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Finite Model Property for Modal Ideal Paraconsistent Four-valued Logic

Norihiro Kamide
Teikyo University,
Department of Information and Electronic Engineering,
Toyosatodai 1-1, Utsunomiya, Tochigi 320-8551, Japan.
drnkamide08@kpd.biglobe.ne.jp

Yoni Zohar
Stanford University,
Computer Science Department,
353 Serra Mall, Stanford, CA 94305, USA.
yoniz@cs.stanford.edu

***Abstract*—A modal extension M4CC of Arieli, Avron, and Zamansky's ideal paraconsistent four-valued logic 4CC is introduced as a Gentzen-type sequent calculus. The completeness theorem with respect to a Kripke semantics for M4CC is proved. The finite model property for M4CC is shown by modifying the completeness proof. The decidability of M4CC is obtained as a corollary.**

I. Introduction

In this study, we introduce a modal extension M4CC of *Arieli, Avron, and Zamansky's ideal paraconsistent four-valued logic* known as 4CC [4]–[6]. We prove the completeness theorem with respect to a Kripke semantics for M4CC, and by modifying the completeness proof, we obtain the finite model property for M4CC. As a corollary, we also obtain the decidability of M4CC.

The proposed logic M4CC is introduced as a Gentzen-type sequent calculus, and is a modal extension of the Gentzen-type sequent calculus EPL which was introduced by Kamide and Zohar in [14], [16]. The calculus EPL was shown in [16] to be theorem-equivalent to the Gentzen-type sequent calculus G_{4CC} which was originally introduced by Arieli and Avron in [4], [5].

The original non-modal logic 4CC is an extension of *Belnap and Dunn's useful four-valued logic* (also called *first-degree entailment logic*) [7], [8], [11], and is a variant of the *logic of logical bilattices* [2], [3]. The logic 4CC is also a specific type of *paraconsistent logics* [20], which have multiple names: they are called *paradefinite logics* by Arieli and Avron [4], [5], *non-alethic logics* by da Costa, and *paranormal logics* by Béziau [9]. Regardless of their names, paradefinite logics incorporate the properties of both *paraconsistency*, which rejects *the principle* $(\alpha \wedge {\sim}\alpha) \rightarrow \beta$ *of explosion*, and *paracompleteness*, which rejects *the law* $\alpha \vee {\sim}\alpha$ *of excluded middle*.

Moreover, 4CC is known to be one of the most important *ideal paraconsistent (or paradefinite) logics* that have natural many-valued semantics. The logic 4CC is maximal relative to classical logic. This means that any attempt to add to it a tautology of classical logic, which is not provable in 4CC, should necessarily end-up with classical logic. For the exact definition and motivation of this property, see [6]. The logic 4CC is also related to *connexive logics* [1], [17], [22], as it has a common characteristic Hilbert-style axiom scheme. For more information on the relationship between 4CC and connexive logics, see [16].

As mentioned above, 4CC is an important ideal paraconsistent (or paradefinite) logic. However, a modal extension of 4CC, which would be much more suitable for actual applications, has not been studied yet. We thus propose in this study the modal extension M4CC of 4CC, and show the completeness theorem with respect to a Kripke semantics for M4CC as well as the finite model property for M4CC. We developed M4CC as a combination of 4CC and the normal modal logic S4, since the combination with S4 gives natural formulations of both Gentzen-type sequent calculus and Kripke semantics. However, we can also combine 4CC and one of the other normal modal logics such as K. Similar proof method can also be used for showing the completeness and finite model property for some such extensions, imposing some appropriate modifications.

The structure of this paper is summarized as follows. In Section II, we introduce M4CC, and address some remarks on M4CC. In Section III, we introduce a Kripke semantics for M4CC, and show the soundness theorem with respect to this semantics. In Section IV, we prove the completeness theorem for M4CC by constructing a canonical model. In Section V, by slightly modifying the completeness proof, we show the finite model property for M4CC, and as a corollary, M4CC is shown to be decidable. In Section VI, we conclude this paper, and address related works on modal extensions of many-valued logics.

II. Sequent calculus

Formulas of *modal ideal paraconsistent four-valued logic* are constructed from countably many propositional variables by the logical connectives $\wedge$ (conjunction), $\vee$ (disjunction), $\rightarrow$ (implication), $\sim$ (paraconsistent negation) and $-$ (conflation), $\Box$ (box) and $\Diamond$ (diamond). In what follows, we use small letters $p, q, ...$ to denote propositional variables, Greek small letters $\alpha, \beta, ...$ to denote formulas, and Greek capital letters $\Gamma, \Delta, ...$ to represent finite (possibly empty) sets of formulas. Let A be a set of symbols (i.e., alphabet). Then, the notation $A^{\star}$ is used to represent the set of all words of finite length of the alphabet A. For any $\sharp \in \{\sim, -, \Box, \Diamond\}^{\star}$, we use an expression $\sharp\Gamma$ to denote the set $\{\sharp\gamma \mid \gamma \in \Gamma\}$. We use the

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symbol Φ to denote the set of all propositional variables, the symbol Φ^* to denote the set of all formulas, and the symbols $\Phi^{\sim}$ and Φ^{-} to denote the sets $\{\sim p \mid p \in \Phi\}$ and $\{-p \mid p \in \Phi\}$, respectively. Let U be a set of formulas. Then, we use an expression $U_{\Box}$ to denote $\{\gamma \mid \Box\gamma \in U\} \cup \{\sim\gamma \mid \sim\Box\gamma \in U\} \cup \{-\gamma \mid -\Box\gamma \in U\}$. We use the symbol $\equiv$ to denote the equality of symbols. A *sequent* is an expression of the form $\Gamma \Rightarrow \Delta$. we use an expression $\alpha \Leftrightarrow \beta$ to represent the abbreviation of the sequents $\alpha \Rightarrow \beta$ and $\beta \Rightarrow \alpha$. An expression $L \vdash S$ means that a sequent S is provable in a sequent calculus L. If L of $L \vdash S$ is clear from the context, we omit L in it.

