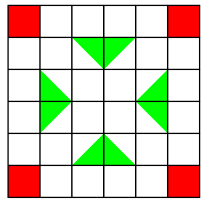
Problem. (Peacefully coexisting polygons.)

Imagine that each point of the unit square is a soldier, and that soldiers capture like queens in chess; that is, they always move in directions that are multiples of 45°, relative to the x-axis.

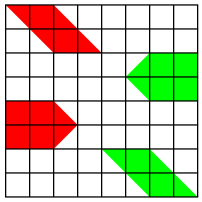
Each of two armies encamps its soldiers in polygons. The diagram shows that it's possible for two armies, each of total area less than $4/36 = 1/9$, to coexist peacefully: No soldier can capture another.

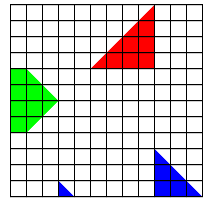


(a) Prove that, in fact, two armies whose total area each exceeds .14 can be peacefully encamped. Similarly, three armies can coexist even if each of them has an area greater than .05.

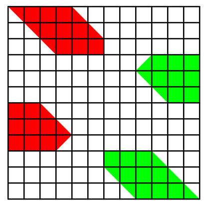
(b*) What is the maximum area achievable in such encampments of two or three armies?

Solution. As of June 1, 2015, the author's best encampments had areas 9/64 and 8/144, respectively:





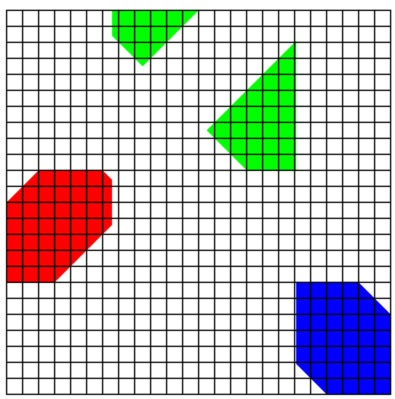
On June 15, 2015, Dick Hess found a way to improve the first one to $21/144 \approx .14583$:



On July 11, 2015, while going through some old notes, I stumbled across the fact that Stephen Ainley had already published Hess's solution(!), on page 31 of his book *Mathematical Puzzles* (1977).

On July 16, 2015, I received an undated letter from Troy Retter of Decatur, Georgia, stating that he believes an upper bound of $(1-\sqrt{1/3}\,)^2 \approx .18$ can be proved.

Troy also found a pattern for three armies that raises my lower bound from $1/18 \approx .056$ to at least .065126. Here's one version of his new setup:



The red/green boundary on the horizontal axis is at $(7-\delta)/24$, and the green/blue boundary is at $(18+\epsilon)/24$, where $\delta = (3 - 14\epsilon - \epsilon^2)/4 \approx .409752$ and ϵ is the root of $x^4 + 28x^3 + 174x^2 - 276x + 25$ between 0 and 1. Better values can be obtained by tweaking the other parameters and making them noninteger multiples of 1/24. Can somebody tell me the best boundaries for configurations of this general shape?