The Problems

1. Let $ABCD$ be a unit square, and let $AB_1C_1D_1$ be its image after a 30 degree rotation about point A. The area of the region consisting of all points inside at least one of $ABCD$ and $AB_1C_1D_1$ can be expressed in the form $\frac{a-\sqrt{b}}{c}$, where a, b, c are positive integers, and b shares no perfect square common factor with c. Find $a+b+c$.

2. Eleven nonparallel lines lie on a plane, and their pairwise intersections meet at angles of integer degree. How many possible values are there for the smallest of these angles?

3. In triangle ABC, $BC = 9$. Points P and Q are located on BC such that $BP = PQ = 2$, $QC = 5$. The circumcircle of APQ cuts AB, AC at D, E respectively. If $BD = CE$, then the ratio $\frac{AB}{AC}$ can be expressed in the form $\frac{m}{n}$, where m, n are relatively prime positive integers. Find $m+n$.

4. In triangle ABC, $AB = 6, BC = 9, \angle ABC = 120°$ Let P and Q be points on AC such that BPQ is equilateral. The perimeter of BPQ can be expressed in the form $\frac{m}{\sqrt{n}}$, where m, n are relatively prime positive integers. Find $m+n$.

5. In triangle ABC, $AB = 36, BC = 40, CA = 44$. The bisector of angle A meet BC at D and the circumcircle at E different from A. Calculate the value of DE^2

6. Three points A, B, C are chosen at random on a circle. The probability that there exists a point P inside an equilateral triangle $A_1B_1C_1$ such that $PA_1 = BC, PB_1 = AC, PC_1 = AB$ can be expressed in the form $\frac{m}{n}$, where m, n are relatively prime positive integers. Find $m+n$.

7. In trapezoid $ABCD$, $AB||CD$, and $AB \perp BC$. There is a point P on side AD such that the circumcircle of triangle BPC is tangent to AD. If $AB = 3, AD = 78, CD = 75$, $CP - BP$ can be expressed in the form $\frac{a\sqrt{b}}{c}$, where a, b, c are positive integers and a, c are relatively prime. Find $a+b+c$.

8. Two circles ω_1, ω_2 have center O_1, O_2 and radius $25, 39$ respectively. The smallest distance between a point on ω_1 with a point on ω_2 is 1. Tangents from O_2 to ω_1 meet ω_1 at S_1, T_1, and tangents from O_1 to ω_2 meet ω_2 at S_2, T_2, such that S_1, S_2 are on the same side of line O_1O_2. O_1S_1 meets O_2S_2 at P and O_1T_1 meets O_2T_2 at Q. The length of PQ can be expressed in the form $\frac{m}{n}$, where m, n are relatively prime positive integers. Find $m+n$.

9. $P - ABCD$ is a right pyramid with square base $ABCD$ edge length 6, and $PA = PB = PC = PD = 6\sqrt{2}$. The probability that a randomly selected point inside the pyramid is at least $\frac{\sqrt{6}}{3}$ units away from each face can be expressed in the form $\frac{m}{n}$ where m, n are relatively prime positive integers. Find $m+n$.

10. Circle ω_1 is defined by the equation $(x-7)^2 + (y-1)^2 = k$, where k is a positive real number. Circle ω_2 passes through the center of ω_1 and its center lies on the line $7x + y = 28$. Suppose that one of the tangent lines from the origin to circles ω_1 and ω_2 meets ω_1 and ω_2 at A_1, A_2 respectively, that $OA_1 = OA_2$, where O is the origin, and that the radius of ω_2 is $\frac{2011}{211}$. What is k?

11. C is on a semicircle with diameter AB and center O. Circle radius r_1 is tangent to OA, OC, and arc AC, and circle radius r_2 is tangent to OB, OC, and arc BC. It is known that $\tan AOC = \frac{24}{7}$. The ratio $\frac{r_2}{r_1}$ can be expressed $\frac{m}{n}$, where m, n are relatively prime positive integers. Find $m+n$.

12. A triangle has the property that its sides form an arithmetic progression, and that the angle opposite the longest side is three times the angle opposite the shortest side. The ratio of the longest side to the shortest side can be expressed as $\frac{a+\sqrt{b}}{c}$, where a, b, c are positive integers, b is not divisible by the square of any prime, and a and c are relatively prime. Find $a+b+c$.

13. In acute triangle ABC, ℓ is the bisector of $\angle BAC$. M is the midpoint of BC. a line through M parallel to ℓ meets AC, AB at E, F, respectively. Given that $AE = 1, EF = \sqrt{3}, AB = 21$, the sum of all possible values of BC can be expressed as $\sqrt{a} + \sqrt{b}$, where a, b are positive integers. What is $a+b$?

14. The point $(10, 26)$ is a focus of a non-degenerate ellipse tangent to the positive x and y axes. the locus of the center of the ellipse lies along graph of, $ax - by + c = 0$, where a, b, c are positive integers with no common factor other than 1. Find $a + b + c$.

15. Two circles ω_1, ω_2 radius $28, 112$ respectively intersect at P, Q. A is on ω_1 and B on ω_2 such that A, P, B are collinear. Tangents to ω_1, ω_2 at A, B respectively meet at T. Suppose $\angle AQT = \angle BQT = 60^\circ$. The length of TQ can be expressed in the form $a\sqrt{b}$ where a, b are positive integers and b is not divisible by the square of any prime. Find $a + b$.