A Gentzen-type sequent calculus M4CC for modal ideal paraconsistent four-valued logic is defined as follows.

*Definition 2.1 (*M4CC): The initial sequents of M4CC are of the following form, for any propositional variable p,

$$p \Rightarrow p \qquad \sim p \Rightarrow \sim p \qquad -p \Rightarrow -p \qquad \sim p, -p \Rightarrow \qquad \Rightarrow \sim p, -p.$$

The structural inference rules of M4CC are of the form:

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \alpha, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\text{cut})$$

$$\frac{\Gamma \Rightarrow \Delta}{\alpha, \Gamma \Rightarrow \Delta}\ (\text{we-left}) \quad \frac{\Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \alpha}\ (\text{we-right}).$$

The non-negated logical inference rules of M4CC are of the form:

$$\frac{\alpha, \beta, \Gamma \Rightarrow \Delta}{\alpha \wedge \beta, \Gamma \Rightarrow \Delta}\ (\wedge\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \wedge \beta}\ (\wedge\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta \quad \beta, \Gamma \Rightarrow \Delta}{\alpha \vee \beta, \Gamma \Rightarrow \Delta}\ (\vee\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha, \beta}{\Gamma \Rightarrow \Delta, \alpha \vee \beta}\ (\vee\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad \beta, \Sigma \Rightarrow \Pi}{\alpha \to \beta, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (\to\text{left}) \quad \frac{\alpha, \Gamma \Rightarrow \Delta, \beta}{\Gamma \Rightarrow \Delta, \alpha \to \beta}\ (\to\text{right})$$

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\Box\alpha, \Gamma \Rightarrow \Delta}\ (\Box\text{left}) \quad \frac{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow \alpha}{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow \Box\alpha}\ (\Box\text{right})$$

$$\frac{\alpha \Rightarrow \Diamond\Gamma, \sim\Box\Sigma, -\Diamond\Pi}{\Diamond\alpha \Rightarrow \Diamond\Gamma, \sim\Box\Sigma, -\Diamond\Pi}\ (\Diamond\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \Diamond\alpha}\ (\Diamond\text{right}).$$

The negated logical inference rules of M4CC are of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{\sim\sim\alpha, \Gamma \Rightarrow \Delta}\ (\sim\sim\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, \sim\sim\alpha}\ (\sim\sim\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{\sim-\alpha, \Gamma \Rightarrow \Delta}\ (\sim-\text{left}) \quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim-\alpha}\ (\sim-\text{right})$$

$$\frac{\sim\alpha, \Gamma \Rightarrow \Delta \quad \sim\beta, \Gamma \Rightarrow \Delta}{\sim(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (\sim\wedge\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \sim\alpha, \sim\beta}{\Gamma \Rightarrow \Delta, \sim(\alpha \wedge \beta)}\ (\sim\wedge\text{right})$$

$$\frac{\sim\alpha, \sim\beta, \Gamma \Rightarrow \Delta}{\sim(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ (\sim\vee\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \sim\alpha \quad \Gamma \Rightarrow \Delta, \sim\beta}{\Gamma \Rightarrow \Delta, \sim(\alpha \vee \beta)}\ (\sim\vee\text{right})$$

$$\frac{\alpha, \sim\beta, \Gamma \Rightarrow \Delta}{\sim(\alpha \to \beta), \Gamma \Rightarrow \Delta}\ (\sim\to\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha \quad \Gamma \Rightarrow \Delta, \sim\beta}{\Gamma \Rightarrow \Delta, \sim(\alpha \to \beta)}\ (\sim\to\text{right})$$

$$\frac{\sim\alpha \Rightarrow \Diamond\Gamma, \sim\Box\Sigma, -\Diamond\Pi}{\sim\Box\alpha \Rightarrow \Diamond\Gamma, \sim\Box\Sigma, -\Diamond\Pi}\ (\sim\Box\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \sim\alpha}{\Gamma \Rightarrow \Delta, \sim\Box\alpha}\ (\sim\Box\text{right})$$

$$\frac{\sim\alpha, \Gamma \Rightarrow \Delta}{\sim\Diamond\alpha, \Gamma \Rightarrow \Delta}\ (\sim\Diamond\text{left}) \quad \frac{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow \sim\alpha}{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow \sim\Diamond\alpha}\ (\sim\Diamond\text{right}).$$

The conflated logical inference rules of M4CC are of the form:

$$\frac{\alpha, \Gamma \Rightarrow \Delta}{--\alpha, \Gamma \Rightarrow \Delta}\ (--\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, \alpha}{\Gamma \Rightarrow \Delta, --\alpha}\ (--\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha}{-\sim\alpha, \Gamma \Rightarrow \Delta}\ (-\sim\text{left}) \quad \frac{\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -\sim\alpha}\ (-\sim\text{right})$$

$$\frac{-\alpha, -\beta, \Gamma \Rightarrow \Delta}{-(\alpha \wedge \beta), \Gamma \Rightarrow \Delta}\ (-\wedge\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, -\alpha \quad \Gamma \Rightarrow \Delta, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha \wedge \beta)}\ (-\wedge\text{right})$$

$$\frac{-\alpha, \Gamma \Rightarrow \Delta \quad -\beta, \Gamma \Rightarrow \Delta}{-(\alpha \vee \beta), \Gamma \Rightarrow \Delta}\ (-\vee\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, -\alpha, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha \vee \beta)}\ (-\vee\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \alpha \quad -\beta, \Sigma \Rightarrow \Pi}{-(\alpha \to \beta), \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (-\to\text{left}) \quad \frac{\alpha, \Gamma \Rightarrow \Delta, -\beta}{\Gamma \Rightarrow \Delta, -(\alpha \to \beta)}\ (-\to\text{right})$$

$$\frac{-\alpha, \Gamma \Rightarrow \Delta}{-\Box\alpha, \Gamma \Rightarrow \Delta}\ (-\Box\text{left}) \quad \frac{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow -\alpha}{\Box\Gamma, \sim\Diamond\Sigma, -\Box\Pi \Rightarrow -\Box\alpha}\ (-\Box\text{right})$$

$$\frac{-\alpha \Rightarrow \Diamond\Gamma, \sim\Box\Sigma, -\Diamond\Pi}{-\Diamond\alpha \Rightarrow \Diamond\Gamma, \sim\Box\Sigma, -\Diamond\Pi}\ (-\Diamond\text{left}) \quad \frac{\Gamma \Rightarrow \Delta, -\alpha}{\Gamma \Rightarrow \Delta, -\Diamond\alpha}\ (-\Diamond\text{right}).$$

