

Parades and Poly-Bernoulli Bijections

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This is a story about a beautiful array of numbers that arises in an astonishing number of interesting combinatorial contexts. It’s a counterexample to the hypothesis that all of the important “special numbers” were discovered long, long ago—because the earliest prominent appearance of this particular array was in 1997. It has, however, been rediscovered several times since then.

The numbers in question, which we shall call $B_{m,n}$ in this note, begin as follows:

$B_{m,n}$	$n = 0$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$
$m = 0$	1	1	1	1	1	1	1	1
$m = 1$	1	2	4	8	16	32	64	128
$m = 2$	1	4	14	46	146	454	1394	4246
$m = 3$	1	8	46	230	1066	4718	20266	85310
$m = 4$	1	16	146	1066	6902	41506	237686	1315666
$m = 5$	1	32	454	4718	41506	329462	2441314	17234438
$m = 6$	1	64	1394	20266	237686	2441314	22934774	202229266
$m = 7$	1	128	4246	85310	1315666	17234438	202229266	2193664790

Notice that we have diagonal symmetry, $B_{m,n} = B_{n,m}$, throughout this table.

The sequences for $m = 0$ and $m = 1$ are familiar. When $m = 2$ the sequence isn’t so well known, although it turns out that Euler mentioned those numbers in Section 216 of the calculus book [11] that he published in 1748(!). Then when $m = 3$ the array begins to break new ground; the historic number 1066 must be there just by coincidence.

In this note we’ll see that $B_{m,n}$ is the number of combinatorial configurations of many different kinds. For example, it’s the number of ways to assign directions to the edges of the complete bipartite graph $K_{m,n}$, in such a way that no oriented cycles arise. It’s also the number of permutations $p_1 p_2 \dots p_{m+n}$ of $\{1, 2, \dots, m+n\}$ for which we have $k - m \leq p_k \leq k + n$, for all k . It’s the number of binary relations \smile between a variable $x \in \{1, \dots, m\}$ and a variable $y \in \{1, \dots, n\}$ such that $x \smile y$ and $x' \smile y'$ implies $\max\{x, x'\} \smile \max\{y, y'\}$. And so on(!).

Furthermore, we’ll see that there are relatively simple *bijections* (one-to-one correspondences) between the objects of each kind: Acyclic orientations correspond to classes of permutations, which correspond to classes of relations, etc. The techniques of contriving such bijections are in fact interesting in themselves.

A new kind of pattern, which we shall call a “parade” of m girls and n boys, turns out to be a particularly insightful way to understand the combinatorial configurations that are enumerated by the numbers $B_{m,n}$.

A few exercises have been included for self-study, with answers at the end.

1. Definitions. Our story begins with Masanobu Kaneko’s paper [16], which introduced a nice generalization of the classic Bernoulli numbers: Let $B_n^{(s)}$ be the sequence of coefficients defined by the formal infinite series

$$\sum_{n \geq 0} B_n^{(s)} \frac{z^n}{n!} = \frac{1}{1 - e^{-z}} \sum_{k \geq 1} \frac{(1 - e^{-z})^k}{k^s}, \tag{1.1}$$

where s is any complex number. Kaneko called $B_n^{(s)}$ a *poly-Bernoulli number*, because $B_n^{(1)}$ is the famous sequence published by Jakob Bernoulli in 1713 (with $B_1^{(1)} = +1/2$), and because $\sum_{k \geq 1} z^k/k^s$ is called the polylogarithm function $\text{Li}_s(z)$. Indeed, when we set $s = 1$ in (1.1) we get the ordinary logarithm, and the right-hand side simplifies to

$$\frac{\ln(1/(1 - (1 - e^{-z})))}{1 - e^{-z}} = \frac{z}{1 - e^{-z}} = 1 + \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \dots, \tag{1.2}$$

which is the exponential generating function that defines *non-poly* Bernoulli numbers. The name ‘poly-Bernoulli’ is a bit of a jawbreaker; let’s refer to $B_n^{(s)}$ as a ‘pB number’, for short.

Kaneko was motivated purely by considerations of abstract number theory, without any hint of applications to combinatorics. The main question that he put to himself at the time was to determine the prime factorization of the denominator of $B_n^{(2)}$, because he knew that the analogous question for $B_n^{(1)}$ has a very interesting answer. (See, for example, exercise 6.54 in [13].)

Students of generatingfunctionology learn that one of the most important formulas is

$$\frac{(e^z - 1)^m}{m!} = \sum_{n \geq 0} \left\{ \begin{matrix} n \\ m \end{matrix} \right\} \frac{z^n}{n!}, \quad (1.3)$$

the exponential generating function for Stirling partition numbers $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$ when m is fixed. (See, for instance, Eq. (7.49) in [13].) Plugging this into (1.1) yields

$$\sum_{n \geq 0} B_n^{(s)} \frac{z^n}{n!} = \sum_{k \geq 0} \frac{(1 - e^{-z})^k}{(k+1)^s} = \sum_{k \geq 0} k! \frac{(-1)^k}{(k+1)^s} \sum_{n \geq 0} \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{(-z)^n}{n!};$$

so we have an explicit way to express every pB number as a sum:

$$B_n^{(s)} = \sum_{k \geq 0} k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\} \frac{(-1)^{n+k}}{(k+1)^s}. \quad (1.4)$$

One consequence of this formula is that $B_n^{(s)}$ turns out to be an integer whenever s is a negative integer. (On the other hand, $B_1^{(s)} = 2^{-s}$ is noninteger whenever $s > 0$.) In fact, the array (0.1) is obtained in this way when $s = -m$: We define

$$B_{m,n} = B_n^{(-m)} = \sum_{k \geq 0} (-1)^{n+k} k! \left\{ \begin{matrix} n \\ k \end{matrix} \right\} (k+1)^m. \quad (1.5)$$

For example, the special case

$$B_{m,2} = -\left\{ \begin{matrix} 2 \\ 1 \end{matrix} \right\} 2^m + 2 \left\{ \begin{matrix} 2 \\ 2 \end{matrix} \right\} 3^m = 2 \cdot 3^m - 2^m \quad (1.6)$$

gives the numbers 1, 4, 14, 46, ... that appear in the column for $n = 2$ in (0.1).

But wait a minute. Formula (1.5) sure doesn't look symmetrical in m and n ; yet we know from Table (0.1) that $B_{n,m}$ is actually *equal* to $B_{m,n}$, at least when m and n are small. Such all-pervasive symmetry simply cannot be a fluke! And indeed, there's an elegant way to verify that symmetry does hold, for all m and for all n , by examining a *bivariate* generating function, $G(w, z) = \sum_{m,n \geq 0} B_{m,n} \frac{w^m}{m!} \frac{z^n}{n!}$, namely the generating function for $B_{m,n}$ that is simultaneously exponential in *both* parameters:

$$\begin{aligned} G(w, z) &= \sum_{m \geq 0} \frac{w^m}{m!} \sum_{n \geq 0} B_{m,n} \frac{z^n}{n!} = \sum_{m \geq 0} \frac{w^m}{m!} \sum_{k \geq 0} (1 - e^{-z})^k (k+1)^m = \sum_{k \geq 0} (1 - e^{-z})^k e^{(k+1)w} \\ &= e^w \sum_{k \geq 0} (e^w - e^{w-z})^k = \frac{e^w}{1 - (e^w - e^{w-z})} = \frac{e^w}{e^{w-z} + 1 - e^w}. \end{aligned}$$

We have proved that the bivariate generating function is nicely symmetric in w and z :

$$G(w, z) = \frac{e^{w+z}}{e^w + e^z - e^{w+z}}. \quad (1.7)$$

Things are looking up, because we can now deduce a symmetrical way to compute $B_{m,n}$, in place of the unsymmetrical (1.5). We can write

$$G(w, z) = \frac{e^{w+z}}{1 - (e^w - 1)(e^z - 1)} = \sum_{k \geq 0} e^w (e^w - 1)^k (e^z - 1)^k e^z; \quad (1.8)$$

so we'd like to understand the generating function $e^w (e^w - 1)^k$. (See [29].) The derivative of (1.3) with respect to z reveals the answer to that question:

$$\frac{e^z (e^z - 1)^m}{m!} = \sum_{n \geq 0} \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\} \frac{z^n}{n!}. \quad (1.9)$$

Consequently we have

$$\sum_{m,n \geq 0} B_{m,n} \frac{w^m}{m!} \frac{z^n}{n!} = G(w, z) = \sum_{k \geq 0} \left(k! \sum_{m \geq 0} \left\{ \begin{matrix} m+1 \\ k+1 \end{matrix} \right\} \frac{w^m}{m!} \right) \left(k! \sum_{n \geq 0} \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\} \frac{z^n}{n!} \right). \quad (1.10)$$

And by equating the coefficients of $w^m z^n$ on each side, we obtain the symmetric formula that we seek,

$$B_{m,n} = \sum_{k \geq 0} k!^2 \left\{ \begin{matrix} m+1 \\ k+1 \end{matrix} \right\} \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\}. \quad (1.11)$$

For example, $B_{3,4} = 0!^2 \left\{ \begin{matrix} 4 \\ 1 \end{matrix} \right\} \left\{ \begin{matrix} 5 \\ 1 \end{matrix} \right\} + 1!^2 \left\{ \begin{matrix} 4 \\ 2 \end{matrix} \right\} \left\{ \begin{matrix} 5 \\ 2 \end{matrix} \right\} + 2!^2 \left\{ \begin{matrix} 4 \\ 3 \end{matrix} \right\} \left\{ \begin{matrix} 5 \\ 3 \end{matrix} \right\} + 3!^2 \left\{ \begin{matrix} 4 \\ 4 \end{matrix} \right\} \left\{ \begin{matrix} 5 \\ 4 \end{matrix} \right\} = 1 \cdot 1 \cdot 1 + 1 \cdot 7 \cdot 15 + 4 \cdot 6 \cdot 25 + 36 \cdot 1 \cdot 10 = 1066$.

2. A combinatorial interpretation. So far we've been happily manipulating algebraic formulas, without regard to what those formulas might actually mean. But the Stirling partition numbers $\left\{ \begin{matrix} n \\ m \end{matrix} \right\}$ do have a simple meaning: They're the number of ways to partition a set of n elements into m disjoint nonempty subsets.

Of course the coefficients of $z^n/n!$ in the exponential generating function $e^z - 1$ have an even simpler meaning: They're the number of surjections that map a set of n labeled elements onto a single point, namely 1 when $n > 0$ but 0 when $n = 0$.

The "symbolic method" of combinatorial enumeration now explains why (1.3) is true: When n elements are partitioned into m disjoint nonempty subsets S_1, \dots, S_m , the number of ways to label the elements of those labeled subsets has the exponential generating function $(e^z - 1)^m$. And we divide by $m!$, because all permutations of $\{S_1, \dots, S_m\}$ give the same set partition. (See, for instance, the elegant exposition of the "symbolic method" in [12], formulas II-(13) and II-(14).)

Similarly, (1.9) tells us that $m! \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\}$ is the number of ways to partition an n -element set into disjoint subsets $\{S_0, S_1, \dots, S_m\}$, where S_0 might be empty but the other sets S_1, \dots, S_m must be nonempty. It's true because we can add an $(n+1)$ st "dummy" element, then find all $\left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\}$ partitions of the extended set into $m+1$ nonempty blocks, afterwards letting S_0 be the elements that belong to the same block as the dummy element.

(Equivalently, there are $m! \left\{ \begin{matrix} n+1 \\ m+1 \end{matrix} \right\}$ ways to choose m disjoint nonempty subsets of an n -element set, without necessarily covering all n elements. The uncovered elements are S_0 .)

OK then, what does the nice symmetrical formula (1.11) mean? Its term for k comes from the term $e^w (e^w - 1)^k (e^z - 1)^k e^z$ in (1.8). Therefore we can interpret it as the number of ways to present an m -element set S as a disjoint union $S = S_0 \cup S_1 \cup \dots \cup S_d$ and an n -element set T as a disjoint union $T = T_1 \cup \dots \cup T_d \cup T_{d+1}$, where S_0 and/or T_{d+1} might be empty but the other subsets must be nonempty.

That sounds pretty abstract. Fortunately there's a much more concrete way to describe the same setup, which we shall call a "girls and boys parade." There are m girls $\{g_1, \dots, g_m\}$ and n boys $\{b_1, \dots, b_n\}$, where g_i is younger than g_{i+1} and b_j is younger than b_{j+1} , but we know nothing about the relative ages of g_i and b_j . In how many ways can they all line up in a sequence such that no girl is directly preceded by an older girl and no boy is directly preceded by an older boy?

