Power of a Point

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1 Introduction

Here are some basic facts about power of a point.

1. (Definition of Power) Let P be a point in the plane and ω be a circle with center O and radius R. Then $\text{Pow}_\omega(P) = PO^2 - R^2$ is the *power of P with respect to ω*.

2. (Power of a Point Theorem) Let P be a point in the plane and ω be a circle. A line through P meets ω at A and B, and another one meets it at C and D. Then $PA \cdot PB = PC \cdot PD = \text{Pow}_\omega(P)$ (assuming directed lengths).

3. (Special Case) When one of the lines is tangent to the circle, we have $C = D$ and $PA \cdot PB = PC^2$.

4. (Converse of Power of a Point) Let A, B, C, D be points in a plane, and let AB meet CD at P. Suppose that P is either on both segments AB and CD, or is on neither of them. If $PA \cdot PB = PC \cdot PD$, then A, B, C, D lie on a circle.

5. (Power of a Point on coordinates) Let $(x-a)^2 + (y-b)^2 - c^2 = 0$ be the equation of a circle in the coordinate plane. For a point $P = (x, y)$, we have $\text{Pow}_\omega(P) = (x-a)^2 + (y-b)^2 - c^2$.

6. (Fact) When P is outside ω we have $\text{Pow}_\omega(P) > 0$, when P is on ω we have $\text{Pow}_\omega(P) = 0$, and when P is inside ω we have $\text{Pow}_\omega(P) < 0$

7. (Radical axis) Given two circles ω_1 and ω_2, the locus of points that have equal power to both circles is a line called the *radical axis*.

8. (Radical axis obvious fact 1) If ω_1 and ω_2 intersect, the radical axis is through their intersection points.

9. (Radical axis obvious fact 2) The radical axis of two circles is perpendicular to the line connecting their centers.

10. (Radical center) Let $\omega_1, \omega_2, \omega_3$ be three circles no two of which are concentric. Then the radical axes formed by taking pairs of circles are either concurrent or all parallel.

11. (Radical axes involving circle(s) of radius 0) They exist.

2 Problems

1. Verify facts 2-10.

2. (Classic) Let ω_1, ω_2 be two circles intersecting at points S, T and let points A, B be on ω_1, ω_2 such that AB is tangent to both circles.

   (a) Let line ST intersect segment AB at M. Show that M is the midpoint of AB.

   (b) Show that $AS/AT = BS/BT$.

3. Prove the circumcenter exists using power of a point.

4. Let C be a point on a semicircle of diameter AB and let D be the midpoint of arc AC. Let E be the projection of D onto the line BC and F the intersection of line AE with the semicircle. Prove that BF bisects the line segment DE.

5. Let A, B, C be three points on circle Γ with $AB = BC$. Let the tangents at A and B to Γ meet at D. Let DC meet Γ again at E. Prove that line AE bisects segment BD.

6. (Mock USAJMO 1, 2011) Given two fixed, distinct points B and C on plane $\mathcal{P}$, find the locus of all points A belonging to $\mathcal{P}$ such that the quadrilateral formed by point A, the midpoint of AB, the centroid of $\triangle ABC$, and the midpoint of AC (in that order) can be inscribed in a circle.

7. (USAJMO 2012) Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

8. (IMO 1995) Let A, B, C, D be four distinct points on a line, in that order. The circles with diameters $\overline{AC}$ and $\overline{BD}$ intersect at X and Y. The line XY meets $\overline{BC}$ at Z. Let P be a point on the line XY other than Z. The line CP intersects the circle with diameter AC at C and M, and the line BP intersects the circle with diameter BD at B and N. Prove that the lines AM, DN, XY are concurrent.

9. (Iran 2011) Let ABC be a triangle with altitudes AD, BE, and CF. Let M be the midpoint of BC, and let X and Y be the midpoints of ME and MF, respectively. Let Z be the point on line XY such that $ZA \parallel BC$. Show that $ZA = ZM$.

10. (USAMO 2009) Given circles ω_1 and ω_2 intersecting at points X and Y, let ℓ_1 be a line through the center of ω_1 intersecting ω_2 at points P and Q and let ℓ_2 be a line through the center of ω_2 intersecting ω_1 at points R and S. Prove that if P, Q, R and S lie on a circle then the center of this circle lies on line XY.

11. (IMO 2013) Let ABC be an acute triangle with orthocenter H, and let W be a point on the side BC, lying strictly between B and C. The points M and N are the feet of the altitudes from B and C, respectively. Denote by ω_1 the circumcircle of BWN, and let X be the point on ω_1 such that WX is a diameter of ω_1. Analogously, denote

by ω_2 the circumcircle of triangle CWM, and let Y be the point such that WY is a diameter of ω_2. Prove that X, Y and H are collinear.