Proposition 2.2: The following sequents are provable in cut-free M4CC: For any formulas α and β,

1) $\alpha \Rightarrow \alpha$,
2) $\sim\alpha, -\alpha \Rightarrow$,
3) $\Rightarrow \sim\alpha, -\alpha$,
4) $\sim\sim\alpha \Leftrightarrow \alpha$,
5) $\sim-\alpha \Leftrightarrow -\sim\alpha$,
6) $\sim(\alpha \wedge \beta) \Leftrightarrow \sim\alpha \vee \sim\beta$,
7) $\sim(\alpha \vee \beta) \Leftrightarrow \sim\alpha \wedge \sim\beta$,
8) $\sim(\alpha \to \beta) \Leftrightarrow \alpha \wedge \sim\beta$,
9) $\sim\Box\alpha \Leftrightarrow \Diamond\sim\alpha$,
10) $\sim\Diamond\alpha \Leftrightarrow \Box\sim\alpha$,
11) $--\alpha \Leftrightarrow \alpha$,
12) $-(\alpha \wedge \beta) \Leftrightarrow -\alpha \wedge -\beta$,
13) $-(\alpha \vee \beta) \Leftrightarrow -\alpha \vee -\beta$,
14) $-(\alpha \to \beta) \Leftrightarrow \alpha \to -\beta$,
15) $-\Box\alpha \Leftrightarrow \Box-\alpha$,
16) $-\Diamond\alpha \Leftrightarrow \Diamond-\alpha$,
17) $\sim\alpha \wedge -\alpha \Rightarrow \beta$ (the principle of quasi-explosion),
18) $\Rightarrow \sim\alpha \vee -\alpha$ (the law of quasi-excluded middle).

Proposition 2.3: The following rules are derivable in M4CC:

$$\frac{\Gamma \Rightarrow \Delta, -\alpha}{\sim\alpha, \Gamma \Rightarrow \Delta}\ (\sim\text{left}) \quad \frac{-\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, \sim\alpha}\ (\sim\text{right})$$

$$\frac{\Gamma \Rightarrow \Delta, \sim\alpha}{-\alpha, \Gamma \Rightarrow \Delta}\ (-\text{left}) \quad \frac{\sim\alpha, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta, -\alpha}\ (-\text{right}).$$

Remark 2.4:

1) ($-\to$left) and ($-\to$right) correspond to the Hilbert-style axiom scheme $-(\alpha \to \beta) \leftrightarrow \alpha \to -\beta$, which is a characteristic axiom scheme for *connexive logics* [1], [17], [22].
2) Based on the use of ($\sim-$left), ($\sim-$right), ($-\sim$left), ($-\sim$right), we can define the classical negation $\neg\alpha$ (i.e., the negation of classical logic) by $\sim-\alpha$ and $-\sim\alpha$.
3) The $\{\Box, \Diamond\}$-free fragment of M4CC is theorem-equivalent to the Gentzen-type sequent calculus G_{4CC} which was originally introduced by Arieli and Avron in [4], [5] for the ideal paraconsistent logic 4CC [4]–[6]. See [16] for the detail of the equivalence among related systems.
4) G_{4CC} [4], [5] is obtained from the $\{\Box, \Diamond\}$-free fragment of M4CC by replacing ($p \Rightarrow p$), ($\sim p \Rightarrow \sim p$),

($-p \Rightarrow -p$), ($\sim p, -p \Rightarrow$), ($\Rightarrow \sim p, -p$), ($-\wedge$left), ($-\wedge$right), ($-\vee$left), ($-\vee$right), ($-\to$left), ($-\to$right), ($--$left), ($--$right), ($-\sim$left), and ($-\sim$right) with $\alpha \Rightarrow \alpha$, ($-$left), and ($-$right).

5) The $\{\Box, \Diamond\}$-free fragment of M4CC is theorem-equivalent to the system which is obtained from G_{4CC} by adding ($\sim\alpha, -\alpha \Rightarrow$), ($\Rightarrow \sim\alpha, -\alpha$), ($\sim$left) and ($\sim$right). See [16] for the detail of the equivalence among related systems.

III. Kripke semantics

In what follows, we use the symbol $\neg$ to denote the abbreviation of $\sim-$. We assume the commutativity of $\wedge$ or $\vee$. We have the following fact: For any formulas $\alpha_1, ..., \alpha_m, \beta_1, ..., \beta_n$, $\vdash \alpha_1, ..., \alpha_m \Rightarrow \beta_1, ..., \beta_n$ iff $\vdash \alpha_1 \wedge \cdots \wedge \alpha_m \Rightarrow \beta_1 \vee \cdots \vee \beta_n$. Let Γ be a set $\{\alpha_1, ..., \alpha_m\}$ ($m \geq 0$). Then, we use an expression Γ^* to denote $\alpha_1 \vee \cdots \vee \alpha_m$ if $m \geq 1$, or otherwise $\neg(p \to p)$ where p is a fixed propositional variable. We also use an expression Γ_* to denote $\alpha_1 \wedge \cdots \wedge \alpha_m$ if $m \geq 1$, or otherwise $p \to p$ where p is a fixed propositional variable.

We now introduce a Kripke semantics for M4CC.

Definition 3.1: A structure $\langle M, R\rangle$ is called a Kripke frame if

1) M is a non-empty set,
2) R is a transitive and reflexive binary relation on M.