The answer is $B_{m,n}$. For example, here are the $B_{2,2} = 14$ possible parades of two girls and two boys:

$$\begin{aligned} &g_1 g_2 b_1 b_2, \quad g_1 b_1 g_2 b_2, \quad g_1 b_1 b_2 g_2, \quad g_1 b_2 g_2 b_1, \quad g_2 b_1 g_1 b_2, \quad g_2 b_1 b_2 g_1, \quad g_2 b_2 g_1 b_1, \\ &b_1 g_1 g_2 b_2, \quad b_1 g_1 b_2 g_2, \quad b_1 g_2 b_2 g_1, \quad b_1 b_2 g_1 g_2, \quad b_2 g_1 g_2 b_1, \quad b_2 g_1 b_1 g_2, \quad b_2 g_2 b_1 g_1. \end{aligned} \quad (2.1)$$

To see why this works in general, we merely need to notice that every parade can be written in the form $S_0 T_1 S_1 \dots T_d S_d T_{d+1}$, where $S_0 \cup S_1 \cup \dots \cup S_d$ is a disjoint union of the girls and $T_1 \cup \dots \cup T_d \cup T_{d+1}$ is a disjoint union of the boys; S_0 and/or T_{d+1} might be empty, but the other subsets are nonempty; girls and boys within a subset appear from youngest to oldest. The value of d is the number of times a boy is directly followed by a girl, and we say that d is the *order* of the parade. (The respective values of d in (2.1) are 0, 1, 1, 1, 1, 1, 1, 1, 2, 2, 1, 1, 2, 2.)

In this note we shall let $\mathcal{P}_{m,n}$ be the set of all possible parades that can be formed by m labeled girls and n labeled boys. Of course $\mathcal{P}_{2,2}$ is too small to give a feeling for parades in general; here's a more typical example, taken more or less at random from $\mathcal{P}_{16,20}$:

$$\Pi = b_6 b_{13} g_2 g_4 b_5 b_{16} g_7 b_{15} g_1 g_{10} g_{16} b_{10} g_3 b_2 b_7 b_{14} g_5 g_9 g_{11} b_8 b_9 b_{18} b_{20} g_8 b_3 b_4 g_{14} b_{17} g_{12} b_{11} g_6 g_{13} g_{15} b_1 b_{12} b_{19}. \quad (2.2)$$

It's a parade of order 9.

Inside a computer, there's a nice way to represent a parade as two digit strings $s_0 s_1 \dots s_m$ and $t_0 t_1 \dots t_n$: Girl g_i belongs to set S_{s_i} and boy b_j belongs to set T_{t_j} , where T_0 is identified with T_{d+1} . Thus $s_0 = t_0 = 0$, and the other digits are between 0 and d ; every digit from 1 to d occurs at least once. For example, the parade Π in (2.2) is represented by the digit strings

$$s_0 s_1 \dots s_{16} = 03141592653589793 \quad \text{and} \quad t_0 t_1 \dots t_{20} = 005772156649015328606 \quad (2.3)$$

(so you might guess that it's not a *truly* random example).

A valid digit string can, in turn, be characterized by a permutation σ of $[1 \dots d]$ and a *restricted growth string* $\Sigma = a_0 a_1 \dots a_m$, where a restricted growth string has $a_0 = 0$ and $0 \leq a_{i+1} \leq \max\{a_0, \dots, a_i\} + 1$ for $0 \leq i < m$. (Restricted growth strings are the method of choice for representing set partitions in programs; see [21, §7.2.1.5].) The corresponding digit string has $s_i = a_{i\sigma}$, where $0\sigma = 0$. For example, the permutations and restricted growth strings that correspond to (2.3) are

$$(\sigma = 314592687, \Sigma = 01232456741485951) \quad \text{and} \quad (\tau = 572164938, T = 001223415567041839505). \quad (2.4)$$

Parades turn out to be quite delightful and instructive combinatorial patterns. The more one studies them, the more one agrees with Harold Arlen, when he wrote "I love a parade" in 1931!

3. Ranking and unranking. The patterns of a finite combinatorial class \mathcal{A} can always be listed in some order: $\alpha_0, \alpha_1, \dots, \alpha_{|\mathcal{A}|-1}$. This ordering, when described in high-falutin' mathematical jargon, is a *bijection* between \mathcal{A} and the integers $[0 \dots |\mathcal{A}|) = \{0, 1, \dots, |\mathcal{A}| - 1\}$. The process of looking at a given pattern α_k and discovering its index k in this correspondence is called *ranking*; the inverse problem, which determines the pattern α_k when its index k is given, is called *unranking*.

For example, there are 2^n binary n -tuples $b_1 \dots b_n$, where each b_j is either 0 or 1. So there are $2^n!$ bijections between binary n -tuples and the numbers of the half-open interval $[0 \dots 2^n)$. Most of those bijections are pretty weird and unimportant. But one of them is quite natural and useful, namely to let $b_1 \dots b_n$ correspond to the integer whose representation in the binary number system is $(b_1 \dots b_n)_2$. In particular, the 4-tuple 1101 has rank $(1101)_2 = 13$; conversely, the 4-tuple α_{13} is 1101. (This bijection corresponds to lexicographic order of the n -tuples. A similar one, where $b_1 \dots b_n \leftrightarrow (b_n \dots b_1)_2$, corresponds to "colexicographic order.")

Suppose \mathcal{A} is the disjoint union $\mathcal{A}' \cup \mathcal{A}''$ of classes whose rank functions are rank' and rank'' . Then $|\mathcal{A}| = |\mathcal{A}'| + |\mathcal{A}''|$, and it's natural to define

$$\text{rank}(\alpha) = \begin{cases} \text{rank}'(\alpha), & \text{if } \alpha \in \mathcal{A}'; \\ |\mathcal{A}'| + \text{rank}''(\alpha), & \text{if } \alpha \in \mathcal{A}''. \end{cases} \quad (3.1)$$

Similarly, if \mathcal{A} is representable as a Cartesian product $\mathcal{A}' \times \mathcal{A}''$, we have $|\mathcal{A}| = |\mathcal{A}'| |\mathcal{A}''|$, and we can define

$$\text{rank}((\alpha', \alpha'')) = \text{rank}'(\alpha') |\mathcal{A}''| + \text{rank}''(\alpha''). \quad (3.2)$$

(This rule corresponds to lexicographic order, and to a mixed-radix number system with radices $|\mathcal{A}'|$ and $|\mathcal{A}''|$.) Unranking is easy in both (3.1) and (3.2).

Our principal goal in this note is to discover useful bijections between many combinatorial classes \mathcal{A} for which $|\mathcal{A}| = B_{m,n}$. In particular, we should be able to find a fairly natural bijection between $\mathcal{P}_{m,n}$ and the integers $[0 \dots B_{m,n})$.

For example, what is the rank of the “typical” parade Π in (2.2)? To answer this question, we need to know the ranks of the permutations and restricted growth strings σ , Σ , τ , and T in (2.4).

There are $(9!) \approx 1.6 \times 10^{1859933}$ ways to rank the permutations of $\{1, 2, 3, 4, 5, 6, 7, 8, 9\}$; we shall choose lexicographic order. Then ranks are readily computed from the “inversion table” $C_1 \dots C_9$, where $C_j = |\{i \mid i > j \text{ and } p_i < p_j\}|$. (See exercise 5.1.1–7 in [20].) Indeed, the formula

$$\text{rank}(C_1 \dots C_n) = ((\dots((C_1(n-1) + C_2)(n-2) + \dots) + C_{n-1})1 + C_n = \sum_{j=1}^n C_j(n-j)! \quad (3.3)$$

arises from the recurrence $n! = n(n-1)!$ using (3.2). The inversion tables for σ and τ are respectively 201140010 and 451021200; so their ranks turn out to be 81577 and 187258.

The basic recurrence for Stirling partition numbers,

$$\left\{ \begin{matrix} m+1 \\ d+1 \end{matrix} \right\} = (d+1) \left\{ \begin{matrix} m \\ d+1 \end{matrix} \right\} + \left\{ \begin{matrix} m \\ d \end{matrix} \right\}, \quad (3.4)$$

is based on the fact that a partition of m girls into $d+1$ nonempty blocks either puts the oldest girl into a $(d+1)$ -block partition of the younger ones or into a new block by herself. It leads via (3.1) and (3.2) to a slick way to compute the rank r of any given restricted growth sequence $a_0 a_1 \dots a_n$:

$$\begin{aligned} & \text{“Set } r \leftarrow d \leftarrow 0, \text{ and do the following for } j = 1, \dots, n: \\ & \text{If } a_j > d, \text{ set } d \leftarrow d+1 \text{ and } r \leftarrow r + (d+1) \left\{ \begin{matrix} j \\ d+1 \end{matrix} \right\}; \\ & \text{otherwise set } r \leftarrow r + a_j \left\{ \begin{matrix} j \\ d+1 \end{matrix} \right\}.” \end{aligned} \quad (3.5)$$

And we find $\text{rank}(\Sigma) = 266642187$, $\text{rank}(T) = 29804164155016$ according to this recipe.

Consequently, the rank of (2.2) turns out to be

$$\begin{aligned} & \sum_{k=0}^8 k!^2 \left\{ \begin{matrix} 17 \\ k+1 \end{matrix} \right\} \left\{ \begin{matrix} 21 \\ k+1 \end{matrix} \right\} + ((81577 \left\{ \begin{matrix} 17 \\ 10 \end{matrix} \right\} + 266642187)9! + 187258) \left\{ \begin{matrix} 21 \\ 10 \end{matrix} \right\} + 29804164155016 \\ & = 7792164621781138538938687784201468, \end{aligned} \quad (3.6)$$

because that parade has order 9. (It’s about 0.167% of $B_{16,20} = 4669695431937298929037253789504488502$. These calculations were done by the author’s program RANK-PARADE1, which is downloadable from [22].)

Now let’s try *unranking*. We know from (0.1) that $B_{4,7} = 1315666$. What is the millionth parade that can be formed with 4 girls and 7 boys? In other words, what parade of $\mathcal{P}_{4,7}$ has rank 999999, according to the ranking scheme that we’ve just discussed?

The parades of order d are enumerated by the term for $k = d$ in (1.11); and the numerical values of those terms when $(m, n) = (4, 7)$ are respectively 1, 1905, 96600, 612360, 604800, for $d = 0, 1, 2, 3$, and 4.

Thus $1 + 1905 + 96600 + 612360 = 710866$ is the number of parades of orders 3 or less. But adding another 604800 will take us over a million. So, in accordance with (3.1), we seek the parade that has rank $999999 - 710866 = 289133$ in the set of order-4 parades, of which there are $4! \left\{ \begin{matrix} 5 \\ 5 \end{matrix} \right\} 4! \left\{ \begin{matrix} 8 \\ 5 \end{matrix} \right\} = 604800$. In the latter formula, $4! \left\{ \begin{matrix} 5 \\ 5 \end{matrix} \right\}$ enumerates the possibilities for the ordered partition $S_0 S_1 S_2 S_3 S_4$ of the girls, and $4! \left\{ \begin{matrix} 8 \\ 5 \end{matrix} \right\}$ enumerates the possibilities for the ordered partition $T_1 T_2 T_3 T_4 T_5$ of the boys. Numerically, $4! = 24$, $\left\{ \begin{matrix} 5 \\ 5 \end{matrix} \right\} = 1$, and $\left\{ \begin{matrix} 8 \\ 5 \end{matrix} \right\} = 1050$.

To get the rank-289133 pattern from $24 \cdot 1 \cdot 24 \cdot 1050$ possibilities in accordance with (3.2), we have

$$289133 = ((11 \cdot 1 + 0) \cdot 24 + 11) \cdot 1050 + 383, \quad (3.7)$$

using a mixed-radix system with radices 24, 1, 24, and 1050.

The millionth parade will have the form $S_0 T_1 S_1 T_2 S_2 T_3 S_3 T_4 S_4 T_5$, according to our interpretation above. Formula (3.7) tells us that the ordering of $\{S_1, \dots, S_4\}$ should be the rank-11 permutation of $\{1, 2, 3, 4\}$; those S ’s should come from the rank-0 set partition of 5 girls into 5 parts; the ordering of $\{T_1, T_2, T_3, T_4\}$ should (by coincidence) be the rank-11 permutation of $\{1, 2, 3, 4\}$; and those T ’s should come from the rank-383 set partition of 8 boys into 5 parts.

In general, to get the rank- r permutation of $\{1, 2, \dots, n\}$, in lexicographic order, the recurrence $n! = n(n-1)!$ leads to the following algorithm: “First set $C_{n+1-j} \leftarrow r \bmod j$ and $r \leftarrow \lfloor r/j \rfloor$, for $j = 1, 2, \dots, n$. Then, for j decreasing from n to 1, set $p_j \leftarrow 1 + C_j$, and increase p_i by 1 for all $i > j$ with $p_i \geq p_j$.” For example, when $r = 11$ and $n = 4$, we find $C_1 C_2 C_3 C_4 = 1210$, and the permutation $p_1 p_2 p_3 p_4$ turns out to be 2431.

How should we unrank set partitions into a given number of blocks? Rule (3.5) has an equally slick counterpart, which finds the restricted growth string $a_0 a_1 \dots a_n$ that has a given rank r and a given maximal element d :

$$\begin{aligned} &\text{“Set } i \leftarrow d, \text{ and do the following for } j = n, n-1, \dots, 0: \\ &\text{If } r < (i+1)\binom{j}{i+1}, \text{ set } a_j = \lfloor r / \binom{j}{i+1} \rfloor \text{ and } r \leftarrow r \bmod \binom{j}{i+1}; \\ &\text{otherwise set } a_j \leftarrow i, r \leftarrow r - (i+1)\binom{j}{i+1}, \text{ and } i \leftarrow i-1.” \end{aligned} \quad (3.8)$$

In particular, from the $\binom{5}{5} = 1$ set partitions when $m = 4$ and $d = 4$, we want the one for $r = 0$, which has the restricted growth string $a_0 a_1 a_2 a_3 a_4 = 01234$. (Hey, algorithms have to work in trivial cases too.) We apply the permutation 2431 to this, obtaining 02431; that means $S_0 = \{g_0\}$, $S_1 = \{g_4\}$, $S_2 = \{g_1\}$, $S_3 = \{g_3\}$, and $S_4 = \{g_2\}$, except that we’re supposed to remove the “dummy” girl g_0 from S_0 .

