Fair Division of Indivisible Items: Asymptotics and Graph-Theoretic Approaches

Ayumi Igarashi\textsuperscript{1} and Warut Suksompong\textsuperscript{2}
1 University of Tokyo, Japan
2 University of Oxford, UK
IJCAI 2019 Tutorial (Part 2) August 10th, 2019

Some slides credit from Dominik Peters
Fair division of a graph

- Office allocation: Allocate a connected set of rooms to each research group.

How can we divide?
Fair division of a graph

- Land division: Allocate a **connected** set of regions to each country.

How can we divide?
Fair division of a graph

- Scheduling: Allocate a connected set of time slots to each agent.

Discrete version of cake [0,1]

| 12:00 - 13:00 | 13:00 - 14:00 | 14:00 - 15:00 | 15:00 - 16:00 | 16:00 - 17:00 | 17:00 - 18:00 |

How can we divide?
Model [Bouveret et al. 2017]

- An undirected graph $G=(V,E)$
- A set of agents $N = \{1,2,...,n\}$
- A non-negative additive utility function $u_i: V \rightarrow \mathbb{R}_+$

$$u_i(X) = \sum_{v \in X} u_i(v)$$
Model

- A connected allocation is a mapping assigning each player to a disjoint connected subset of the vertices.
Classical fairness notions

A connected allocation is envy-free if no one envies others:
$ u_i(\text{i's bundle}) \geq u_j(\text{j's bundle}) $ for all $i,j$ in $N$

$ u_A(\text{A's bundle}) \geq u_A(\text{B's bundle}) $  

$ u_B(\text{B's bundle}) \geq u_B(\text{A's bundle}) $
Classical fairness notions

- A connected allocation is proportional if each player receives value $\geq$ his proportional share: $u_i(i\text{'s bundle}) \geq u_i(V)/n$ for all $i$ in $N$

$u_A(A\text{'s bundle}) \geq u_A(V)/2$

$u_B(B\text{'s bundle}) \geq u_B(V)/2$
Existence of EF and Prop

- Proportional/envy-free contiguous allocation of a cake \([0,1]\) exists with divisibilities.

<table>
<thead>
<tr>
<th></th>
<th>Existence</th>
<th>Complexity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Envy-freeness</td>
<td>✔</td>
<td>no finite protocol</td>
</tr>
<tr>
<td></td>
<td>[Stromquist, 1980]</td>
<td>[Stromquist, 2008]</td>
</tr>
<tr>
<td>Proportionality</td>
<td>✔</td>
<td>polytime</td>
</tr>
<tr>
<td></td>
<td>[Dubins and Spanier, 1961]</td>
<td>[Dubins and Spanier, 1961]</td>
</tr>
</tbody>
</table>
Approximate fairness

- Proportional/envy-free allocation may not exist with indivisibilities $\rightarrow$ Relaxations?

Consider an instance of two players and one item.
Approximate fairness

• Proportional/envy-free allocation may not exist with *indivisibilities* → Relaxations?

• Budish (2011) proposed the following two concepts:
  - Maximin share (MMS)
  - Envy-freeness up to one good (EF1)
Approximate fairness

- Proportional/envy-free allocation may not exist with indivisibilities → Relaxations?

- Budish (2011) proposed the following two concepts:
  - Maximin share (MMS)
  - Envy-freeness up to one good

![Diagram showing the relationships between fairness concepts: MMS, Prop, EF1, and the relations between fairness concepts.]

Relations between fairness concepts
Maximin share

Maximin share [Budish, 2011]: the best utility each agent would receive if she had to cut and choose the last.
Maximin share

- Maximin share [Budish, 2011]: the best utility each agent would receive if she had to cut and choose the last.
Maximin share

Maximin share (MMS) : $u_i$ (i’s bundle) $\geq$ MMS$_i$ for all i in N

MMS$_i$ = $\max \{ \min_j u_i(P_j) \mid P_1, \ldots, P_n: a connected partition of G \}$
Unrestricted setting: MMS

- Identified special condition on the existence of MMS. Extensive experiments did not find any counter example [Bouveret and Lemaître, 2014].