Definition 3.2: A *paraconsistent valuation* $\models^*$ on a Kripke frame $\langle M, R\rangle$ is a mapping from the set $\Phi \cup \Phi^{\sim} \cup \Phi^{-}$ to the power set 2^M of M such that

(*) $\quad x \in \models^* (-p)$ iff $x \notin \models^* (\sim p)$,
$\quad x \in \models^* (\sim p)$ iff $x \notin \models^* (-p)$.

We will write $x \models^* p$, $x \models^* \sim p$, and $x \models^* -p$ for $x \in \models^* (p)$, $x \in \models^* (\sim p)$, and $x \in \models^* (-p)$, respectively. We will also use the same notation as $x \models^* \alpha$ for an extended paraconsistent valuation for any formula α. The paraconsistent valuation $\models^*$ is extended to the mapping from the set of all formulas to 2^M by:

1) $x \models^* \alpha \wedge \beta$ iff $x \models^* \alpha$ and $x \models^* \beta$,
2) $x \models^* \alpha \vee \beta$ iff $x \models^* \alpha$ or $x \models^* \beta$,
3) $x \models^* \alpha \to \beta$ iff $x \models^* \alpha$ implies $x \models^* \beta$,
4) $x \models^* \Box\alpha$ iff $\forall y \in M$ [xRy implies $y \models^* \alpha$],
5) $x \models^* \Diamond\alpha$ iff $\exists y \in M$ [xRy and $y \models^* \alpha$],
6) $x \models^* \sim\sim\alpha$ iff $x \models^* \alpha$,
7) $x \models^* \sim-\alpha$ iff $x \not\models^* \alpha$,
8) $x \models^* \sim(\alpha \wedge \beta)$ iff $x \models^* \sim\alpha$ or $x \models^* \sim\beta$,
9) $x \models^* \sim(\alpha \vee \beta)$ iff $x \models^* \sim\alpha$ and $x \models^* \sim\beta$,
10) $x \models^* \sim(\alpha \to \beta)$ iff $x \models^* \alpha$ and $x \models^* \sim\beta$,
11) $x \models^* \sim\Box\alpha$ iff $\exists y \in M$ [xRy and $y \models^* \sim\alpha$],
12) $x \models^* \sim\Diamond\alpha$ iff $\forall y \in M$ [xRy implies $y \models^* \sim\alpha$],
13) $x \models^* --\alpha$ iff $x \models^* \alpha$,
14) $x \models^* -\sim\alpha$ iff $x \not\models^* \alpha$,
15) $x \models^* -(\alpha \wedge \beta)$ iff $x \models^* -\alpha$ and $x \models^* -\beta$,
16) $x \models^* -(\alpha \vee \beta)$ iff $x \models^* -\alpha$ or $x \models^* -\beta$,
17) $x \models^* -(\alpha \to \beta)$ iff $x \models^* \alpha$ implies $x \models^* -\beta$,
18) $x \models^* -\Box\alpha$ iff $\forall y \in M$ [xRy implies $y \models^* -\alpha$],
19) $x \models^* -\Diamond\alpha$ iff $\exists y \in M$ [xRy and $y \models^* -\alpha$].

Definition 3.3: A *paraconsistent Kripke model* is a structure $\langle M, R, \models^*\rangle$ such that

1) $\langle M, R\rangle$ is a Kripke frame,
2) $\models^*$ is a paraconsistent valuation on $\langle M, R\rangle$.

A formula α is *true* in a paraconsistent Kripke model $\langle M, R, \models^*\rangle$ iff $x \models^* \alpha$ for any $x \in M$, and is *M4CC-valid* (in a Kripke frame) iff it is true for every paraconsistent valuation $\models^*$ (on the Kripke frame). A sequent $\Gamma \Rightarrow \Delta$ is called *M4CC-valid* (denoted as M4CC $\models \Gamma \Rightarrow \Delta$) iff the formula $\Gamma_* \to \Delta^*$ is M4CC-valid.

Theorem 3.4: In Definition 3.2, the requirement $(*)$, together with clauses 13–19, can be replaced with the following requirement: For any formula α,

(**) $\quad x \models^* -\alpha$ iff $x \not\models^* \sim\alpha$,
$\quad x \models^* \sim\alpha$ iff $x \not\models^* -\alpha$.

Proof.

($\Longrightarrow$) : We prove that $(**)$ holds in every paraconsistent valuation $\models^*$ on every Kripke frame $\langle M, R\rangle$, every $x \in M$ and every formula α. We do so by induction on α. We show only some cases for the first condition of (**).

1) Case $\alpha \equiv p \in \Phi$: If α is a propositional variable, then $(**)$ directly follows from $(*)$.
2) Case$\alpha \equiv \alpha_1 \wedge \alpha_2$: If $\alpha \equiv \alpha_1 \wedge \alpha_2$, then by clause 15 we have $x \models^* -\alpha$ iff both $x \models^* -\alpha_1$ and $x \models^* -\alpha_2$. By the induction hypothesis, the latter holds iff $x \not\models^* \sim\alpha_1$ and $x \not\models^* \sim\alpha_2$, which by clause 8 holds iff $x \not\models^* \sim\alpha$.
3) Case $\alpha \equiv \sim\beta$: If $\alpha \equiv \sim\beta$, then by clause 14 we have $x \models^* -\alpha$ iff $x \not\models^* \beta$, which, by clause 6 holds iff $x \not\models^* \sim\sim\beta = \sim\alpha$.
4) Case $\alpha \equiv -\beta$: If $\alpha \equiv -\beta$, then by clause 13 we have that $x \models^* -\alpha$ iff $x \models^* \beta$, which by clause 7 holds iff $x \not\models \sim-\beta \equiv \sim\alpha$.
5) Case $\alpha \equiv \Box\beta$: If $\alpha \equiv \Box\beta$, then by clause 18 we have $x \models^* -\alpha$ iff for every $y \in M$, xRy implies $y \models^* -\beta$. By the induction hypothesis, the latter holds iff for every $y \in M$, xRy implies $y \not\models^* \sim\beta$. By clause 11, this holds iff $x \not\models^* \sim\Box\beta \equiv \sim\alpha$.