Proceeding similarly for the boys, the set partition of rank 383 when $n = 7$ and $d = 4$ turns out to have the restricted growth string 01123242. Apply the boys’ permutation 2431, to get the digit string 02243414; hence $T_1 = \{b_6\}$, $T_2 = \{b_1, b_2\}$, $T_3 = \{b_4\}$, $T_4 = \{b_3, b_5, b_7\}$, $T_5 = \{b_0\}$; and b_0 is removed from T_5 , which is the special set T_{d+1} .

Ta da: The millionth parade with four girls and seven boys is defined by $S_0 T_1 S_1 T_2 S_2 T_3 S_3 T_4 S_4 T_5$, so it is

$$b_6 g_4 b_1 b_2 g_1 b_4 g_3 b_3 b_5 b_7 g_2. \quad (3.9)$$

Unfortunately, this example doesn’t illustrate the general case in which S_0 and/or T_{d+1} are nonempty. We do get an example with nonempty S_0 if we interchange m and n , asking instead for the millionth parade when $m = 7$ and $n = 4$. That one turns out to be

$$g_5 b_1 g_6 b_4 g_1 b_3 g_3 g_7 b_2 g_2 g_4; \quad (3.10)$$

it comes from the partition of rank 0 and the permutation of rank 5 for the boys, together with the partition of rank 497 and the permutation of rank 11 for the girls.

The main advantage of a bijection between a combinatorial class \mathcal{A} and the integers $[0..|\mathcal{A}|)$ is that it allows us to generate *uniformly random patterns* from \mathcal{A} . For example, given a random number in the interval $[0..B_{m,n})$, we can compute the corresponding parade by performing $O((m+n) \log(m+n))$ arithmetic operations on numbers that have $O((m+n) \log(m+n))$ bits, using the procedure above.

It’s easy to understand the very first parade, according to this bijection: We set $d = 0$ and use the all-0 restricted growth string and the empty permutation for both girls and boys. The result is $g_1 g_2 \dots g_m b_1 b_2 \dots b_n$.

Similarly, the very last parade is only slightly more difficult to describe. Suppose $m \leq n$. Then the construction yields $d = m$; the restricted growth strings are $012\dots d$ for the girls and $0^{n-m+1}12\dots d$ for the boys; both permutations are $d\dots 21$. Hence the final parade is $b_n g_m b_{n-1} g_{m-1} \dots b_{n+1-m} g_1 b_1 \dots b_{n-m}$. When $m > n$, it is $g_1 \dots g_{m-n} b_n g_m b_{n-1} g_{m-1} \dots b_1 g_{m+1-n}$.

Notice that the ordering of the 14 parades in (2.1) does *not* correspond to our bijection. (They appear there in lexicographic order, assuming that $g_1 < g_2 < b_1 < b_2$.) If we had listed them in the order of our bijection, for ranks $r = 0, 1, \dots, 13$, the respective values of d would have been $(0, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 2, 2)$. That listing would in fact have been

$$\begin{aligned} &g_1 g_2 b_1 b_2, \quad g_2 b_1 g_1 b_2, \quad g_2 b_1 b_2 g_1, \quad g_2 b_2 g_1 b_1, \quad b_1 g_1 g_2 b_2, \quad b_1 b_2 g_1 g_2, \quad b_2 g_1 g_2 b_1, \\ &g_1 b_1 g_2 b_2, \quad g_1 b_1 b_2 g_2, \quad g_1 b_2 g_2 b_1, \quad b_1 g_1 b_2 g_2, \quad b_2 g_1 b_1 g_2, \quad b_1 g_2 b_2 g_1, \quad b_2 g_2 b_1 g_1. \end{aligned} \quad (3.11)$$

Every parade with 1 girl and n boys corresponds naturally to the subset of boys that precedes the girl. When those parades are listed in order of rank, the subsets arise in the somewhat bizarre order

$$\begin{aligned} &\emptyset, \{b_1\}, \{b_1 b_2\}, \{b_2\}, \{b_1 b_3\}, \{b_1 b_2 b_3\}, \{b_2 b_3\}, \{b_3\}, \\ &\{b_1 b_4\}, \{b_1 b_2 b_4\}, \{b_2 b_4\}, \{b_1 b_3 b_4\}, \{b_1 b_2 b_3 b_4\}, \{b_2 b_3 b_4\}, \{b_3 b_4\}, \{b_4\}, \end{aligned} \quad (3.12)$$

which corresponds to the permutation $f(r)$ of positive integers defined by

$$f(1) = 1; \quad f(2^k + j) = 2^k + f(j+1) \text{ for } 0 \leq j < 2^k - 1 \text{ and } f(2^{k+1} - 1) = 2^k, \quad \text{for all } k > 0. \quad (3.13)$$

The parades for m girls and 1 boy arise in essentially the same order, but with respect to the subsets of girls that *follow* the boy: $\emptyset, \{g_1\}, \{g_1 g_2\}, \{g_2\}, \{g_1, g_3\}$, etc.

4. A recurrence for pB numbers. Taking a different tack, we can use the elegant bivariate generating function $G(w, z)$ in (1.7) to deduce an unexpected relationship between the numbers on adjacent rows of (0.1). Notice first that differentiation with respect to w has a simple “shift-up” effect on the coefficients:

$$\frac{\partial}{\partial w} G(w, z) = \frac{\partial}{\partial w} \sum_{m, n \geq 0} B_{m, n} \frac{w^m}{m!} \frac{z^n}{n!} = \sum_{m, n \geq 0} B_{m, n} \frac{w^{m-1}}{(m-1)!} \frac{z^n}{n!} = \sum_{m, n \geq 0} B_{m+1, n} \frac{w^m}{m!} \frac{z^n}{n!}. \quad (4.1)$$

(In this formula, $1/(-1)! = 0$.) Similarly, $\frac{\partial}{\partial z} G(w, z) = \sum_{m, n \geq 0} B_{m, n+1} \frac{w^m}{m!} \frac{z^n}{n!}$. Furthermore

$$\frac{\partial}{\partial w} \frac{e^{w+z}}{e^w + e^z - e^{w+z}} = \frac{e^{w+2z}}{(e^w + e^z - e^{w+z})^2} = e^{-w} G(w, z)^2. \quad (4.2)$$

We also have

$$G(w, z) = \frac{1}{e^{-z} + e^{-w} - 1}; \quad (4.3)$$

whence it follows that

$$e^{-z} G(w, z)^2 + e^{-w} G(w, z)^2 - G(w, z) = G(w, z)^2. \quad (4.4)$$

By equating the coefficients of $w^m z^n$ on both sides of this equation, we see that

$$B_{m, n+1} + B_{m+1, n} - B_{m, n} = \binom{n}{0} B_{m, n+1} + \binom{n}{1} B_{m, n} + \binom{n}{2} B_{m, n-1} + \cdots + \binom{n}{n} B_{m, 1}. \quad (4.5)$$

Therefore the pB numbers can be computed row by row, using a recurrence that’s totally different from any of our previous formulas:

$$B_{0, n} = 1; \quad B_{m+1, n} = B_{m, n} + \binom{n}{1} B_{m, n} + \binom{n}{2} B_{m, n-1} + \cdots + \binom{n}{n} B_{m, 1}, \quad (4.6)$$

valid for all $m, n \geq 0$. For example,

$$B_{3, 4} = B_{2, 4} + \binom{4}{1} B_{2, 4} + \binom{4}{2} B_{2, 3} + \binom{4}{3} B_{2, 2} + \binom{4}{4} B_{2, 1} = 146 + 4 \cdot 146 + 6 \cdot 46 + 4 \cdot 14 + 1 \cdot 4 = 1066. \quad (4.7)$$

(Masanobu Kaneko [17] derived the recurrence (4.6) shortly after he had discovered the pB numbers.)

OK, we know from algebra and calculus that the number of parades of girls and boys satisfies the recurrence (4.6). Can we also find a purely *combinatorial* explanation for that fact?

Yes! There’s obviously only one possible parade when no girls are present; hence $B_{0, n} = 1$. So suppose we have a parade Π with $m + 1$ girls and n boys; we want to represent it uniquely as one of the parades represented by the right-hand side of (4.6). We’ll say that $\Pi \in \mathcal{P}_{m+1, n}$ is of type T if T is the block of boys that immediately follows the oldest girl, g_{m+1} . For example, if $m = 3$, the parade in (3.9) has type $\{b_1 b_2\}$.

The number of parades of type \emptyset is $B_{m, n}$, because such parades arise if and only if g_{m+1} comes last; conversely, g_{m+1} can safely be appended to any parade that has m girls.

Otherwise we shall show that the number of parades of type T is $B_{m, n+1-|T|}$; and this will establish (4.6), because there are $\binom{n}{t}$ types T with $|T| = t$. Let b_μ be the oldest boy in T . Remove $T \setminus \{b_\mu\}$ from the set of boys, and give the remaining boys the new names $b'_1, \dots, b'_{n-(t-1)}$ (youngest to oldest). Then map $\Pi \mapsto \Pi'$ by renaming the boys, and by replacing the subsequence ‘ $g_{m+1} T$ ’ of Π by $b'_{\mu-(t-1)}$, which is the new name of b_μ . (That boy may have to sort himself into his proper place among other boys.)

We’ve thereby mapped every $(m + 1, n)$ -parade of type $T \neq \emptyset$ into an $(m, n + 1 - |T|)$ -parade; and the mapping is clearly invertible. For example, if $T = \{b_2 b_3 b_6\}$, the parade $\Pi' = b'_1 b'_4 b'_5 g_1 g_4 b'_3 g_2 b'_2 g_5 b'_6 g_3$ could have come only from $\Pi = b_1 b_7 g_6 b_2 b_3 b_6 g_1 g_4 b_5 g_2 b_4 g_5 b_8 g_3$. (Further explanation is below.)

5. A recursive ranking scheme. Now let’s turn the tables and assign ranks to objects that satisfy (4.6), instead of assigning ranks to objects that are enumerated by (1.11) as we did in §3. Once again, every parade in $\mathcal{P}_{m, n}$ will be assigned a number between 0 and $B_{m, n} - 1$ inclusive.

In accordance with the additive rule (3.1), we'll give the smallest $B_{m,n}$ ranks to parades of type \emptyset . The next $\binom{n}{1}B_{m,n}$ ranks in (4.6) will go to the parades enumerated by $\binom{n}{1}B_{m,n}$; and so on. In general the right-hand side of (4.6) has $n+1$ terms, $t_0 + t_1 + t_2 + \dots + t_n$, where $t_0 = B_{m,n}$ and $t_k = \binom{n}{k}B_{m,n+1-k}$ when $k > 0$. If we're unranking, the parade of rank r will belong to those that correspond to term t_k , where k is found as follows: "Set $k \leftarrow 0$. While $r \geq t_k$, set $r \leftarrow r - t_k$ and $k \leftarrow k + 1$."

But let's do ranking first. What is the recursive rank of the parade $\Pi = b_6g_4b_1b_2g_1b_4g_3b_3b_5b_7g_2$ in (3.9), whose rank was 999999 under the old scheme? This parade of type $\{b_1b_2\}$ is mapped into

$$\begin{aligned}\Pi' &= b_1b_5g_1b_3g_3b_2b_4b_6g_2 \text{ of type } \{b_2b_4b_6\}, \text{ which is mapped into} \\ \Pi'' &= b_1b_3g_1b_2b_4g_2 \text{ of type } \emptyset, \text{ which is mapped into} \\ \Pi''' &= b_1b_3g_1b_2b_4 \text{ of type } \{b_2b_4\}, \text{ which is mapped into} \\ \Pi'''' &= b_1b_2b_3.\end{aligned}\tag{5.1}$$

Consequently

$$\begin{aligned}\text{rank}(\Pi) &= B_{3,7} + \binom{7}{1}B_{3,7} + 0B_{3,6} + \text{rank}(\Pi'); \\ \text{rank}(\Pi') &= B_{2,6} + \binom{6}{1}B_{2,6} + \binom{6}{2}B_{2,5} + 14B_{2,4} + \text{rank}(\Pi''); \\ \text{rank}(\Pi'') &= \text{rank}(\Pi'''); \\ \text{rank}(\Pi''') &= B_{0,4} + \binom{4}{1}B_{0,4} + 4B_{0,3} + \text{rank}(\Pi'''); \\ \text{rank}(\Pi''') &= 0.\end{aligned}\tag{5.2}$$

So $\text{rank}(\Pi''') = 9$, $\text{rank}(\Pi'') = 9$, $\text{rank}(\Pi') = 18621$, and $\text{rank}(\Pi) = 701101$. (The coefficients in '0 $B_{3,6}$ ', '14 $B_{2,4}$ ', '4 $B_{0,3}$ ' come from ranking the types: The rank of $\{b_1b_2\}$ among 2-subsets of $\{b_1, \dots, b_7\}$ is 0; the rank of $\{b_2b_4b_6\}$ among 3-subsets of $\{b_1, \dots, b_6\}$ is 14; the rank of $\{b_2b_4\}$ among 2-subsets of $\{b_1, \dots, b_4\}$ is 4. Formula (A.2) in the Appendix below explains how such ranks are readily computed.)

