Some problems I have proposed

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1. **International Mathematical Olympiad 2017 Shortlist, Problem C3**

   Sir Alex plays the following game on a row of 9 cells. Initially, all cells are empty. In each move, Sir Alex is allowed to perform exactly one of the following two operations:

   (1) Choose any number of the form 2^j, where j is a non-negative integer, and put it into an empty cell.

   (2) Choose two (not necessarily adjacent) cells with the same number in them; denote that number by 2^j. Replace the number in one of the cells with 2^{j+1} and erase the number in the other cell.

   At the end of the game, one cell contains the number 2^n, where n is a given positive integer, while the other cells are empty. Determine the maximum number of moves that Sir Alex could have made, in terms of n.

2. **International Mathematical Olympiad 2017 Shortlist, Problem N3**

   Determine all integers $n \geq 2$ with the following property: for any integers $a_1, a_2, \ldots, a_n$ whose sum is not divisible by n, there exists an index $1 \leq i \leq n$ such that none of the numbers

   $$a_i, a_i + a_{i+1}, \ldots, a_i + a_{i+1} + \cdots + a_{i+n-1}$$

   is divisible by n. (We let $a_i = a_{i-n}$ when $i > n$.)

3. **Asian Pacific Mathematics Olympiad 2017, Problem 1**

   We call a 5-tuple of integers *arrangeable* if its elements can be labeled a, b, c, d, e in some order so that $a - b + c - d + e = 29$. Determine all 2017-tuples of integers $n_1, n_2, \ldots, n_{2017}$ such that if we place them in a circle in clockwise order, then any 5-tuple of numbers in consecutive positions on the circle is arrangeable.

4. **Asian Pacific Mathematics Olympiad 2017, Problem 5**
   (with Pakawut Jiradilok)

   Let n be a positive integer. A pair of n-tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ with integer entries is called an *exquisite pair* if

   $$|a_1 b_1 + \cdots + a_n b_n| \leq 1.$$

   Determine the maximum number of distinct n-tuples with integer entries such that any two of them form an exquisite pair.

5. **International Mathematical Olympiad 2016 Shortlist, Problem N1**

   For any positive integer k, denote the sum of digits of k in its decimal representation by $S(k)$. Find all polynomials $P(x)$ with integer coefficients such that for any positive integer $n \geq 2016$, the integer $P(n)$ is positive and

   $$S(P(n)) = P(S(n)).$$

6. **Asian Pacific Mathematics Olympiad 2016, Problem 3**

   Let AB and AC be two distinct rays not lying on the same line, and let ω be a circle with center O that is tangent to ray AC at E and ray AB at F. Let R be a point on segment EF. The line through O parallel to EF intersects the line AB at P. Let N be the intersection of lines PR and AC, and let M be the intersection of line AB and the line through R parallel to AC. Prove that line MN is tangent to ω.

7. **Asian Pacific Mathematics Olympiad 2016, Problem 4**

   The country Dreamland consists of 2016 cities. The airline Starways wants to establish some one-way flights between pairs of cities in such a way that each city has exactly one flight out of it. Find the smallest positive integer k such that no matter how Starways establishes its flights, the cities can always be partitioned into k groups so that from any city it is not possible to reach another city in the same group by using at most 28 flights.

8. **Asian Pacific Mathematics Olympiad 2015, Problem 1**

   Let ABC be a triangle, and let D be a point on side BC. A line through D intersects side AB at X and ray AC at Y. The circumcircle of triangle BXD intersects the circumcircle ω of triangle ABC again at point $Z \neq B$. The lines ZD and ZY intersect ω again at V and W, respectively. Prove that $AB = VW$.