- Intricate counter example with a number of goods exponential in the number of players [Procaccia and Wang, 2014]

- Reduced the number of goods to linear in the number of players [Kurokawa et al., 2016].
Moving-knife algorithm
[Bouveret et al. 2017]

Moving-knife procedures that achieve proportionality in cake-cutting produce MMS, when the graph is a path.
Moving-knife algorithm
[Bouveret et al. 2017]

Moving-knife procedures that achieve proportionality in cake-cutting produce MMS, when the graph is a path.
Moving-knife algorithm
[Bouveret et al. 2017]

Moving-knife procedures that achieve proportionality in cake-cutting produce MMS, when the graph is a path.
Moving-knife algorithm
[Bouveret et al. 2017]

Moving-knife procedures that achieve proportionality in cake-cutting produce MMS, when the graph is a path.
Moving-knife algorithm
[Bouveret et al. 2017]

Moving-knife procedures that achieve proportionality in cake-cutting produce MMS, when the graph is a path.

Maximin share for the reduced instance does not decrease.
MMS existence

- Theorem [Bouveret et al. 2017] MMS exists on trees and can be computed in polynomial time.

Cut a minimal subtree guaranteeing MMS for some player.

→ Recurse on the remaining instance.

Discrete version of last diminisher
Theorem [Bouveret et al. 2017] MMS may not exist on a single cycle of 8 vertices with 4 players.

Create two types of players whose MMS partitions intersect with each other.
MMS: other work

- Lonc and Truszczynski [2018]:
  1/2-approximation for MMS in the case of cycles.

- Igarashi and Peters [2019]:
  A connected allocation satisfying MMS and Pareto-optimality exists when the graph is a tree.

- NP-hard to compute even with binary additive valuations and even on a path.
  → polytime solvable for non-nested valuations.
MMS: open questions

- Complete characterisation of graphs guaranteeing MMS.

- The complexity of deciding the existence of a connected MMS.
  - Checking whether a given allocation is MMS is polytime solvable for a cycle.

- Existence of a connected MMS allocation of goods and bads.
  - Related works [Aziz et al., 2019; Bouveret et al. 2019]
Envy-freeness up to one good

- Envy-freeness need not exist $\rightarrow$ Relaxations?
- Budish [2011]: Envy-freeness up to one good
  - For each $i, j$ in $N$ there is a good $o^*$ in $j$'s bundle with
    \[
    u_i(\text{i's bundle}) \geq u_i(\text{j's bundle} \setminus \{o^*\})
    \]
Envy-freeness up to one good

- Envy-freeness need not exist $\rightarrow$ Relaxations?
- Budish [2011]: Envy-freeness up to one good
  - For each $i,j$ in $N$ there is a good $o^*$ in $j$'s bundle with

\[ u_i(\text{ i's bundle } ) \geq u_i(\text{ j's bundle } \setminus \{o^*\} ) \]

Not EF1
Envy-freeness up to one good

Without connectivity constraints, EF1 always exists

- Envy-graph algorithm [Lipton et al., 2004]
- Round-robin procedure [Caragiannis et al., 2016]
- Maximum Nash welfare [Caragiannis et al., 2016]

Theorem [Bilò et al. 2019; (a) and (d) appear also in Oh et al. 2019].
EF1 exists on a path

(a) when there are 2 agents (cut-and-choose); or
(b) when there are 3 agents (Stromquist's procedure); or
(c) when there are 4 agents (Sperner's lemma); or
(d) when valuations are identical (≈ leximin)
EF1 for two agents

- Discrete version of cut and choose protocol

Divisible cake:

1. Alice divides the cake into two equally-valued pieces
2. Bob chooses preferred piece and receives it
3. Alice receives other piece
EF1 for two agents

- Discrete version of cut and choose protocol

\( v \) is an agent \( i \)'s lumpy tie if

\[ v \succcurlyeq_i v \]

and

\[ i \succcurlyeq v \]
EF1 for two agents

- Discrete version of cut and choose protocol
  - Alice selects her lumpy tie $v$ and hides it
  - Bob selects either the left or right piece
  - Alice receives $v$ and the remaining piece

- This is EF1. This works for all graphs with Hamiltonian path — any others?
Theorem [Bilò et al. 2019]

For every connected graph $G$, the followings are equivalent:

1. $G$ admits a bipolar numbering.
2. $G$ guarantees EF1 for two agents.

Fig. 1. Non-traceable graphs with bipolar numberings.
EF1 for more agents.

Theorem [Bilò et al. 2019; (a) and (d) appear also in Oh et al. 2019]. EF1 exists on a path

(a) when there are 2 agents (cut-and-choose); or
(b) when there are 3 agents (Stromquist's procedure); or
(c) when there are 4 agents (Sperner's lemma); or
(d) when valuations are identical (≈ leximin)

By Sperner's lemma, EF2 always exists [Bilò et al. 2019]
Existence extends to graphs with Hamiltonian path
Existence does not require additive valuations
Sperner's Lemma

A combinatorial analog of the Brouwer/Kakutani fixed point theorem
Sperner's Lemma

1. Color the corners with distinct colors.
2. Color every vertex of edge with the two colors of the endpoints.

→ a colorful triangle.
Sperner's Lemma and EF1

Let us first give a high-level illustration. An allocation is EF2 if for any agent, either the allocation is EF1, or there are two goods that the agent has two of. For two or three agents, we have seen algorithms that are guaranteed to obtain an envy-free division of the cake, and round it to get an allocation of the items. Suksompong followed this approach and showed that the result is an allocation where any agent envies the bundle of agent most-prefers. For example, the top vertex will be labelled as "in-"

...
EF1: open question

- Characterisation of graphs that guarantee EF1 allocation beyond 2 agents.
- The complexity of finding an EF1 allocation with binary additive valuations.
- The complexity of finding an EF2 allocation.
Fair division over a social network

- Envy-freeness requires that no agent envies any other agent. In many situations, agents often do not even know each other.

- Local envy-freeness [Abebe et al. 2017, Bei et al. 2017]: No agent envies his/her neighbor in a social network.

Graphs represent envy-relations.
Fair division over a social network

- Envy-freeness requires that no agent envies any other agent. In many situations, agents often do not even know each other.

- Local envy-freeness [Abebe et al. 2017, Bei et al. 2017]: No agent envies his/her neighbor in a social network.

- Divisible items [Abebe et al. 2017, Bei et al. 2017]

- Indivisible items [Bredereck et al., 2018]
Local fairness

- Combination of two models?
  Given a connected allocation, one can induce a social network.

- Local fairness?
  EF1, MMS, etc.
References

• Aziz, Caragiannis, Igarashi, and Walsh. Fair allocation of indivisible goods and chores. IJCAI 2019.
• Bei, Qiao, and Zhang. Networked fairness in cake cutting. IJCAI 2017.
• Bilò, Caragiannis, Flammini, Igarashi, Monaco, Peters, Vinci, and Zwicker. Almost envy-free allocation with connected bundles. ITCS 2019.
• Bouveret, Cechlárová, Elkind, Igarashi, Peters. Fair division of a graph. IJCAI 2017.
• Bouveret and Lemaître. Characterizing conflicts in fair division of indivisible goods using a scale of criteria. AAMAS 2014.
• Oh, Procaccia, and Suksompong. Fairly allocating many goods with few queries. AAAI 2019.
• Lonc and Truszczynski. Maximin share allocations on cycles. IJCAI 2018.