When the same method is applied to the "typical" parade (2.2), which is of type $\{b_{10}\}$, we find that Π' has type $\{b_1b_{12}b_{19}\}$, Π'' has type $\{b_{15}\}$ (where b_{15} was originally b_{17}), and so on. The recursive rank turns out to be 1491392338417882718739839722665904161, about 32% of $B_{16,20}$. Middle of the road.

The recursive *un*ranking procedure is another good test of these methods, so let's study it next. What is the millionth element of $\mathcal{P}_{4,7}$ according to this *new* ranking scheme? For that problem we have $m = 3$, $n = 7$, and the terms (t_0, \dots, t_7) are (85310, 597170, 425586, 165130, 37310, 4830, 322, 8). Hence the parade of rank $r = 999999$ leads to $k = 2$; it will be the parade of rank $999999 - 85310 - 597170 = 317519$ that corresponds to term $t_2 = \binom{7}{2}B_{3,6}$.

In accordance with the multiplicative rule (3.2), we now find $317519 = 15 \cdot B_{3,6} + 13529$. Algorithm (A.1) in the Appendix below tells us that the 2-subset of $\{b_1, \dots, b_7\}$ that has rank 15 is $\{b_1b_7\}$. So we want the type $\{b_1b_7\}$ parade of $\mathcal{P}_{4,7}$ that maps into the rank 13529 parade of $\mathcal{P}_{3,6}$.

Let $\Pi_{m,n,r}$ be the parade of recursive rank r in $\mathcal{P}_{m,n}$. We've just concluded that $\Pi_{4,7,999999}$ is the parade of type $\{b_1b_7\}$ that maps to $\Pi_{3,6,13529}$; we shall say that " $\Pi_{4,7,999999} = \Pi_{3,6,13529}$ extended by $\{b_1b_7\}$."

It turns out that, similarly,

$$\begin{aligned}\Pi_{3,6,13529} &= \Pi_{2,5,139} \text{ extended by } \{b_3b_5\}; \\ \Pi_{2,5,139} &= \Pi_{1,5,11} \text{ extended by } \{b_4\}; \\ \Pi_{1,5,11} &= \Pi_{0,4,0} \text{ extended by } \{b_3b_4\}.\end{aligned}\tag{5.3}$$

Now $\Pi_{0,4,0}$ is $b_1b_2b_3b_4$. So we can go backward in (5.3) and determine

$$\begin{aligned}\Pi_{1,5,11} &= b_1b_2b_5g_1b_3b_4; \\ \Pi_{2,5,139} &= b_1b_2b_5g_1b_3g_2b_4; \\ \Pi_{3,6,13529} &= b_1b_2b_6g_1b_4g_2g_3b_3b_5; \\ \Pi_{4,7,999999} &= b_2b_3g_4b_1b_7g_1b_5g_2g_3b_4b_6.\end{aligned}\tag{5.4}$$

The algorithm that leads from (5.3) to (5.4), which is somewhat delicate, is described in the Appendix below. Incidentally, when girls and boys are reversed in this example, we find

$$\Pi_{7,4,999999} = g_3b_2b_3g_2g_7b_4g_1g_5g_6b_1g_4.\tag{5.5}$$

(That computation involves extending $\Pi_{3,4,847}$ by \emptyset .)

Notice that this procedure leads to an interesting characterization: *Every parade can be built up uniquely by starting with a parade that has no girls and repeatedly extending it, adding one girl at a time. Each nonempty extension from n' boys to n boys is guided by an $(n - n' + 1)$ -element subset of $\{b_1, \dots, b_n\}$.*

The fourteen parades of $\mathcal{P}_{2,2}$ are ranked in the following order by this recursive scheme, in contrast to (2.1) and (3.11):

$$\begin{aligned} & b_1b_2g_1g_2, b_2g_1b_1g_2, b_1g_1b_2g_2, g_1b_1b_2g_2, b_2g_2b_1g_1, b_2g_1g_2b_1, g_2b_1g_1b_2, \\ & g_1b_2g_2b_1, b_1g_2b_2g_1, g_2b_2g_1b_1, b_1g_1g_2b_2, g_1b_1g_2b_2, g_2b_1b_2g_1, g_1g_2b_1b_2. \end{aligned} \quad (5.6)$$

In general, the first parade in this recursive ranking of $\mathcal{P}_{m,n}$ is clearly $b_1b_2 \dots b_n g_1g_2 \dots g_m$, because we start with all the boys and append all the girls, one by one.

The very last parade, on the other hand, is obtained when we extend the last parade of $\mathcal{P}_{m-1,1}$ by $\{b_1, \dots, b_n\}$. So it is $g_1g_2 \dots g_m b_1b_2 \dots b_n$.

When the 2^n parades of $\mathcal{P}_{1,n}$ are ranked recursively, the subsets of boys that follow the girl appear in increasing order of their size, and in colexicographic order within each size. For example, the eight parades of $\mathcal{P}_{1,3}$ are

$$b_1b_2b_3g_1, b_2b_3g_1b_1, b_1b_3g_1b_2, b_1b_2g_1b_3, b_3g_1b_1b_2, b_2g_1b_1b_3, b_1g_1b_2b_3, g_1b_1b_2b_3. \quad (5.7)$$

It's the same as the order of subsets made by the girls who *precede* the boy in $\mathcal{P}_{n,1}$; for example,

$$b_1g_1g_2g_3, g_1b_1g_2g_3, g_2b_1g_1g_3, g_1g_2b_1g_3, g_3b_1g_1g_2, g_1g_3b_1g_2, g_2g_3b_1g_1, g_1g_2g_3b_1. \quad (5.8)$$

6. Automorphisms. There are many one-to-one mappings of $\mathcal{P}_{m,n}$ into itself; in fact, the total number is $B_{m,n}!$, which is huge. The vast majority of them are completely arbitrary and of no interest whatsoever. But some of them are particularly important, because they're "natural" and easy to understand. Indeed, we obtain $m!n!$ natural automorphisms by using any permutation $\sigma_1 \dots \sigma_m$ of $[1 \dots m]$ to rename the girls and any permutation $\tau_1 \dots \tau_n$ of $[1 \dots n]$ to rename the boys; after replacing g_j by g_{σ_j} and b_k by b_{τ_k} , for all j and k , adjacent girls and adjacent boys can bubblesort themselves and form a new parade.

For example, suppose $m = 4$, $n = 7$, $\sigma_1 \dots \sigma_4 = 3142$, and $\tau_1 \dots \tau_7 = 5721643$. Then the parade $b_6g_4b_1b_2g_1b_4g_3b_3b_5b_7g_2$ of (3.9) becomes $b_4g_2b_5b_7g_3b_1g_4b_2b_6b_3g_1$ before sorting, and $b_4g_2b_5b_7g_3b_1g_4b_2b_3b_6g_1$ afterwards.

Another straightforward automorphism reflects the parade, left-to-right. Then (3.9) becomes the pseudo-parade $g_2b_7b_5b_3g_3b_4g_1b_2b_1g_4b_6$ before sorting, and $g_2b_3b_5b_7g_3b_4g_1b_1b_2g_4b_6$ afterwards.

Furthermore, the number of basic automorphisms increases by a further factor of $2!^23!^2 \dots \min\{m, n\}!^2$ when we realize that we can specify, for each k , two independent permutations of $[1 \dots k]$ that can be applied respectively to the blocks $S_1 \dots S_k$ and $T_1 \dots T_k$ that occur in parades of order k .

This multiplicity of automorphisms means that every bijection between $\mathcal{P}_{m,n}$ and another class of combinatorial patterns leads to many further bijections. Some of those bijections will, of course, be much more intuitive and/or interesting than others.

7. Acyclic bipartite orientations. It's high time to fulfill the promise that was made in the introduction, namely to discuss other combinatorial patterns that are enumerated by the pB numbers.

Let $\mathcal{O}_{m,n}$ be the set of all ways to assign a direction to each of the mn edges of the complete bipartite graph $K_{m,n}$, in such a way that the resulting digraph has no oriented cycles.

It turns out that $|\mathcal{O}_{m,n}| = B_{m,n}$. For example, when $m = n = 2$, the four edges $u_1 - v_1 - u_2 - v_2 - u_1$ of $K_{2,2}$ form a 4-cycle; and exactly two of the 16 ways to orient those edges will turn that cycle into an oriented cycle. That leaves $14 = B_{2,2}$ acyclic orientations.

Indeed, there's an appealing bijection between $\mathcal{P}_{m,n}$ and $\mathcal{O}_{m,n}$: We can associate the vertices of $K_{m,n}$ with m girls in one part and n boys in the other. Given any parade in $\mathcal{P}_{m,n}$, we orient the edge $g_i - b_j$ by simply saying that $g_i \rightarrow b_j$ if and only if b_j follows g_i in the parade. The resulting digraph is obviously acyclic.

Conversely, this mapping from $\mathcal{P}_{m,n}$ to $\mathcal{O}_{m,n}$ is invertible. Given any acyclic orientation, we want to show that it corresponds to a unique parade. At least one vertex must be a "source," having no predecessor; and the sources must either be all girls or all boys. Place all the sources first in the parade; remove them from the digraph; and repeat the argument.

Any orientation of $K_{m,n}$ can be represented conveniently as an $m \times n$ matrix (a_{ij}) of 0s and 1s, where $a_{ij} = 1$ if and only if $u_i \rightarrow v_j$. For example, the acyclic orientation of $K_{4,7}$ that corresponds under our bijection to (3.9), the millionth parade of $\mathcal{P}_{4,7}$, is

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}. \quad (7.1)$$

It's easy to reconstruct (3.9) from this. *Hint:* A girl source is a row of 1s; a boy source is a column of 0s.

(The fact that $|\mathcal{O}_{m,n}| = B_{m,n}$ was discovered by Peter Cameron, Celia Glass, Kamilla Rekvényi, and Robert Schumacher [8].)

8. Doubly bounded permutations. Let $\mathcal{V}_{m,n}$ be the set of all permutations $p_1 p_2 \dots p_{m+n}$ of $[1 \dots m+n]$ with the property that

$$j - m \leq p_j \leq j + n \quad \text{for } 1 \leq j \leq m + n. \quad (8.1)$$

Guess what? $|\mathcal{V}_{m,n}| = B_{m,n}$. For example, when $m = n = 2$, the permutation $p_1 p_2 p_3 p_4$ must have $p_1 \neq 4$ and $p_4 \neq 1$. Of the 24 possibilities, we must throw out the 6 with $p_1 = 4$ and the 6 with $p_4 = 1$; but we threw 4231 and 4321 out twice, so exactly $14 = B_{2,2}$ remain.

The number of permutations that we seek is the *permanent* of a nicely structured $(m+n) \times (m+n)$ matrix $Q_{m,n}$, illustrated here for $m = 4$ and $n = 7$:

$$Q_{m,n} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} J_{m,n} & S_m \\ S_n^T & J_{n,m} \end{pmatrix}. \quad (8.2)$$

Here $J_{m,n}$ is an $m \times n$ matrix of all 1s; S_m is a lower-triangular $m \times m$ matrix with 1s on and below the diagonal, but with 0s above. The value of $|\mathcal{V}_{m,n}| = \text{per}(Q_{m,n})$ is the number of ways we can place rooks on the 1s of $Q_{m,n}$, with no two rooks in the same row or the same column.

Suppose we place k rooks in $J_{m,n}$, the submatrix at the upper left. Then $m - k$ rooks must be placed in S_m , and $n - k$ rooks in S_n^T . Consequently there are k rooks also in $J_{n,m}$.

And now—aha—the number of ways to place $m - k$ nonattacking rooks on the 1s of S_m is exactly the quantity $\{m+1\}_{k+1}$ that appears in formula (3.4)! This remarkable fact, discovered by Irving Kaplansky and John Riordan in 1946 [19], comes to us accompanied by a splendid *bijection* between the restricted growth strings $a_0 a_1 \dots a_m$ with maximum element k and the placements of $m - k$ rooks, discovered by Edward Bender in 1969: If $j > 0$ and a_j exceeds $\max\{a_0, \dots, a_{j-1}\}$, we put no rook into row j ; otherwise we place a rook in column $i + 1$ of row j , where $i < j$ is maximum such that $a_i = a_j$. (See exercise 5.1.3–19 in [20].)