($\Longleftarrow$) : We prove that in every paraconsistent valuation $\models^*$ on every Kripke frame $\langle M, R\rangle$, every $x \in M$ and every formula α, clauses $(*)$ and 13–19 hold, provided that $(**)$ holds. We explicitly show $(*)$, 13, 14, 15 and 18, leaving the rest to the reader.

$(*)$: $(*)$ is a particular instance of $(**)$ for the case of propositional variables.

(13): Using $(**)$ and 7, $x \models^* --\alpha$ iff $x \not\models^* \sim-\alpha$ iff $x \models^* \alpha$.

(14): Using $(**)$ and 6, $x \models^* -\sim\alpha$ iff $x \not\models^* \sim\sim\alpha$ iff $x \not\models \alpha$.

(15): Using $(**)$ and 8, $x \models^* -(\alpha_1 \wedge \alpha_2)$ iff $x \not\models^* \sim(\alpha_1 \wedge \alpha_2)$ iff $x \not\models^* \sim\alpha_1$ and $x \not\models^* \sim\alpha_2$ iff $x \models^* -\alpha_1$ and $x \models^* -\alpha_2$.

(18): Using $(**)$ and 11, $x \models^* -\Box\alpha$ iff $x \not\models^* \sim\Box\alpha$ iff $y \not\models^* \sim\alpha$ for every $y \in M$ such that xRy, iff $y \models^* -\alpha$ for every $y \in M$ such that xRy. ∎

In particular, we have the following corollary.

Corollary 3.5: For any paraconsistent valuation $\models^*$ on a Kripke frame $\langle M, R\rangle$, any $x \in M$, and any formula α,

1) $x \models^* {\sim}\alpha$ iff $x \not\models^* -\alpha$,
2) $\models^* ({\sim}\alpha) \cap \models^* (-\alpha) = \emptyset$.

Remark 3.6: Using Corollary 3.5, we can show that the following formulas are M4CC-valid:

1) $({\sim}\alpha \wedge -\alpha)\rightarrow\beta$,
2) ${\sim}\alpha \vee -\alpha$.

Theorem 3.7 (Soundness for M4CC*):* For any sequent S, if M4CC $\vdash S$, then M4CC $\models S$.

Proof. We prove this theorem by induction on the proofs P of S in M4CC. We distinguish the cases according to the last inference of P, and show only the following characteristic cases. Let $\mathbf{M}$ be a Kripke frame $\langle M, R\rangle$.

1) Case ${\sim}p, -p \Rightarrow$: The last inference of P is of the form: ${\sim}p, -p \Rightarrow$. We show M4CC $\models {\sim}p, -p \Rightarrow$ (i.e., $({\sim}p \wedge -p)\rightarrow\neg(q\rightarrow q)$ is M4CC-valid). We show $x \models^* ({\sim}p \wedge -p)\rightarrow\neg(q\rightarrow q))$ (i.e., $x \not\models^* {\sim}p \wedge -p$ or $x \models^* \neg(q\rightarrow q)$) for any paraconsistent valuation $\models^*$ on $\mathbf{M}$ and any $x \in M$. To show this, it is sufficient to show the fact that $x \not\models^* {\sim}p \wedge -p$ (i.e., $x \not\models^* {\sim}p$ or $x \not\models^* -p$), since we have $x \models^* \neg(q\rightarrow q)$. This fact can be shown as follows. If $x \models^* {\sim}p$, then we obtain $x \not\models^* -p$ by Corollary 3.5. If $x \models^* -p$, then we obtain $x \not\models^* {\sim}p$ by Corollary 3.5. Thus, in both cases, we obtain the required fact.
2) Case $\Rightarrow {\sim}p, -p$: The last inference of P is of the form: $\Rightarrow {\sim}p, -p$. We show M4CC $\models\ \Rightarrow {\sim}p, -p$ (i.e., $(q\rightarrow q)\rightarrow({\sim}p \vee -p)$ is M4CC-valid). We show $x \models^* (q\rightarrow q)\rightarrow({\sim}p \vee -p)$ (i.e., $x \not\models^* (q\rightarrow q) = f$ or $x \models^* {\sim}p\vee -p$) for any paraconsistent valuation $\models^*$ on $\mathbf{M}$ and any $x \in M$. To show this, it is sufficient to show the fact that $x \models^* {\sim}p \vee -p$ (i.e., $x \models^* {\sim}p$ or $x \models^* -p$), since we have $x \models^* q\rightarrow q$. This fact can be shown as follows. If $x \not\models^* {\sim}p$, then we obtain $x \models^* -p$ by Corollary 3.5. If $x \not\models^* -p$, then we obtain $x \models^* {\sim}p$ by Corollary 3.5. Thus, in both cases, we obtain the required fact. ∎

IV. Completeness

Definition 4.1: Let $U, V \subseteq \Phi^*$. A pair (U, V) is called *consistent* if for any $\alpha_1, ..., \alpha_m \in U$ and any $\beta_1, ..., \beta_n \in V$ (m and n are arbitrary fixed integers and $m, n \geq 0$), the sequent $\alpha_1, ..., \alpha_m \Rightarrow \beta_1, ..., \beta_n$ is not provable in M4CC. A pair (U, V) is called *maximal consistent* if the following conditions hold:

1) (U, V) is consistent,
2) $U \cup V = \Phi^*$.

We then obtain the following lemma by a standard way.

Lemma 4.2: Suppose that a pair (U_0, V_0) is consistent. Then, there exist $U, V \in \Phi^*$ such that $U_0 \subseteq U$, $V_0 \subseteq V$, and (U, V) is maximal consistent.

Definition 4.3: A structure $\langle M_L,\ R_L,\ \models_L\rangle$ is called a *canonical paraconsistent Kripke model* if

1) $M_L := \{U(\subseteq \Phi^*) \mid (U, \Phi^* - U) \text{ is maximal consistent}\}$,
2) $U_1 R_L U_2$ for any $U_1, U_2 \in M_L$ is defined as $(U_1)_\Box \subseteq (U_2)_\Box$,
3) $U \models_L p$, $U \models_L {\sim}p$, and $U \models_L -p$ for any $U \in M_L$ and any propositional variable p are defined as $p \in U$, ${\sim}p \in U$, and $-p \in U$, respectively, with the condition: ${\sim}p \notin U$ or $-p \notin U$.