9. **Asian Pacific Mathematics Olympiad 2015, Problem 4**
   (with Pakawut Jiradilok)

   Let n be a positive integer. Consider $2n$ distinct lines on the plane, no two of which are parallel. Of the $2n$ lines, n are colored blue, the other n are colored red. Let $\mathcal{B}$ be the set of all points on the plane that lie on at least one blue line, and $\mathcal{R}$ the set of all points on the plane that lie on at least one red line. Prove that there exists a circle that intersects $\mathcal{B}$ in exactly $2n - 1$ points, and also intersects $\mathcal{R}$ in exactly $2n - 1$ points.

10. **Asian Pacific Mathematics Olympiad 2015, Problem 5**
    (with Pakawut Jiradilok)

    Determine all sequences $a_0, a_1, a_2, \ldots$ of positive integers with $a_0 \geq 2015$ such that for all integers $n \geq 1$:

    (i) a_{n+2} is divisible by a_n;

    (ii) $|s_{n+1} - (n+1)a_n| = 1$, where $s_{n+1} = a_{n+1} - a_n + a_{n-1} - \cdots + (-1)^{n+1}a_0$.

11. **Asian Pacific Mathematics Olympiad 2014, Problem 2**

    Let $S = \{1, 2, \ldots, 2014\}$. For each non-empty subset $T \subseteq S$, one of its members is chosen as its *representative*. Find the number of ways to assign representatives to all non-empty subsets of S so that if a subset $D \subseteq S$ is a disjoint union of non-empty subsets $A, B, C \subseteq S$, then the representative of D is also the representative of at least one of A, B, C.

12. **Asian Pacific Mathematics Olympiad 2014, Problem 3**

    Find all positive integers n such that for any integer k there exists an integer a for which $a^3 + a - k$ is divisible by n.

13. **International Mathematical Olympiad 2013, Problem 4**
    (with Potcharapol Suteparuk)

    Let ABC be an acute-angled triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote by ω_2 the circumcircle of CWM, and let Y be the point on ω_2 such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

14. **United States of America Mathematical Olympiad 2013, Problem 3**

    Let n be a positive integer. There are $\frac{n(n+1)}{2}$ marks, each with a black side and a white side, arranged into an equilateral triangle, with the biggest row containing n marks. Initially, each mark has the black side up. An *operation* is to choose a line parallel to one of the sides of the triangle, and flipping all the marks on that line. A configuration is called *admissible* if it can be obtained from the initial configuration by performing a finite number of operations. For each admissible configuration C, let $f(C)$ denote the smallest number of operations required to obtain C from the initial configuration. Find the maximum value of $f(C)$, where C varies over all admissible configurations.

15. **International Mathematical Olympiad 2012 Shortlist, Problem C1**

    Several positive integers are written in a row. Iteratively, Alice chooses two adjacent numbers x and y such that $x > y$ and x is to the left of y, and replaces the pair (x, y) by either $(y + 1, x)$ or $(x - 1, x)$. Prove that she can perform only finitely many such iterations.

16. **International Mathematical Olympiad 2012 Shortlist, Problem N1**

    Call admissible a set A of integers that has the following property:

    If $x, y \in A$ (possibly $x = y$) then $x^2 + kxy + y^2 \in A$ for every integer k.

    Determine all pairs m, n of nonzero integers such that the only admissible set containing both m and n is the set of all integers.

17. **United States of America Junior Mathematical Olympiad 2012, Problem 5**

    For distinct positive integers $a, b < 2012$, define $f(a, b)$ to be the number of integers k with $1 \leq k < 2012$ such that the remainder when ak divided by 2012 is greater than that of bk divided by 2012. Let S be the minimum value of $f(a, b)$, where a and b range over all pairs of distinct positive integers less than 2012. Determine S.

18. **International Mathematical Olympiad 2011 Shortlist, Problem A2**

    Determine all sequences $(x_1, x_2, \ldots, x_{2011})$ of positive integers such that for every positive integer n there is an integer a with

$$x_1^n + 2x_2^n + \cdots + 2011x_{2011}^n = a^{n+1} + 1.$$