Bender's bijection is illustrated for $m = 3$ and $k = 1$ in the following seven cases, where each possibility for the rooks is shown above its corresponding restricted growth string:

$$\begin{array}{ccccccc} \begin{pmatrix} \blacksquare & 0 & 0 \\ 1 & \blacksquare & 0 \\ 1 & 1 & 1 \end{pmatrix} & \begin{pmatrix} \blacksquare & 0 & 0 \\ 1 & 1 & 0 \\ 1 & \blacksquare & 1 \end{pmatrix} & \begin{pmatrix} \blacksquare & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & \blacksquare \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ \blacksquare & 1 & 0 \\ 1 & 1 & \blacksquare \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ \blacksquare & 1 & 0 \\ 1 & \blacksquare & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & \blacksquare & 0 \\ \blacksquare & 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & \blacksquare & 0 \\ 1 & 1 & \blacksquare \end{pmatrix} \\ 0001 & 0010 & 0011 & 0100 & 0101 & 0110 & 0111 \end{array}. \quad (8.3)$$

Once we've placed the rooks into S_m and S_n^T , we're left with $k \times k$ submatrices of $J_{m,n}$ and $J_{n,m}$ where rooks can still be placed. Since there are $k!$ ways to complete the job, we've proved that $|\mathcal{V}_{m,n}|$ is the sum (1.11), which is $B_{m,n}$.

In fact, this argument also provides us with a simple bijection. For example, we know that the millionth parade (3.9) has the restricted growth sequence 01234 for the girls, together with the permutation 2431; and it has the restricted growth sequence 01123242 for the boys, together with the permutation 2431. Using Bender’s bijection, and transposing the boys’ placement in S_n in order to cover S_n^T , the corresponding rook placement in $Q_{4,7}$ turns out to be

$$\begin{pmatrix} 1 & 1 & \blacksquare & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & \blacksquare & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & \blacksquare & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ \blacksquare & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \blacksquare & 1 & 1 \\ 0 & \blacksquare & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & \blacksquare \\ 0 & 0 & 0 & 1 & \blacksquare & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & \blacksquare & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & \blacksquare & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & \blacksquare & 1 & 1 & 1 \end{pmatrix}. \quad (8.4)$$

(Instead of transposing the boys’ placement, we could have rotated it by 180° ; it’s unclear which alternative is better.) Given the placements in (8.4), the millionth doubly bounded permutation in $\mathcal{V}_{4,7}$ is

$$p_1 \dots p_{11} = 3 \ 6 \ 4 \ 1 \ 9 \ 2 \ 11 \ 5 \ 10 \ 7 \ 8. \quad (8.5)$$

Exercise 8.1. What permutation $p_1 \dots p_{36}$ of $\mathcal{V}_{16,20}$ corresponds to the “typical” parade (2.2) of $\mathcal{P}_{16,20}$?

(The number of permutations satisfying (8.1) was found in 1974 by Katalin Vesztergombi [33], who actually solved a much more general problem, as we shall see below. Stéphane Launois [24] noticed in 2007 that her formula matches (1.5); his paper was apparently the first publication to point out that pB numbers can have combinatorial significance. The bijection between $\mathcal{V}_{m,n}$ and $\mathcal{P}_{m,n}$ mentioned here is based on L. Lovász’s solution to a similar problem; see [25], Problem 4.36.)

Notice, by the way, that the *inverses* of the permutations in $\mathcal{V}_{m,n}$ are the permutations in $\mathcal{V}_{n,m}$, because $Q_{m,n}^T = Q_{n,m}$.

9. Weak-excedance-first permutations. Let $\mathcal{E}_{m,n}$ be the set of all permutations $q_1 \dots q_{m+n}$ for which (i) $q_j \geq j$ for $1 \leq j \leq m$, and (ii) $q_j \leq j$ for $m < j \leq m+n$. (Condition (i) is called a “weak excedance,” in contrast to the condition ‘ $q_j > j$ ’, which is simply called an *excedance*. Condition (ii) is a “non-excedance.”) For example, the elements of $\mathcal{E}_{2,2}$ are

$$1234, 1324, 1423, 1432, 2314, 2413, 2431, 3214, 3412, 3421, 4213, 4231, 4312, 4321. \quad (9.1)$$

Such permutations were introduced by Beáta Bényi and Péter Hajnal [5, §3.2].

We can count them by evaluating the permanent of a suitable $(m+n) \times (m+n)$ matrix, which looks like this in the special case $m=4$ and $n=7$:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} S_m^T & J_{m,n} \\ J_{n,m} & S_n \end{pmatrix}. \quad (9.2)$$

Aha! It’s just a jumbled-up version of $Q_{m,n}$ in (8.2), containing S_m^T , $J_{m,n}$, $J_{n,m}$, and S_n as submatrices.

Consequently every permutation $q_1 \dots q_{m+n}$ of $\mathcal{E}_{m,n}$ is in bijection with the permutation

$$p_1 \dots p_{m-1} p_m p_{m+1} \dots p_{m+n-1} p_{m+n} = \bar{q}_m \dots \bar{q}_2 \bar{q}_1 \bar{q}_{m+n} \dots \bar{q}_{m+2} \bar{q}_{m+1} \quad (9.3)$$

of $\mathcal{V}_{m,n}$, where $\bar{q} = m + n + 1 - q$. By (8.5), the millionth weak-excedance-first permutation in $\mathcal{E}_{4,7}$ is

$$q_1 \dots q_{11} = 11\ 8\ 6\ 9\ 4\ 5\ 2\ 7\ 1\ 10\ 3. \quad (9.4)$$

10. Lonesum matrices. The row sums $R = (r_1, \dots, r_m)$ and column sums $S = (s_1, \dots, s_n)$ of an $m \times n$ matrix (a_{ij}) , namely

$$r_i = \sum_{j=1}^n a_{ij} \quad \text{and} \quad s_j = \sum_{i=1}^m a_{ij}, \quad (10.1)$$

are important in many applications, especially when all of the matrix entries a_{ij} are 0 or 1. Herbert Ryser [28] found necessary and sufficient conditions for the existence of at least one 0–1 matrix whose row and column sums match a given pair (R, S) . He also showed that any two 0–1 matrices with the *same* (R, S) can be transformed into each other by a sequence of “switches,” where every switch changes a 2×2 submatrix from $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or vice versa. (Such a switch clearly leaves all row and column sums unchanged.)

Therefore a 0–1 matrix is *uniquely* determined by its R and S sequences if and only if it is “ $\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\}$ -free”; that is, if and only if none of its 2×2 submatrices have the forms $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. For example, the matrix (7.1) is $\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$ -free; so it’s the only one with $R = (4, 0, 3, 6)$ and $S = (1, 1, 3, 2, 3, 0, 3)$.

Let $\mathcal{L}_{m,n}$ be the set of all $m \times n$ matrices of 0s and 1s whose row and column sums determine them uniquely. Chad Brewbaker [7], calling such matrices “lonesum,” proved that $|\mathcal{L}_{m,n}| = B_{m,n}$. (His paper was the *second* publication that presented pB numbers in a combinatorial context.)

We have almost proved his theorem already, because it’s easy to see that *an orientation of a complete bipartite graph is acyclic if and only if it has no oriented 4-cycle*. For if the shortest oriented cycle has length k , the value of k must be even; and we can assign labels to the vertices so that the cycle has the form

$$u_1 \rightarrow v_1 \rightarrow u_2 \rightarrow v_2 \rightarrow \dots \rightarrow u_{k/2} \rightarrow v_{k/2} \rightarrow u_1. \quad (10.2)$$

There’s a contradiction if $k > 4$, because both $v_2 \rightarrow u_1$ and $u_1 \rightarrow v_2$ would give a shorter cycle.

It follows that an $m \times n$ matrix of 0s and 1 is lonesum if and only if it is one of the matrices such as (7.1) that describes an acyclic orientation of $K_{m,n}$. (An oriented 4-cycle $u_i \rightarrow v_j \rightarrow u_{i'} \rightarrow v_{j'} \rightarrow u_i$ would show up in the matrix as $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ or $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in rows $\{i, i'\}$ and columns $\{j, j'\}$.)

The bijection we used for $\mathcal{O}_{m,n}$ therefore works also for $\mathcal{L}_{m,n}$. The $\{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\}$ -free matrix (7.1) is the millionth lonesum matrix of $\mathcal{L}_{4,7}$. It is easily reconstructed from its row sums $(4, 0, 3, 6)$ and column sums $(1, 1, 3, 2, 3, 0, 3)$.

(It’s not difficult to see that lonesum matrices are precisely the matrices that can be transformed by row and column permutation to the Ferrers diagram for an integer partition, with all the 1s concentrated at the top and left, because any permutation of R and/or S preserves lonesumness. In a Ferrers diagram, the row and column sums appear in nonincreasing order, and they’re “conjugate” partitions of their sum.)

11. Strongly Γ -free matrices. A $\{\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\}$ -free matrix is called “strongly Γ -free,” because $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ looks like the letter Γ and $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ is another matrix of the form $\begin{pmatrix} 1 & 1 \\ 1 & * \end{pmatrix}$. We shall let $\mathcal{G}_{m,n}$ be the set of all $m \times n$ matrices of 0s and 1s that are strongly Γ -free, and (surprise?) we shall prove that $|\mathcal{G}_{m,n}| = B_{m,n}$. It’s obvious that $|\mathcal{G}_{2,2}| = 14 = B_{2,2}$.

(The much larger class of $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ -free matrices, which was called simply “ Γ -free” by Anna Lubiw [26], also has important applications to combinatorial optimization. There are $(n+3)3^{n-1}$ Γ -free matrices when $m = 2$, and 1725320 of them when $m = n = 5$.)

The strong Γ -free constraint makes it easy to evaluate $|\mathcal{G}_{m,n}|$ by showing that recurrence (4.6) holds. Clearly $|\mathcal{G}_{0,n}| = 1$ and $|\mathcal{G}_{1,n}| = 2^n$, because a matrix with fewer than 2 rows has no 2×2 submatrices.

If we're given an arbitrary matrix of $\mathcal{G}_{m+1,n}$, suppose there are exactly t 1s in its top row. If $t = 0$, the remaining m rows are a perfectly general matrix of $\mathcal{G}_{m,n}$. Otherwise the first $t - 1$ of those 1s must have nothing but 0s below them; and if we remove those $t - 1$ columns, we obtain a perfectly general matrix of $\mathcal{G}_{m,n+1-t}$. Recurrence (4.6) is valid because the 1s of the top row can appear in $\binom{n}{t}$ columns.

It's also easy to convert that argument to a bijection with parades. Let $\Gamma_{m,n,r}$ be the strongly Γ -free 0–1 matrix of rank r , computed according to the recurrence. We simply let $\Gamma_{m,n,r}$ correspond to $\Pi_{m,n,r}$, the parade of recursive rank r that was defined constructively in §5 above.

For example, the millionth matrix in $\mathcal{G}_{4,7}$ is $\Gamma_{4,7,999999}$, which corresponds to $\Pi_{4,7,999999}$. From (5.4), we have

$$\begin{aligned} \Gamma_{1,5,11} &= (0 \ 0 \ 1 \ 1 \ 0); & \Gamma_{2,5,139} &= \begin{pmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}; & \Gamma_{3,6,13529} &= \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}; \\ \Gamma_{4,7,999999} &= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix}. \end{aligned} \tag{11.1}$$

The boys that follow a particular girl in the parade correspond to 1s in a particular row of the matrix.

Conversely, we've seen that the millionth parade (3.9) has recursive rank 701101. So it corresponds to

$$\Gamma_{4,7,701101} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \tag{11.2}$$

by the calculations in (5.1) and (5.2). These Gamma-avoiding matrices follow the recursion (4.6) so closely, we can regard $\Gamma_{m,n,r}$ as a natural way to represent the parade $\Pi_{m,n,r}$.

Exercise 11.1. What parade corresponds to the following matrix of $\mathcal{G}_{4,7}$?

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Exercise 11.2. What strongly Γ -free 16×20 matrix corresponds to the “typical” parade Π of (2.2)?

Exercise 11.3. What's the maximum number of 1s in an element of $\mathcal{G}_{m,n}$?

(Bijections between $\mathcal{G}_{m,n}$ and parade-like arrangements were first constructed by Bényi and Hajnal [4], then simplified by Bényi and Nagy [6]; but those bijections were more complicated than the one above.)

Notice, by the way, that strongly Γ -free matrices are also in bijection with “strongly L -free matrices,” namely the matrices that are $\left\{ \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ -free, under the obvious operation of top-to-bottom reflection. And of course there are similar bijections with matrices that are $\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ -free, or $\left\{ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ -free.

12. Γ -and- L -free matrices. Let $\mathcal{Q}_{m,n}$ be the set of all $m \times n$ matrices of 0s and 1s that are $\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ -free. Chad Brewbaker, after completing his explorations of lonesum matrices, began to suspect that such matrices might be another pB class, and he communicated this question to Beáta Bényi and Péter Hajnal. They found [5] that, yes indeed, $|\mathcal{Q}_{m,n}| = B_{m,n}$, because the same recurrence, (4.6), is satisfied— but with m and n reversed.

Suppose a matrix of $\mathcal{Q}_{m,n+1}$ has an all-zero first column, or only one 1 in that column. Then its remaining columns can be any element of $\mathcal{Q}_{m,n}$. On the other hand, if the matrix has $t > 1$ rows that begin with 1, those rows must be identical; so there are $\binom{m}{t}$ times $|\mathcal{Q}_{m+1-t,n}|$ matrices for every such t .

That argument leads to a very simple bijection from $\mathcal{Q}_{m,n}$ to $\mathcal{G}_{m,n}$: We simply work from left to right. In any column with more than one 1, zero out the entries to the right of all but the bottommost 1. And to go back, go from right to left, copying entries from the right of the bottommost 1.