Lemma 4.4: Suppose $U \in M_L$. We have:

1) If $\alpha_1, ..., \alpha_m \in U$ and M4CC $\vdash \alpha_1, ..., \alpha_m \Rightarrow \beta$, then $\beta \in U$.
2) For any formula α, either $\alpha \in U$ or $\neg\alpha \in U$.

Proof. Let V be $\Phi^* - U$. Then, (U, V) is maximal consistent.

1) Suppose $\beta \notin U$ and $\alpha_1, ..., \alpha_m \in U$. We have $\beta \in V$. On the other hand, we have $\vdash \alpha_1, ..., \alpha_m \Rightarrow \beta$ as an assumption, and hence (U, V) is not consistent. This contradicts for the assumption $U \in M_L$. Thus, $\beta \in U$.
2) Since $\vdash \alpha, \neg\alpha \Rightarrow$, we have no case that $\alpha \in U$ and $\neg\alpha \in U$. Suppose $\alpha \notin U$ and $\neg\alpha \notin U$. Then we have that $\alpha \in V$ and $\neg\alpha \in V$. This contradicts for the fact $\vdash\ \Rightarrow \alpha, \neg\alpha$. Thus, either $\alpha \in U$ or $\neg\alpha \in U$. ∎

Lemma 4.5: Let $U \in M_L$. For any formulas α and β,

1) $\alpha \wedge \beta \in U$ iff $\alpha \in U$ and $\beta \in U$,
2) $\alpha \vee \beta \in U$ iff $\alpha \in U$ or $\beta \in U$,
3) $\alpha\rightarrow\beta \in U$ iff $\alpha \notin U$ or $\beta \in U$,
4) $\Box\alpha \in U$ iff $\forall W \in M_L\ [U R_L W$ implies $\alpha \in W]$,
5) $\Diamond\alpha \in U$ iff $\exists W \in M_L\ [U R_L W$ and $\alpha \in W]$,
6) ${\sim}{\sim}\alpha \in U$ iff $\alpha \in U$,
7) ${\sim}-\alpha \in U$ iff $\alpha \notin U$ (i.e., $\neg\alpha \in U$ iff $\alpha \notin U$),
8) ${\sim}(\alpha \wedge \beta) \in U$ iff ${\sim}\alpha \in U$ or ${\sim}\beta \in U$,
9) ${\sim}(\alpha \vee \beta) \in U$ iff ${\sim}\alpha \in U$ and ${\sim}\beta \in U$,
10) ${\sim}(\alpha\rightarrow\beta) \in U$ iff $\alpha \in U$ and ${\sim}\beta \in U$,
11) ${\sim}\Box\alpha \in U$ iff $\exists W \in M_L\ [U R_L W$ and ${\sim}\alpha \in W]$,
12) ${\sim}\Diamond\alpha \in U$ iff $\forall W \in M_L\ [U R_L W$ implies ${\sim}\alpha \in W]$,
13) $--\alpha \in U$ iff $\alpha \in U$,
14) $-{\sim}\alpha \in U$ iff $\alpha \notin U$,
15) $-(\alpha \wedge \beta) \in U$ iff $-\alpha \in U$ and $-\beta \in U$,
16) $-(\alpha \vee \beta) \in U$ iff $-\alpha \in U$ or $-\beta \in U$,
17) $-(\alpha\rightarrow\beta) \in U$ iff $\alpha \in U$ implies $-\beta \in U$,
18) $-\Box\alpha \in U$ iff $\forall W \in M_L\ [U R_L W$ implies $-\alpha \in W]$,
19) $-\Diamond\alpha \in U$ iff $\exists W \in M_L\ [U R_L W$ and $-\alpha \in W]$.

Proof. We show only the following cases.

(13): ($\Longrightarrow$): Suppose $--\alpha \in U$. Since we have $\vdash --\alpha \Rightarrow \alpha$, we obtain $\alpha \in U$ by Lemma 4.4 (1). ($\Longleftarrow$): Suppose $\alpha \in U$. Since we have $\vdash \alpha \Rightarrow --\alpha$, we obtain $--\alpha \in U$ by Lemma 4.4 (1).

(17): ($\Longrightarrow$): Suppose $-(\alpha\rightarrow\beta) \in U$ and $\alpha \in U$. Since we have $\vdash \alpha, -(\alpha\rightarrow\beta) \Rightarrow -\beta$, we obtain $-\beta \in U$ by Lemma 4.4 (1). ($\Longleftarrow$): Suppose $-(\alpha\rightarrow\beta) \notin U$. We obtain $\neg-(\alpha\rightarrow\beta) \in U$ by Lemma 4.4 (2). Since we have $\vdash \neg-(\alpha\rightarrow\beta) \Rightarrow \alpha$ and $\vdash \neg-(\alpha\rightarrow\beta) \Rightarrow \neg-\beta$, we obtain $\alpha \in U$ and $\neg-\beta \in U$ by Lemma 4.4 (1). Therefore, $\alpha \in U$ and $-\beta \notin U$.

(18): ($\Longrightarrow$): Suppose $-\Box\alpha \in U$, $U R_L W$ and $W \in M_L$. Then, we have $-\alpha \in U_\Box \subseteq W_\Box$, and hence $\Box-\alpha \in W$ or $-\Box\alpha \in W$. In the former case, by using Lemma 4.4 (1) and the fact $\vdash \Box-\alpha \Rightarrow -\alpha$, we obtain $-\alpha \in W$. In the latter case, by using Lemma 4.4 (1) and the fact $\vdash -\Box\alpha \Rightarrow -\alpha$, we obtain $-\alpha \in W$.

