, A. Paschke, and D. Roman, editors, *Proceedings of the 9th International Symposium of RuleML*, volume 9202 of *LNCS*, pages 3–12. Springer, 2015.

[25] G. Gentzen. Investigations into logical deduction, 1934. In German. An English translation appears in 'The Collected Works of Gerhard Gentzen', edited by M. E. Szabo, North-Holland, 1969.

[26] Timorthy L. Hinrichs and Michael R. Genesereth. Herbrand logic. Technical report, Stanford University, 2006.

[27] JK. Lambert. The definition of E! in free logic. In *Abstracts: The International Congress for Logic, Methodology and Philosophy of Science*, 1960.

[28] J. W. Lloyd. *Foundations of logic programming*. Springer Science & Business Media, 2012.

[29] M. Machover. *Set theory, logic and their limitations*. Cambridge University Press, 1996.

[30] G. Metcalfe, N. Olivetti, and D. M. Gabbay. *Proof theory for fuzzy logics*, volume 36. Springer Science & Business Media, 2008.

[31] F. Poggiolesi. *Gentzen calculi for modal propositional logic*, volume 32. Springer Science & Business Media, 2010.

[32] João Rasga, Walter Carnielli, and Cristina Sernadas. Interpolation via translations. *Mathematical Logic Quarterly*, 55(5):515–534, 2009.

[33] A. Robinson and A. Voronkov. *Handbook of automated reasoning*, volume 1. Elsevier, 2001.

[34] R. M. Robinson. An essentially undecidable axiom system. In *Proceedings of the international Congress of Mathematics*, volume 1, pages 729–730, 1950.

[35] Reuben N. S. Rowe and James Brotherston. Automatic Cyclic Termination Proofs for Recursive Procedures in Separation Logic. In *Proceedings of the 6th ACM SIGPLAN Conference on Certified Programs and Proofs, CPP 2017, Paris, France, January 16–17, 2017*, pages 53–65, 2017. doi:10.1145/3018610.3018623.

[36] Peter Schroeder-Heister. Definitional reflection and the completion. In Roy Dyckhoff, editor, *Extensions of Logic Programming: 4th International Workshop, ELP '93 St Andrews, U.K., March 29–April 1, 1993 Proceedings*, pages 333–347, Berlin, Heidelberg, 1994. Springer Berlin Heidelberg.

[37] K. Schütte. Beweistheoretische erfassung der unendlichen induktion in der zahlentheorie. *Mathematische Annalen*, 122:369–389, 1950/51.

[38] G. Sundholm. A completeness proof for an infinitary tense-logic. *Theoria*, 43(1):47–51, 1977.

[39] Gadi Tellez and James Brotherston. Automatically Verifying Temporal Properties of Pointer Programs with Cyclic Proof. In *Proceedings of the 26th International Conference on Automated Deduction, CADE 26, Gothenburg, Sweden, August 6–11, 2017*, pages 491–508, 2017. doi:10.1007/978-3-319-63046-5_30.

[40] H. Wansing. Sequent systems for modal logics. In Dov M. Gabbay and Franz Guenthner, editors, *Handbook of Philosophical Logic, 2nd edition*, volume 8, pages 61–145. Springer, 2002.