For example, the element of $\mathcal{Q}_{7,4}$ that corresponds to the transpose of the matrix $\Gamma_{4,7,999999}$ in (11.1) is

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}. \quad (12.1)$$

(Notice that transposition is a bijection between $\mathcal{G}_{m,n}$ and $\mathcal{G}_{n,m}$, while top-to-bottom reflection is an automorphism of $\mathcal{Q}_{m,n}$.)

Of course $\mathcal{Q}_{m,n}$ is bijectively equivalent to $m \times n$ matrices that are $\left\{\begin{pmatrix} 11 \\ 01 \end{pmatrix}, \begin{pmatrix} 01 \\ 11 \end{pmatrix}\right\}$ -free, $\left\{\begin{pmatrix} 00 \\ 01 \end{pmatrix}, \begin{pmatrix} 01 \\ 00 \end{pmatrix}\right\}$ -free, or $\left\{\begin{pmatrix} 00 \\ 10 \end{pmatrix}, \begin{pmatrix} 10 \\ 00 \end{pmatrix}\right\}$ -free, as well as to the $n \times m$ matrices that are $\left\{\begin{pmatrix} 01 \\ 11 \end{pmatrix}, \begin{pmatrix} 10 \\ 11 \end{pmatrix}\right\}$ -free, $\left\{\begin{pmatrix} 11 \\ 01 \end{pmatrix}, \begin{pmatrix} 11 \\ 10 \end{pmatrix}\right\}$ -free, $\left\{\begin{pmatrix} 10 \\ 00 \end{pmatrix}, \begin{pmatrix} 01 \\ 00 \end{pmatrix}\right\}$ -free, or $\left\{\begin{pmatrix} 00 \\ 10 \end{pmatrix}, \begin{pmatrix} 00 \\ 01 \end{pmatrix}\right\}$ -free.

13. Lonesum matrices redux. In §10 and §11, we've constructed bijections from $\mathcal{L}_{m,n}$ to $\mathcal{P}_{m,n}$ to $\mathcal{G}_{m,n}$. Thus, we know how to start with an $m \times n$ matrix that's lonesum and find a corresponding $m \times n$ matrix that's strongly Γ -free. It's instructive now to study the *composition* of those bijections, because the resulting process can be understood in terms of matrices alone, without reference to the intermediate parades that gave us the original insights.

Given a matrix $\Gamma \in \mathcal{G}_{m,n}$, we shall find a corresponding matrix $\Lambda \in \mathcal{L}_{m,n}$, where the correspondence is reversible. (In fact, if Γ happens to be $\Gamma_{m,n,r}$, which is the strongly Γ -free matrix of rank r , then Λ will be $\Lambda_{m,n,r}$, the top-to-bottom reflection of the matrix that corresponds via the bijection of §10 to the parade $\Pi_{m,n,r}$, which was defined in §5! But we won't need to "look under the hood" at that machinery, nor will we even need to know anything about parades when defining this bijection.)

To start, if $m = 1$ we simply let $\Lambda = \Gamma$. Suppose therefore that $m > 1$. Let Γ' be the bottom $m - 1$ rows of Γ , and let Λ' be the matrix that corresponds to Γ' . We'll give a rule that tells how to obtain Λ by putting an appropriate new row above Λ' , and by making a simple adjustment to Λ' itself.

The construction depends, of course, on the top row of Γ . If that top row is entirely zero, the top row of Λ will also be zero. Otherwise let Γ have 1s in columns $j_1 < \dots < j_t$ of its top row. We know that columns j_1, \dots, j_{t-1} of Γ' will all be zero. (By induction, those columns of Λ' will also be zero.)

If column j_t of Λ' is all 1s, we simply let the top row of Λ be the top row of Γ . Otherwise let r_λ be the maximum row sum over all rows that have 0 in column j_t of Λ' ; and let the top row of Λ be the top row of Γ *plus* row λ of Λ' . (All rows with the same sum must be identical, by the lonesum property.)

Finally, modify Λ by changing columns j_1, \dots, j_{t-1} so that they all are copies of column j_t .

For example, the matrices Λ obtained from $\Gamma_{4,7,999999}$ and $\Gamma_{4,7,701101}$ in (11.1) and (11.2) are

$$\Lambda_{4,7,999999} = \begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \Lambda_{4,7,701101} = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{pmatrix}. \quad (13.1)$$

To go back from Λ to Γ , we just need to identify j_1, \dots, j_t . Of all the columns with 1 in the top row, they're the ones whose column sum is smallest.

Exercise 13.1. What lonesum matrix corresponds to the plurisum matrix in exercise 11.1?

Exercise 13.2. What lonesum matrix corresponds to the "typical" matrix in the answer to exercise 11.2?

Exercise 13.3. What matrix Γ corresponds to Λ when Λ is a Ferrers diagram? (A Ferrers diagram has 1 in column j of row i if and only if $j \leq p_i$, where $p_1 \geq \dots \geq p_m$ is a given sequence of nonnegative integers.)

Exercise 13.4. For how many matrices does this bijection between $\mathcal{L}_{m,n}$ and $\mathcal{G}_{m,n}$ yield $\Lambda = \Gamma$?

Exercise 13.5. Let $\Lambda \in \mathcal{L}_{m,n}$ correspond to $\Gamma \in \mathcal{G}_{m,n}$ as above. True or false: (a) Row i of Λ is zero if and only if row i of Γ is zero. (b) Column j of Λ is zero if and only if column j of Γ is zero. (c) If matrices $\hat{\Lambda}$ and $\hat{\Gamma}$ are obtained from Λ and Γ by deleting all of the zero rows and all of the zero columns, then $\hat{\Lambda} = \hat{\Gamma}$.

14. Max-closed relations. Our final example comes from yet another branch of discrete mathematics, the *binary relations* between two linearly ordered sets X and Y . Any relation ‘ \smile ’ between $X = \{x_1, \dots, x_m\}$ and $Y = \{y_1, \dots, y_n\}$ is equivalent to an $m \times n$ matrix, whose entry in row i and column j is 1 if x_i and y_j satisfy the relation (written ‘ $x_i \smile y_j$ ’), but it’s 0 if they do not (‘ $x_i \not\smile y_j$ ’).

We assume that the elements are linearly ordered, with $x_1 < \dots < x_m$ and $y_1 < \dots < y_n$. The relation is called “max-closed” when it satisfies the condition

$$x_i \smile y_j \text{ and } x_{i'} \smile y_{j'} \text{ implies } x_{\max\{i,i'\}} \smile y_{\max\{j,j'\}}. \quad (14.1)$$

Max-closed relations were introduced in 1995 by Jeavons and Cooper [15], who observed that constraint satisfaction problems can be solved efficiently whenever they involve only max-closed constraints. (In the special case $m = n = 2$, a constraint satisfaction problem is a Boolean satisfiability problem, and max-closed constraints correspond to so-called “dual Horn clauses.”)

Definition (14.1) puts us back into familiar territory, because it amounts to saying that a matrix defines a max-closed relation if and only if the matrix is $\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \right\}$ -free.

Let $\mathcal{M}_{m,n}$ be the set of all max-closed relations between ordered domains of sizes m and n . We shall prove that $|\mathcal{M}_{m,n}| = B_{m,n}$ by constructing a bijection between $\mathcal{M}_{m,n}$ and $\mathcal{G}_{m,n}$, as suggested by Ira Gessel.

Gessel’s bijection is, in fact, amazingly simple, once you’ve seen it. Take any matrix in $\mathcal{M}_{m,n}$ and rotate it by 180° . This gives a $\left\{ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \right\}$ -free matrix (which is equivalent to a *min-closed* relation). Now repeatedly take any 2×2 submatrix that has the form $\begin{pmatrix} 1 & 1 \\ 1 & * \end{pmatrix}$ and change it to $\begin{pmatrix} 0 & 1 \\ 1 & * \end{pmatrix}$. The resulting matrix is strongly Γ -free(!).

(One can see, in fact, that each matrix is entirely characterized by the positions of the bottommost 1 in each column and the positions of the rightmost 1 in each row. Those positions are the same in both matrices.)

For example, the matrices of $\mathcal{G}_{4,7}$ in (11.1) and (11.2) correspond to the min-closed relations

$$\begin{pmatrix} 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}, \quad (14.2)$$

which correspond, in turn, to the max-closed relations

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix} \quad (14.3)$$

in $\mathcal{M}_{m,n}$.

Exercise 14.1. What max-closed 16×20 matrix corresponds to the “typical” parade Π of (2.2)?

Max-closed binary relations are also equivalent to another well-studied class of combinatorial patterns, called *permutation tableaux*, in the cases where the tableau is a rectangular matrix. In this context, Einar Steingrímsson and Lauren Williams have devised an interesting “zig-zag” bijection between $\mathcal{M}_{m,n}$ and $\mathcal{E}_{m,n}$; see [32] and the exposition in [20, exercise 5.1.4–45].

15. Taking stock. We’ve now met eight basic classes of combinatorial objects that all are enumerated by the pB numbers. In alphabetic order of their class names, they are

$$\begin{aligned} &\mathcal{E}_{m,n}, \text{ weak-excedance-free permutations (§9);} \\ &\mathcal{G}_{m,n}, \text{ strongly } \Gamma\text{-free matrices (§11);} \\ &\mathcal{L}_{m,n}, \text{ lonesum matrices (§10, §13);} \\ &\mathcal{M}_{m,n}, \text{ max-closed relations (§14);} \\ &\mathcal{O}_{m,n}, \text{ acyclic bipartite orientations (§7);} \\ &\mathcal{P}_{m,n}, \text{ parades (§2–§6);} \\ &\mathcal{Q}_{m,n}, \Gamma\text{-and-L-free matrices (§12);} \\ &\mathcal{V}_{m,n}, \text{ doubly bounded permutations (§8).} \end{aligned} \quad (15.1)$$

The matrices, orientations, and relations are of size $m \times n$; the parades and permutations are of length $m + n$. We know how to map every element of any one class bijectively to a corresponding element of any other class, at least in principle. So there’s much more to explore!

16. Companion numbers. Two other symmetrical arrays of numbers turn out to be intimately related to the pB numbers (0.1), and they too play important roles in our story. They're called $C_{m,n}$ and $D_{m,n}$, and they look like this:

$$\begin{array}{rcccccccc}
C_{m,n} & n=0 & n=1 & n=2 & n=3 & n=4 & n=5 & n=6 & n=7 \\
m=0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
m=1 & 1 & 3 & 7 & 15 & 31 & 63 & 127 & 255 \\
m=2 & 1 & 7 & 31 & 115 & 391 & 1267 & 3991 & 12355 \\
m=3 & 1 & 15 & 115 & 675 & 3451 & 16275 & 72955 & 316275 \\
m=4 & 1 & 31 & 391 & 3451 & 25231 & 164731 & 999391 & 5767051 \\
m=5 & 1 & 63 & 1267 & 16275 & 164731 & 1441923 & 11467387 & 85314915 \\
m=6 & 1 & 127 & 3991 & 72955 & 999391 & 11467387 & 116914351 & 1096832395 \\
m=7 & 1 & 255 & 12355 & 316275 & 5767051 & 85314915 & 1096832395 & 12764590275
\end{array} \tag{16.1}$$

$$\begin{array}{rcccccccc}
D_{m,n} & n=0 & n=1 & n=2 & n=3 & n=4 & n=5 & n=6 & n=7 \\
m=0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
m=1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
m=2 & 0 & 1 & 5 & 13 & 29 & 61 & 125 & 253 \\
m=3 & 0 & 1 & 13 & 73 & 301 & 1081 & 3613 & 11593 \\
m=4 & 0 & 1 & 29 & 301 & 2069 & 11581 & 57749 & 268381 \\
m=5 & 0 & 1 & 61 & 1081 & 11581 & 95401 & 673261 & 4306681 \\
m=6 & 0 & 1 & 125 & 3613 & 57749 & 673261 & 6487445 & 55213453 \\
m=7 & 0 & 1 & 253 & 11593 & 268381 & 4306681 & 55213453 & 610093513
\end{array} \tag{16.2}$$

Their bivariate exponential generating functions are respectively

$$\sum_{m,n \geq 0} C_{m,n} \frac{w^m z^n}{m! n!} = \frac{e^{w+z}}{(e^w + e^z - e^{w+z})^2}; \quad \sum_{m,n \geq 0} D_{m,n} \frac{w^m z^n}{m! n!} = \frac{1}{e^w + e^z - e^{w+z}}. \tag{16.3}$$

Thus if $H(w, z)$ is the generating function for the D array, the corresponding generating function for the C array turns out to be $G(w, z)H(w, z)$, where $G(w, z)$ is the generating function (1.7) for the B array. Another noteworthy generating function [23] is

$$\sum_{m,n \geq 1} C_{m-1,n-1} \frac{w^m z^n}{m! n!} = \ln \frac{1}{e^w + e^z - e^{w+z}}. \tag{16.4}$$

Furthermore, since $e^w + e^z - e^{w+z} = 1 - (e^w - 1)(e^z - 1)$, these generating functions lead to the explicit formulas

$$C_{m,n} = \sum_{k \geq 0} k! (k+1)! \left\{ \begin{matrix} m+1 \\ k+1 \end{matrix} \right\} \left\{ \begin{matrix} n+1 \\ k+1 \end{matrix} \right\}; \quad D_{m,n} = \sum_{k \geq 0} k!^2 \left\{ \begin{matrix} m \\ k \end{matrix} \right\} \left\{ \begin{matrix} n \\ k \end{matrix} \right\}. \tag{16.5}$$