($\Longleftarrow$): We show the contraposition. Suppose $-\Box\alpha \notin U$. Then (*): $(U_{\Box}, \{-\alpha\})$ is consistent (this fact will be proved later). By using Lemma 4.2, we have that there exists a maximal consistent pair (W, V) such that $U_{\Box} \subseteq W$ and $\{-\alpha\} \subseteq V$. Then, we have $W \in M_L$ and $-\alpha \notin W$. Moreover we have (**): $U_{\Box} \subseteq W$ implies $U_{\Box} \subseteq W_{\Box}$ (this fact will be proved later). Therefore, we have the required fact that there exists $W \in M_L$ such that UR_LW and $-\alpha \notin W$. We show the remained fact (*). Suppose that $(U_{\Box}, \{-\alpha\})$ is not consistent. Then, there exist $\beta_1, ..., \beta_n, -\delta_1, ..., -\delta_o \in U_{\Box}$ (and $\Box\beta_1, ..., \Box\beta_n$, $-\Box\delta_1, ..., -\Box\delta_o \in U$) such that $\vdash \beta_1, ..., \beta_n, -\delta_1, ..., -\delta_o \Rightarrow -\alpha$. Applying ($\Box$left), ($-\Box$left), and ($-\Box$right) to this sequent, we obtain $\vdash \Box\beta_1, ..., \Box\beta_n, -\Box\delta_1, ..., -\Box\delta_o \Rightarrow -\Box\alpha$. By Lemma 4.4 (1), and $\Box\beta_1, ..., \Box\beta_n, -\delta_1, ..., -\Box\delta_o \in U$, we obtain $-\Box\alpha \in U$. This contradicts for the assumption $-\Box\alpha \notin U$. Therefore, $(U_{\Box}, \{-\alpha\})$ is consistent. We show the remained fact (**): $U_{\Box} \subseteq W$ implies $U_{\Box} \subseteq W_{\Box}$. Suppose $\gamma \in U_{\Box}$. Then, $\Box\gamma \in U$ or $-\Box\beta \in U$ with $\gamma \equiv -\beta$. By Lemma 4.4 (1) and the facts $\vdash \Box\gamma \Rightarrow \Box\Box\gamma$ and $\vdash -\Box\beta \Rightarrow \Box-\Box\beta$, we obtain $\Box\Box\gamma \in U$ or $\Box-\Box\beta \in U$. In the latter case, by using Lemma 4.4 (1) and the fact $\vdash \Box-\Box\beta \Rightarrow \Box\Box-\beta$, we also obtain $\Box\Box-\beta \in U$ (i.e., $\Box\Box\gamma \in U$). Thus, we have $\Box\gamma \in U_{\Box} \subseteq W$ by the assumption. Therefore, $\gamma \in W_{\Box}$. ∎

Proposition 4.6: Let $U \in M_L$. For any formula α,

1) $\sim\alpha \in U$ iff $-\alpha \notin U$,
2) $-\alpha \in U$ iff $\sim\alpha \notin U$.

Proof. By induction on α. ∎

Lemma 4.7: Let $\langle M_L, R_L, \models_L\rangle$ be the canonical paraconsistent Kripke model defined in Definition 4.3. For any formula γ and any $U \in M_L$,

$$U \models_L \gamma \text{ iff } \gamma \in U.$$

Proof. We prove this lemma by induction on γ. Since the base step is obvious from the definition of the canonical paraconsistent Kripke model, we show only the following cases in the induction step.

1) Case $\gamma \equiv -(\alpha\to\beta)$: ($\Longrightarrow$): Suppose $U \models_L -(\alpha\to\beta)$ (i.e., [$U \models_L \alpha$ implies $U \models_L -\beta$]). Then, we obtain [$\alpha \in U$ implies $-\beta \in U$] by induction hypothesis. By Lemma 4.5, we obtain $-(\alpha\to\beta) \in U$. ($\Longleftarrow$): Suppose $-(\alpha\to\beta) \in U$. Then, we have [$\alpha \in U$ implies $-\beta \in U$] by Lemma 4.5. By induction hypothesis, we obtain [$U \models_L \alpha$ implies $U \models_L -\beta$] Therefore, $U \models_L -(\alpha\to\beta)$.
2) Case $\gamma \equiv -\Box\beta$: ($\Longrightarrow$): Suppose $U \models_L -\Box\beta$ (i.e., $\forall W \in M_L$ [$U_{\Box} \subseteq W_{\Box}$ implies $W \models_L -\beta$]. Then, by induction hypothesis, we have $\forall W \in M_L$ [$U_{\Box} \subseteq W_{\Box}$ implies $-\beta \in W$]. Thus, we obtain $-\Box\beta \in U$ by Lemma 4.5. ($\Longleftarrow$): Obvious by using Lemma 4.5. ∎

Theorem 4.8 (Completeness for M4CC*):* For any sequent S, if M4CC $\models S$, then M4CC $\vdash S$.

Proof. Let S be $\Gamma \Rightarrow \Delta$. We prove the contraposition using the canonical paraconsistent Kripke model. Namely, we prove the following statement: If $\Gamma \Rightarrow \Delta$ is not provable in M4CC, then $\exists U \in M_L$ [$U \not\models_L \Gamma_*\to\Delta^*$]. Let $\Gamma \equiv \{\alpha_1, ..., \alpha_m\}$, $\Delta \equiv \{\beta_1, ..., \beta_n\}$ and $m, n \geq 0$. Suppose that $\Gamma \Rightarrow \Delta$ is not provable in M4CC. By using Lemma 4.2, we have that there exists a maximal consistent pair (U, V) such that $\{\alpha_1, ..., \alpha_m\} \subseteq U$ and $\{\beta_1, ..., \beta_n\} \subseteq V$. Then, we have $U \in M_L$. By Lemma 4.7, we obtain that $U \models_L \alpha_i$ ($i = 1, ..., m$) and $U \not\models_L \beta_j$ ($j = 1, ..., n$). Thus, we have $U \not\models_L (\alpha_1 \wedge \cdots \wedge \alpha_m)\to(\beta_1 \vee \cdots \vee \beta_n)$ (i.e., $\exists U \in M_L$ [$U \not\models_L \Gamma_*\to\Delta^*$]). ∎

V. Finite Model Property

In what follows, slightly modifying the proof of Theorem 4.8, we show the following finite model property for M4CC.