12. (IMO 2009) Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides $\overline{CA}$ and $\overline{AB}$, respectively. Let K, L, and M be the midpoints of the segments BP, CQ, and PQ, respectively, and let Γ be the circle passing through K, L and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

3 Extra Practice

13. (Alex Song) Let ABC be an acute triangle with centroid G and $\angle A > 60°$. Let the second intersections of the circumcircles of AGB and AGC with side BC be L and K. Show that L and K lie on segment BC.

14. (Euler's Relation) In a triangle with circumcenter O, incenter I, circumradius R, and inradius r, prove that $OI^2 = R(R - 2r)$

15. (Own) Let A be a point outside circle ω with center O. Line ℓ passes through A and meets ω at X, Y. Let ℓ_1 be another line through A, not necessarily intersecting ω, and let M be on ℓ_1 such that $OM \perp \ell_1$. Let $D \neq M$ be the other intersection of ℓ_1 and the circumcircle of XYM. Find the locus of D as ℓ_1 varies.

16. (USAJMO 2012) Given a triangle ABC, let P and Q be points on segments $\overline{AB}$ and $\overline{AC}$, respectively, such that $AP = AQ$. Let S and R be distinct points on segment $\overline{BC}$ such that S lies between B and R, $\angle BPS = \angle PRS$, and $\angle CQR = \angle QSR$. Prove that P, Q, R, S are concyclic.

17. (ELMO 2012) In acute triangle ABC, let D, E, F denote the feet of the altitudes from A, B, C, respectively, and let ω be the circumcircle of $\triangle AEF$. Let ω_1 and ω_2 be the circles through D tangent to ω at E and F, respectively. Show that ω_1 and ω_2 meet at a point P on BC other than D.

18. (USAMO 1990) An acute-angled triangle ABC is given in the plane. The circle with diameter $\overline{AB}$ intersects altitude $\overline{CC'}$ and its extension at points M and N, and the circle with diameter $\overline{AC}$ intersects altitude $\overline{BB'}$ and its extensions at P and Q. Prove that the points M, N, P, Q lie on a common circle.

19. (TSTST 2011) Acute triangle ABC is inscribed in circle ω. Let H and O denote its orthocenter and circumcenter, respectively. Let M and N be the midpoints of sides AB and AC, respectively. Rays MH and NH meet ω at P and Q, respectively. Lines MN and PQ meet at R. Prove that $OA \perp RA$.

20. (Russia 2011) The perimeter of $\triangle ABC$ is 4. On rays AB, AC, points X, Y are chosen, respectively such that $AX = AY = 1$. Segments BC and XY intersect at point M. Prove that the perimeter of one of the triangles ABM and ACM is 2.

21. (USAMO 1997) Let ABC be a triangle. Take points D, E, F on the perpendicular bisectors of $\overline{BC}$, $\overline{CA}$, $\overline{AB}$ respectively. Show that the lines through A, B, C perpendicular to $\overline{EF}$, $\overline{FD}$, $\overline{DE}$ respectively are concurrent.

22. Let ABC be an acute-angled triangle, and let P and Q be two points on its side BC. Construct a point C_1 in such a way that the convex quadrilateral $APBC_1$ is cyclic, $QC_1 \parallel CA$, and the points C_1 and Q lie on opposite sides of the line AB. Construct a point B_1 in such a way that the convex quadrilateral $AQCB_1$ is cyclic, $PB_1 \parallel BA$, and the points B_1 and P lie on opposite sides of the line AC. Prove that the points B_1, C_1, P, and Q lie on a circle.

23. (MOP 1998) Let ω_1 and ω_2 be two circles of the same radius, intersecting at A and B. Let O be the midpoint of AB. Let CD be a chord of ω_1 passing through O, and let the segment CD meet ω_2 at P. Let EF be a chord of ω_2 passing through O, and let the segment EF meet ω_1 at Q. Prove that AB, CQ, EP are concurrent.

24. (TSTST 2017) Let ABC be a triangle with circumcircle Γ, circumcenter O, and orthocenter H. Assume that $AB \neq AC$. Let M and N be the midpoints of sides AB and AC, respectively, and let E and F be the feet of the altitudes from B and C in $\triangle ABC$, respectively. Let P be the intersection point of line MN with the tangent line to Γ at A. Let Q be the intersection point, other than A, of Γ with the circumcircle of $\triangle AEF$. Let R be the intersection point of lines AQ and EF. Prove that $PR \perp OH$.

25. (Japan 2011) Given an acute triangle ABC with the midpoint M of BC. Draw the perpendicular HP from the orthocenter H of ABC to AM. Show that $AM \cdot PM = BM^2$.

26. (IMO Shortlist 2009) Let ABC be a triangle. The incircle of ABC touches the sides AB and AC at the points Z and Y, respectively. Let G be the point where the lines BY and CZ meet, and let R and S be points such that the two quadrilaterals $BCYR$ and $BCSZ$ are parallelogram. Prove that $GR = GS$.

27. Prove Brianchon's Theorem using radical axis: Let $ABCDEF$ be a convex hexagon with an inscribed circle. Prove that lines AD, BE and CF concur.