A bit of fooling around reveals that C 's make B 's, and D 's make C 's, in simple ways:

$$B_{m,n} = C_{m-1,n} + C_{m,n-1} + [m=n=0]; \tag{16.6}$$

$$C_{m,n} = D_{m+1,n} + D_{m,n+1} + D_{m,n} - [m+1=n=0] - [m=n+1=0]. \tag{16.7}$$

(The value of $B_{m,n}$, $C_{m,n}$, or $D_{m,n}$ is zero whenever $m < 0$ or $n < 0$.) There also are elegant relations involving binomial coefficients:

$$B_{m+1,n} = \binom{n}{0} C_{m,n} + \binom{n}{1} C_{m,n-1} + \cdots + \binom{n}{n} C_{m,0}; \tag{16.8}$$

$$C_{m,n-1} = \binom{n}{0} D_{m,n} + \binom{n}{1} D_{m,n-1} + \cdots + \binom{n}{n} D_{m,0} - [m=n=0]. \tag{16.9}$$

And there are recurrence relations, analogous to (4.6) but simpler:

$$C_{0,n} = [n \geq 0]; \quad C_{m+1,n} = \binom{n+1}{1} C_{m,n} + \binom{n+1}{2} C_{m,n-1} + \binom{n+1}{3} C_{m,n-2} + \cdots + \binom{n+1}{n+1} C_{m,0}; \tag{16.10}$$

$$D_{0,n} = [n=0]; \quad D_{m+1,n} = \binom{n+1}{1} D_{m,n} + \binom{n+1}{2} D_{m,n-1} + \binom{n+1}{3} D_{m,n-2} + \cdots + \binom{n+1}{n+1} D_{m,0}. \tag{16.11}$$

We will see that all of these relations have nice combinatorial explanations. (Formula (16.6) is equivalent to Equation (9) in a paper [2] by Arakawa and Kaneko, written in 1999; of course their reasoning at the time was based on analytic number theory, not combinatorics. Formulas (16.7)–(16.11) were discovered combinatorially by Beáta Bényi and Peter Hajnal [5, Observation 1, Theorem 5, and Theorem 18]. In fact, [5] was a pioneering paper in which the C and D arrays were introduced and their significant connections to the pB numbers $B_{m,n}$ were first revealed.)

Before proceeding further, let's pause to observe that (16.6) proves a nonobvious property of the pB numbers:

$$\sum_{k=0}^n (-1)^k B_{k,n-k} = [n=0]. \quad (16.12)$$

Proof: The sum $(C_{-1,n} + C_{0,n-1}) - (C_{0,n-1} + C_{1,n-2}) + \cdots + (-1)^{n-1} (C_{n-2,1} + C_{n-1,0}) + (-1)^n (C_{n-1,0} + C_{n,-1})$ telescopes to zero when $n > 0$. (Relation (16.12) is due to Arakawa and Kaneko [3]; it was discovered independently by Bényi and Hajnal [4], who gave a combinatorial proof.)

What kinds of combinatorial patterns are enumerated by $C_{m,n}$? We shall see that there are lots and lots of them. For example, $C_{m,n}$ is the number of permutations $p_0 p_1 \dots p_{m+n}$ of the $m+n+1$ numbers $[0 \dots m+n]$ that have the doubly bounded property

$$j - m \leq p_j \leq j + n \quad \text{for } 0 \leq j \leq m+n. \quad (16.13)$$

(It just like (8.1), but now there's one more element.)

The main applications of $C_{m,n}$ are, however, to parades and to the other combinatorial patterns that we've been studying. For example, it turns out that *the number of parades with m girls and n boys that begin with a girl is $C_{m-1,n}$, and the number of parades that begin with a boy is $C_{m,n-1}$* . That's why $B_{m,n} = C_{m-1,n} + C_{m,n-1}$, except when $m = n = 0$ (formula (16.6)).

Exercise 16.1. Prove, by a direct counting argument, that the number of parades of m girls and n boys that begin with a girl is $\sum_{k \geq 0} k! (k+1)! \binom{m}{k+1} \binom{n+1}{k+1}$. (This is $C_{m-1,n}$, by (16.5).)

One nice way to prove the fact just stated is to use an indirect argument. Suppose $X_{m,n}$ is the number of parades in $\mathcal{P}_{m,n}$ that begin with a girl. Then the number of parades that have exactly k boys before the first girl is $\binom{n}{k} X_{n-k}$, because $n-k$ boys appear after that girl, and there are $\binom{n}{k}$ ways to choose the initial block of k boys. Consequently

$$B_{m,n} = \binom{n}{0} X_{m,n} + \binom{n}{1} X_{m,n-1} + \binom{n}{2} X_{m,n-2} + \cdots + \binom{n}{n} X_{m,0}. \quad (16.14)$$

This system of equations can be solved for the unknown X 's in terms of the B 's; in fact, we have

$$X_{m,n} = \binom{n}{0} B_{m,n} - \binom{n}{1} B_{m,n-1} + \binom{n}{2} B_{m,n-2} - \cdots + (-1)^n \binom{n}{n} B_{m,0}, \quad (16.15)$$

by well-known properties of the "binomial transform." We need not know this explicit solution (16.15), however; all we need to know is that (16.14) defines the X 's uniquely. Because formula (16.8) tells us that $X_{m,n} = C_{m-1,n}$ is a solution to (16.14).

In a similar way we can interpret the elements $D_{m,n}$ of array (16.2), by showing that *the number of parades with m girls and n boys that begin with a girl and end with a boy is $D_{m,n}$* . Indeed, if $Y_{m,n}$ of the $C_{m,n-1}$ parades that end with a boy also begin with a girl, then $\binom{n}{k} Y_{m,n-k}$ of those parades must have k boys before the first girl. Hence $C_{m,n-1} = \sum_{k \geq 0} \binom{n}{k} Y_{m,n-k}$; and we must have $Y_{m,n} = D_{m,n}$ because of (16.9). (See the answer to exercise 16.1, for comments about the case $m = n = 0$.)

And by applying the bijections that we know, between $\mathcal{P}_{m,n}$ and seven other classes of combinatorial patterns, we can deduce that the C and D numbers are important for those classes as well.

17. Counting significant subclasses. Let's therefore seize the opportunity to exploit interesting subclasses of the eight classes in (15.1). Beginning with $\mathcal{P}_{m,n}$, the class of all m -girl- n -boy parades, let $\mathcal{P}_{m,n}^{\leftarrow}$ be the subclass of parades that start with a girl, and let $\mathcal{P}_{m,n}^{\rightarrow}$ be the subclass of parades that end with a girl; let $\mathcal{P}_{m,n}^{\leftarrow}$ be the subclass of parades that end with a boy, and let $\mathcal{P}_{m,n}^{\rightarrow}$ be the subclass of parades that start with a boy. Also let $\mathcal{P}_{m,n}^{\times} = \mathcal{P}_{m,n}^{\leftarrow} \cap \mathcal{P}_{m,n}^{\rightarrow}$, $\mathcal{P}_{m,n}^{\times} = \mathcal{P}_{m,n}^{\leftarrow} \cap \mathcal{P}_{m,n}^{\leftarrow}$, $\mathcal{P}_{m,n}^{\times} = \mathcal{P}_{m,n}^{\rightarrow} \cap \mathcal{P}_{m,n}^{\rightarrow}$, and $\mathcal{P}_{m,n}^{\times} = \mathcal{P}_{m,n}^{\leftarrow} \cap \mathcal{P}_{m,n}^{\rightarrow}$ be the doubly constrained subclasses. We have just proved that

$$|\mathcal{P}_{m,n}^{\leftarrow}| = |\mathcal{P}_{m,n}^{\rightarrow}| = C_{m-1,n}; \quad |\mathcal{P}_{m,n}^{\leftarrow}| = |\mathcal{P}_{m,n}^{\rightarrow}| = C_{m,n-1}; \quad |\mathcal{P}_{m,n}^{\times}| = |\mathcal{P}_{m,n}^{\times}| = D_{m,n}. \quad (17.1)$$

Exercise 17.1. Describe $\mathcal{P}_{m,n}^{\leftarrow}$, which is the intersection $\mathcal{P}_{m,n}^{\leftarrow} \cap \mathcal{P}_{m,n}^{\rightarrow}$, and compute $|\mathcal{P}_{m,n}^{\leftarrow}|$.

OK. But what about $|\mathcal{P}_{m,n}^{\times}|$, the number of parades that both start and end with a girl? Good question; the answer turns out to be a bit subtle. If g_m is first, she can be followed by any element of $\mathcal{P}_{m-1,n}^{\times}$. Otherwise the parade can be mapped bijectively into an element of $\mathcal{P}_{m-1,n+1}^{\times}$ (!). The idea is to replace g_m by b_{n+1} ; and if she wasn't last in the parade, to move the entire block of boys that followed her all the way to the end. (This bijection comes from [5]; another proof follows by considering permanents.) Thus we have

$$|\mathcal{P}_{m,n}^{\times}| = D_{m-1,n} + D_{m-1,n+1}. \quad (17.2)$$

(Adding $|\mathcal{P}_{m,n}^{\times}|$ proves (16.7).) A symmetrical argument, interchanging girls and boys, yields

$$|\mathcal{P}_{m,n}^{\times}| = D_{m,n-1} + D_{m+1,n-1}. \quad (17.3)$$

And even more is true, because we can specify the sizes of the first and/or last blocks: *The number of m -girl- n -boy parades that begin with a block of k girls is $\binom{m}{k} |\mathcal{P}_{m-k,n}^{\rightarrow}|$. The number of parades that begin with a block of k girls and end with a block of l boys is $\binom{m}{k} \binom{n}{l} |\mathcal{P}_{m-k,n-l}^{\times}|$. The number of parades that begin with a block of k girls and end with a block of l girls is $\binom{m}{k} \binom{m-k}{l} |\mathcal{P}_{m-k-l,n}^{\times}|$, if $n > 0$. And so on.*

Turning now to other classes, the bijection between parades and acyclic bipartite orientations in §7 maps $\mathcal{P}_{m,n}^{\leftarrow}$ into $\mathcal{O}_{m,n}^{\leftarrow}$, the set of acyclic orientations whose sources are girls. Similarly, the orientations of $\mathcal{O}_{m,n}^{\rightarrow}$ are those whose sources are boys; those of $\mathcal{O}_{m,n}^{\leftarrow}$ are those whose sinks are girls; those of $\mathcal{O}_{m,n}^{\rightarrow}$ are those whose sinks are boys.

Exercise 17.2. How many acyclic orientations of $K_{m,n}$ have a unique source vertex and a unique sink vertex?

By the simple bijection between \mathcal{O} and \mathcal{L} in §10, $\mathcal{L}_{m,n}^{\leftarrow}$ is the set of lonesum matrices that have at least one row of all 1s; that's equivalent, by the lonesum property, to having no all-0 columns. So there are $C_{m-1,n}$ such lonesum matrices.

Similarly, $\mathcal{L}_{m,n}^{\rightarrow}$ contains the lonesum matrices with at least one all-0 row, or equivalently no all-1 columns. (They are the complements of the matrices in $\mathcal{L}_{m,n}^{\leftarrow}$.)

Most interesting, perhaps, is $\mathcal{L}_{m,n}^{\times}$, the class of lonesum matrices for which every row and column is nonzero, of which there are $D_{m,n}$.

Exercise 17.3. How many lonesum matrices have at least one all-0 row and at least one all-1 row?

Exercise 17.4. What does the *order* of a parade tell us about the number of distinct row and column sums in the corresponding lonesum matrix?

If we choose a lonesum matrix at random, each entry is equally likely to be 0 or 1, because the class of lonesum matrices is closed under complementation. Thus the average row sum of a matrix in $\mathcal{L}_{m,n}$ is $n/2$. What is the probability, $p_{m,n}(r)$, that a given row sum is exactly r , when $0 \leq r \leq n$? This probability distribution is the same for each of the m rows; and by our bijection between $\mathcal{L}_{m,n}$ and $\mathcal{P}_{m,n}$, we have

$$p_{m,n}(r) = P_{m,n}(r)/B_{m,n}, \quad (17.4)$$

where $P_{m,n}(r)$ is the number of parades in which girl g_m precedes exactly r boys.

There are $\binom{n}{r}$ ways to choose the boys that follow g_m ; we shall focus on one of those choices. If k girls precede g_m in such a parade, there are $B_{k,n-r}$ ways to arrange the boys and girls that precede g_m , for each of the $\binom{m-1}{k}$ to choose those girls. And there are $C_{m-1-k,r-1}$ ways to arrange the boys that follow her, because she is never followed by a girl. Hence

$$P_{m,n}(r) = \begin{cases} [r=n], & \text{if } m = 0; \\ B_{m-1,n}, & \text{if } m > 0 \text{ and } r = 0; \\ \binom{n}{r} \sum_{k=0}^{m-1} \binom{m-1}{k} B_{k,n-r} C_{m-1-k,r-1}, & \text{otherwise.} \end{cases} \quad (17.5)$$

For example, the counts $(P_{3,4}(0), \dots, P_{3,4}(4)) = (146, 252, 270, 252, 146)$ tell us how many of the 1066 parades of $\mathcal{P}_{3,4}$ yield various row sums of $\mathcal{L}_{3,4}$. (It is combinatorially obvious, but *not* algebraically obvious, that $P_{m,n}(r) = P_{m,n}(n-r)$!)