Theorem 5.1 (Finite model property for M4CC*):* For any sequent S, S is M4CC-valid in any finite Kripke frame iff M4CC $\vdash S$.

Proof. We give a sketch of the only if part of the proof of this theorem by constructing a canonical finite paraconsistent Kripke model.

Prior to give the sketch of the proof, we need to introduce the notion of quasi-subformula. A *quasi-subformula* of a formula α is defined by the following conditions:

1) a subformula of α is a quasi-subformula of α,
2) if α is of the form $\sharp(\beta \wedge \gamma)$, $\sharp(\beta \vee \gamma)$ or $\sharp(\beta\to\gamma)$ with $\sharp \in \{\sim, -\}$, then quasi-subformulas of $\sharp\beta$ and $\sharp\gamma$ are quasi-subformulas of α,
3) if α is of the form $\sharp\sharp\beta$, $\sharp\Box\beta$, or $\sharp\Diamond\beta$ with $\sharp \in \{\sim, -\}$, then quasi-subformulas of $\sharp\beta$ are quasi-subformulas of α,
4) if α is of the form $\sim-\beta$, then quasi-subformulas of $\sim\beta$ are quasi-subformulas of α,
5) if α is of the form $-\sim\beta$, then quasi-subformulas of $-\beta$ are quasi-subformulas of α.

We use an expression $\Phi(\delta)$ to denote the set of all quasi-subformulas of δ.

Let $S \equiv \Gamma \Rightarrow \Delta$ be an unprovable sequent in M4CC, and δ be $\Gamma_*\to\Delta^*$. For any $U, V \subseteq \Phi(\delta)$, a pair (U, V) is called $\Phi(\delta)$*-maximal consistent* if

1) (U, V) is consistent,
2) $U \cup V = \Phi(\delta)$.

Then, we can obtain the following statement which is a modified version of Lemma 4.2:

> For any $U_0, V_0 \subseteq \Phi(\delta)$, if a pair (U_0, V_0) is consistent, then there exist U and V such that $U_0 \subseteq U$, $V_0 \subseteq V$, and (U, V) is $\Phi(\delta)$-maximal consistent.

We define a finite canonical paraconsistent Kripke model $\langle M_F, R_F, \models_F\rangle$ by:

1) $M_F := \{U(\subseteq \Phi(\delta)) \mid (U, \Phi(\delta) - U)$ is $\Phi(\delta)$-maximal consistent$\}$,
2) $U_1R_FU_2$ for any $U_1, U_2 \in M_F$ is defined as $(U_1)_{\Box} \subseteq (U_2)_{\Box}$,

3) $U \models_F p$, $U \models_F \sim p$, and $U \models_F -p$ for any $U \in M_F$ and any propositional variable $p \in \Phi(\delta)$ are defined as $p \in U$, $\sim p \in U$, and $-p \in U$, respectively, with the condition: $\sim p \notin U$ or $-p \notin U$.

We remark that M_F is finite, because $\Phi(\delta)$ is finite.

We can show some similar results to Lemmas 4.4 and 4.5 with some appropriate modifications. The items (1) - (19) in Lemma 4.5 are modified by adding the condition: The left-hand side formula in the item is in $\Phi(\delta)$. For example, (8) in Lemma 4.5 must modify as (8′): for any $\sim(\alpha \rightarrow \beta) \in \Phi(\delta)$, $\sim(\alpha \rightarrow \beta) \in U$ iff $\alpha \in U$ and $\sim\beta \in U$. Then, we can show a modified version of Lemma 4.7 by using the finite canonical paraconsistent Kripke model just defined above, and using this modified lemma, we can prove this theorem. ∎

We can obtain the following as a corollary of Theorem 5.1.

Corollary 5.2 (Decidability for M4CC): M4CC is decidable.

VI. Conclusions and Related Works

In this study, we introduced a modal extension M4CC of Arieli, Avron, and Zamansky's ideal paraconsistent four-valued logic 4CC [4]–[6]. For M4CC, we proved the Kripke-completeness theorem as well as the finite model property. As a corollary, we obtained the decidability result for M4CC. In the remainder of this paper, we address some related works on some modal extensions of many-valued logics.

The idea of extending many-valued logics to modal many-valued logics is not new. Some traditional results in this respect are found, for example, in [12], [13]. Nevertheless, the modal extensions of many-valued logics have not yet been studied intensively. Some many-valued modal logics over finite residuated lattices were studied by Bou et al. in [10], with a special attention to some basic classes of Kripke frames and their axiomatizations. One may refer also to [18], [19], [21], where some modal extensions of Belnap and Dunn's useful four-valued logic have been studied, and certain properties of such logics from proof-theoretic, semantic, and algebraic viewpoints have also been analyzed. Some three- and four-valued modal logics, which are extensions of Belnap and Dunn's four-valued logic and its three-valued variant, were introduced by Odintsov and Wansing in [19], by providing them with the sound and complete tableau calculi, Kripke semantics, and modal algebras with twist structures. A family of four-valued modal logics, which are modal extensions of Belnap and Dunn's four-valued logic, was studied by Rivieccio et al. in [21], by considering the many-valued Kripke structures and their counterpart modal algebras in the sense of the topological duality theory. A Belnapian version BK of the least normal modal logic K with the addition of strong negation was introduced by Odintsov and Speranski in [18], and a systematic study of the lattices of logics containing BK was carried out by them. A modal multilattice logic was studied by Kamide and Shramko in [15], and a Kripke semantics for it, which are simple and compatible with the standard Kripke semantics for S4, was developed by them. Our Kripke semantics for M4CC is rather similar to such a Kripke semantics, but the proof is different. The proof of the completeness theorem presented in [15] was an embedding-based indirect proof, but our proof for the completeness theorem for M4CC was the direct proof that can also show the finite model property.

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