The Problems

1. Let \(ABCD\) be a unit square, and let \(AB_1C_1D_1\) be its image after a 30 degree rotation about point \(A\). The area of the region consisting of all points inside at least one of \(ABCD\) and \(AB_1C_1D_1\) can be expressed in the form \(\frac{a+\sqrt{b}}{c}\), where \(a, b, c\) are positive integers, and \(b\) shares no perfect square common factor with \(c\). Find \(a + b + c\).

2. Eleven nonparallel lines lie on a plane, and their pairwise intersections meet at angles of integer degree. How many possible values are there for the smallest of these angles?

3. In triangle \(ABC\), \(BC = 9\). Points \(P\) and \(Q\) are located on \(BC\) such that \(BP = PQ = 2\), \(QC = 5\). The circumcircle of \(APQ\) cuts \(AB, AC\) at \(D, E\) respectively. If \(BD = CE\), then the ratio \(\frac{AD}{AC}\) can be expressed in the form \(\frac{m}{n}\), where \(m, n\) are relatively prime positive integers. Find \(m + n\).

4. In triangle \(ABC\), \(AB = 6, BC = 9, \angle ABC = 120^\circ\). Let \(P\) and \(Q\) be points on \(AC\) such that \(BPQ\) is equilateral. The perimeter of \(BPQ\) can be expressed in the form \(\frac{m}{\sqrt{n}}\), where \(m, n\) are relatively prime positive integers. Find \(m + n\).

5. In triangle \(ABC\), \(AB = 36, BC = 40, CA = 44\). The bisector of angle \(A\) meet \(BC\) at \(D\) and the circumcircle at \(E\) different from \(A\). Calculate the value of \(DE^2\)

6. Three points \(A, B, C\) are chosen at random on a circle. The probability that there exists a point \(P\) inside an equilateral triangle \(A_1B_1C_1\) such that \(PA_1 = BC, PB_1 = AC, PC_1 = AB\) can be expressed in the form \(\frac{m}{n}\), where \(m, n\) are relatively prime positive integers. Find \(m + n\).

7. In trapezoid \(ABCD\), \(AB\parallel CD\), and \(AB \perp BC\). There is a point \(P\) on side \(AD\) such that the circumcircle of triangle \(BPC\) is tangent to \(AD\). If \(AB = 3, AD = 78, CD = 75, CP - BP\) can be expressed in the form \(\frac{m}{\sqrt{n}}\), where \(a, b, c\) are positive integers and \(a, c\) are relatively prime. Find \(a + b + c\).

8. Two circles \(\omega_1, \omega_2\) have center \(O_1, O_2\) and radius 25, 39 respectively. The smallest distance between a point on \(\omega_1\) with a point on \(\omega_2\) is 1. Tangents from \(O_2\) to \(\omega_1\) meet \(\omega_1\) at \(S_1, T_1\), and tangents from \(O_1\) to \(\omega_2\) meet \(\omega_2\) at \(S_2, T_2\), such that \(S_1, S_2\) are on the same side of line \(O_1O_2\). \(O_1S_1\) meets \(O_2S_2\) at \(P\) and \(O_1T_1\) meets \(O_2T_2\) at \(Q\). The length of \(PQ\) can be expressed in the form \(\frac{m}{n}\), where \(m, n\) are relatively prime positive integers. Find \(m + n\).

9. \(P - ABCD\) is a right pyramid with square base \(ABCD\) edge length 6, and \(PA = PB = PC = PD = 6\sqrt{2}\). The probability that a randomly selected point inside the pyramid is at least \(\frac{\sqrt{5}}{2}\) units away from each face can be expressed in the form \(\frac{m}{n}\), where \(m, n\) are relatively prime positive integers. Find \(m + n\).

10. Circle \(\omega_1\) is defined by the equation \((x - 7)^2 + (y - 1)^2 = k\), where \(k\) is a positive real number. Circle \(\omega_2\) passes through the center of \(\omega_1\) and its center lies on the line \(7x + y = 28\). Suppose that one of the tangent lines from the origin to circles \(\omega_1, \omega_2\) meet \(\omega_1, \omega_2\) at \(A_1, A_2\) respectively, that \(OA_1 = OA_2\), where \(O\) is the origin, and that the radius of \(\omega_2\) is \(\frac{2011}{201}\). What is \(k\)?

11. \(C\) is on a semicircle with diameter \(AB\) and center \(O\). Circle radius \(r_1\) is tangent to \(OA, OC\), and arc \(AC\), and circle radius \(r_2\) is tangent to \(OB, OC\), and arc \(BC\). It is known that \(\tan AOC = \frac{24}{7}\). The ratio \(\frac{r_2}{r_1}\) can be expressed in the form \(\frac{m}{n}\), where \(m, n\) are relatively prime positive integers. Find \(m + n\).

12. A triangle has the property that its sides form an arithmetic progression, and that the angle opposite the longest side is three times the angle opposite the shortest side. The ratio of the longest side to the shortest side can be expressed as \(\frac{a + \sqrt{b}}{c}\), where \(a, b, c\) are positive integers, \(b\) is not divisible by the square of any prime, and \(a\) and \(c\) are relatively prime. Find \(a + b + c\).

13. In acute triangle \(ABC\), \(\ell\) is the bisector of \(\angle BAC\). \(M\) is the midpoint of \(BC\), a line through \(M\) parallel to \(\ell\) meets \(AC, AB\) at \(E, F\) respectively. Given that \(AE = 1, EF = \sqrt{3}, AB = 21\), the sum of all possible values of \(BC\) can be expressed as \(\sqrt{a} + \sqrt{b}\), where \(a, b\) are positive integers. What is \(a + b\)?
14. The point $(10, 26)$ is a focus of a non-degenerate ellipse tangent to the positive $x$ and $y$ axes. The locus of the center of the ellipse lies along the graph of $ax - by + c = 0$, where $a, b, c$ are positive integers with no common factor other than 1. Find $a + b + c$.

15. Two circles $\omega_1, \omega_2$ radius 28, 112 respectively intersect at $P, Q$. $A$ is on $\omega_1$ and $B$ on $\omega_2$ such that $A, P, B$ are collinear. Tangents to $\omega_1, \omega_2$ at $A, B$ respectively meet at $T$. Suppose $\angle AQT = \angle BQT = 60^\circ$. The length of $TQ$ can be expressed in the form $a\sqrt{b}$ where $a, b$ are positive integers and $b$ is not divisible by the square of any prime. Find $a + b$. 