Four of the eight classes in (15.1), namely $\mathcal{G}_{m,n}$, $\mathcal{L}_{m,n}$, $\mathcal{M}_{m,n}$, and $\mathcal{Q}_{m,n}$ consist of $m \times n$ matrices. We gave a simple bijection between the parades of $\mathcal{P}_{m,n}$ and the strongly Γ -free matrices of $\mathcal{G}_{m,n}$ in §11; and we gave simple bijections $\mathcal{G}_{m,n} \leftrightarrow \mathcal{Q}_{m,n}$, $\mathcal{G}_{m,n} \leftrightarrow \mathcal{L}_{m,n}$, and $\mathcal{G}_{m,n} \leftrightarrow \mathcal{M}_{m,n}$ in §12, §13, §14, respectively. The latter bijections have the nice property that they preserve all-zero rows and all-zero columns.

Therefore the eight enumerations of (17.1), (17.2), and (17.3) in terms of C and D numbers all are valid for $\mathcal{G}_{m,n}$, $\mathcal{L}_{m,n}$, $\mathcal{M}_{m,n}$, and $\mathcal{Q}_{m,n}$ as well as for $\mathcal{P}_{m,n}$, where

$$\begin{aligned} &\text{the matrices of } \mathcal{G}_{m,n}^{\leftarrow} \text{ are those with no all-0 columns;} \\ &\text{the matrices of } \mathcal{G}_{m,n}^{\rightarrow} \text{ are those with no all-0 rows;} \\ &\text{the matrices of } \mathcal{G}_{m,n}^{\leftarrow\leftarrow} \text{ are those with at least one all-0 row;} \\ &\text{the matrices of } \mathcal{G}_{m,n}^{\rightarrow\rightarrow} \text{ are those with at least one all-0 column.} \end{aligned} \tag{17.6}$$

It's a bit of a surprise that $|\mathcal{G}_{m,n}^{\leftarrow}| = |\mathcal{G}_{m,n}^{\leftarrow\leftarrow}|$, and that $|\mathcal{G}_{m,n}^{\rightarrow\rightarrow}| = |\mathcal{G}_{m,n}^{\rightarrow}|$.

The “core” matrices of $\mathcal{G}_{m,n}$ are those of $\mathcal{G}_{m,n}^{\times}$, namely the matrices Γ with at least one 1 in every row and in every column. Such matrices have a very close connection to the corresponding parades of $\mathcal{P}_{m,n}^{\times}$, namely the parades Π that begin with a girl and end with a boy. For example, the topmost 1 in column j of Γ appears in row i if and only if g_{m+1-i} is the closest girl to b_j in Π . (See exercise 11.2 and its answer.)

Exercise 17.5. If $\Gamma \in \mathcal{G}_{m,n}^{\times}$ corresponds to $\Pi \in \mathcal{P}_{m,n}^{\times}$, prove that the number of 1s in Γ is equal to $m+n-l$, where l is the number of *left-to-right minima* among the girls in Π , namely the number of girls that are younger than any girl who has appeared previously in the parade.

18. Significant subclasses of permutations. We've now seen that the C and D numbers give useful enumerations for six of the eight pB classes listed in (15.1). Of course we can apply similar methods to the remaining two classes, because of our bijection $\mathcal{P}_{m,n} \leftrightarrow \mathcal{V}_{m,n}$ between parades and doubly bounded permutations in §8, as well as our bijection $\mathcal{V}_{m,n} \leftrightarrow \mathcal{E}_{m,n}$ in §9 between those permutations and the permutations that begin with m weak excedances. (Recall that $\mathcal{V}_{m,n}$ is the class defined by

$$j - m \leq p_j \leq j + n \quad \text{for } 1 \leq j \leq m + n. \tag{18.1}$$

where $p_1 \dots p_{m+n}$ is a permutation of $\{1, \dots, m+n\}$.) Then $|\mathcal{V}_{m,n}^{\leftarrow}| = C_{m-1,n}$, and so on.

But those results are strange, because $\mathcal{V}_{m,n}^{\leftarrow}$ is a very peculiar subclass of $\mathcal{V}_{m,n}$: It consists of the permutations that satisfy not only (18.1) but also the further condition $p_{n+1}^{-1} \leq m$. (A girl comes first in parade Π if and only if a rook is placed in the left column of the S_m quadrant of matrix $Q_{m,n}$ in (8.2).)

We get a much more interesting and meaningful subclass of $\mathcal{V}_{m,n}$ by strengthening the upper bound in (18.1), replacing that condition by

$$j - m \leq p_j < j + n \quad \text{for } 1 \leq j \leq m + n. \tag{18.2}$$

The permutations in this subclass are those whose rook placements avoid the *diagonal* of submatrix S_m in $Q_{m,n}$, *not* the elements of S_m 's left column. Hence the bijection of §8 matches them to parades of $\mathcal{P}_{m,n}$ that satisfy m additional conditions: (i) g_1 doesn't begin the parade; (ii) g_{i+1} doesn't immediately follow g_i in the parade, for $1 \leq i < m$.

By evaluating the associated permanent, we can deduce that there are exactly $C_{m-1,n}$ such parades! Furthermore, there are $C_{m-1,n}$ lonesum matrices of size $m \times n$ such that (i) row 1 isn't all 1s; (ii) row $i+1$ isn't equal to row i , for $1 \leq i < m$. And there are $D_{m,n}$ if also (iii) column 1 isn't all 1s; (iv) column $j+1$ isn't equal to column j , for $1 \leq j < n$. That's not uninteresting; but it's not really what we want.

Therefore let's change the bijection, so that $\mathcal{V}_{m,n}^{\leftarrow}$ will correspond to a more natural kind of parade. Even though our bijection was based on Bender's elegant construction, we've seen that it leads to weird consequences; hence we want to replace his method by a new rook placement scheme. In order to represent any given restricted growth string $a_0 a_1 \dots a_m$ by a set of nonattacking rooks in the lower triangular matrix S_m , we shall proceed as follows, for $1 \leq i \leq m$: (i) If $a_i = 0$, place a rook in column i of row i . (ii) Otherwise if

$a_i \notin \{a_1, \dots, a_{i-1}\}$, put no rook in row i . (iii) Otherwise, if $j < i$ is maximum such that $a_i = a_j$, put a rook in column j of row i . For example, the seven rook placements in (8.3) now have the following new look:

$$\begin{array}{ccccccc} \begin{pmatrix} \blacksquare & 0 & 0 \\ 1 & \blacksquare & 0 \\ 1 & 1 & 1 \end{pmatrix} & \begin{pmatrix} \blacksquare & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & \blacksquare \end{pmatrix} & \begin{pmatrix} \blacksquare & 0 & 0 \\ 1 & 1 & 0 \\ 1 & \blacksquare & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & \blacksquare & 0 \\ 1 & 1 & \blacksquare \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ 1 & \blacksquare & 0 \\ \blacksquare & 1 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ \blacksquare & 1 & 0 \\ 1 & 1 & \blacksquare \end{pmatrix} & \begin{pmatrix} 1 & 0 & 0 \\ \blacksquare & 1 & 0 \\ 1 & \blacksquare & 1 \end{pmatrix} \\ 0001 & 0010 & 0011 & 0100 & 0101 & 0110 & 0111 \end{array} \quad (18.3)$$

Yes, it works! When the upper triangular matrix S_n^T in $Q_{m,n}$ is treated in the same way as S_m , the permutations that satisfy (18.1) have four significant subclasses:

$$\begin{aligned} & \text{the permutations of } \mathcal{V}_{m,n}^{\leftarrow} \text{ are those with } p_j < j + n \text{ for all } j; \\ & \text{the permutations of } \mathcal{V}_{m,n}^{\rightarrow} \text{ are those with } p_j > j - m \text{ for all } j; \\ & \text{the permutations of } \mathcal{V}_{m,n}^{\leftarrow} \text{ are those with } p_j = j - m \text{ for at least one } j; \\ & \text{the permutations of } \mathcal{V}_{m,n}^{\rightarrow} \text{ are those with } p_j = j + n \text{ for at least one } j. \end{aligned} \quad (18.4)$$

In cases $\mathcal{V}_{m,n}^{\leftarrow}$ and $\mathcal{V}_{m,n}^{\rightarrow}$, we can also count the number of permutations in which equality holds for exactly k values of j , by looking at the k th term of either formula (16.10) or the analogous formula that arises when m and n are interchanged.

The enumerations of $|\mathcal{V}_{m,n}| = B_{m,n}$, $|\mathcal{V}_{m,n}^{\leftarrow}| = C_{m-1,n}$, $|\mathcal{V}_{m,n}^{\rightarrow}| = C_{m,n-1}$, and $|\mathcal{V}_{m,n}^{\times}| = D_{m,n}$ were completed long ago by Katalin Vesztergombi [33], who actually proved a considerably stronger three-parameter result: *The number of permutations $p_1 \dots p_N$ of $[1 \dots N]$ that satisfy*

$$j - m < p_j < j + n \quad \text{for } 1 \leq j \leq N, \quad (18.5)$$

when $0 \leq m, n \leq N$ and $m + n \geq N$, is $f(m + n - N, N - m, N - n)$, where

$$f(r, s, t) = \sum_{k=0}^s (-1)^{k+s} (r+k)! (r+k)^t \left\{ \begin{matrix} s+1 \\ k+1 \end{matrix} \right\}; \quad \sum_{r,s,t} f(r, s, t) \frac{x^r w^s z^t}{r! s! t!} = \frac{1}{e^w + e^z - (1+x)e^{w+z}}. \quad (18.6)$$

Cases $r = 0, 1$, and 2 yield the D, C , and B arrays, respectively. So we should perhaps really be calling all of these arrays ‘‘Vesztergombi numbers,’’ not pB numbers. (The case $r = 0$ had already been resolved by Kaplansky and Riordan in 1946 [19, §8].)

Finally, the bijection (9.3) connects $\mathcal{V}_{m,n}$ with $\mathcal{E}_{m,n}$, the class of permutations $q_1 \dots q_{m+n}$ such that

$$q_j \geq j, \quad \text{for } 1 \leq j \leq m \quad \text{and} \quad q_j \leq j, \quad \text{for } m+1 \leq j \leq m+n. \quad (18.7)$$

As before, we get significant subclasses:

$$\begin{aligned} & \text{the permutations of } \mathcal{E}_{m,n}^{\leftarrow} \text{ are those with } q_j > j \text{ for } 1 \leq j \leq m; \\ & \text{the permutations of } \mathcal{E}_{m,n}^{\rightarrow} \text{ are those with } q_j < j \text{ for } m+1 \leq j \leq m+n; \\ & \text{the permutations of } \mathcal{E}_{m,n}^{\leftarrow} \text{ are those with } q_j = j \text{ for at least one } j \leq m; \\ & \text{the permutations of } \mathcal{E}_{m,n}^{\rightarrow} \text{ are those with } q_j = j \text{ for at least one } j > m. \end{aligned} \quad (18.8)$$

Notice, for example, that $\mathcal{E}_{m,n}^{\leftarrow}$ is the class of *excedance-first* permutations, where $q_j > j$ if and only if $j \leq m$. (They were first enumerated by Richard Ehrenborg and Einar Steingrímsson in 2000 [10]; see also [9].) And $\mathcal{E}_{m,n}^{\rightarrow}$ is the class of permutations where $q_j \geq j$ if and only if $j \leq m$, etc. We’ve got their number.

Exercise 18.1. How many permutations of $\mathcal{E}_{m,n}$ satisfy not only (18.7) but also (a) $q_j \neq m$, for $1 \leq j \leq m$? (b) the conditions of (a) and also $q_j \neq m+1$, for $m+1 \leq j \leq m+n$?

19. Open problems. The present notes have probably only scratched the surface of a much larger theory that deserves to be explored further. Indeed, the topic seems well suited to REU sessions (Research Experiences for Undergraduates), since none of the results above require graduate-level mathematics.

Here are a few of the main questions that the author didn’t have time to answer as he was preparing this writeup. (He hopes that anybody who is able to answer any of them, even partially, will let him know. *Progress that has been reported to him appears below, at the conclusion of this note.*)

Problem 1. The following identity is a consequence of the generating functions (1.7) and (16.3):

$$C_{m,n} = \sum_{k=0}^m \sum_{l=0}^n \binom{m}{k} \binom{n}{l} B_{k,l} D_{m-k,n-l}. \quad (19.1)$$

Does it have a nice combinatorial proof?

Problem 2. There are $\binom{16}{2} = 120$ ways to choose two distinct 2×2 matrices Φ and Ψ of 0s and 1s. We've seen above, in §10, §11, §12, and §14, that $B_{m,n}$ is the number of binary $m \times n$ matrices that are $\{\Phi, \Psi\}$ -free, for more than a dozen pairs $\{\Phi, \Psi\}$.

On the other hand, that's not true for all such pairs. For example, the number of 2×3 matrices that are $\left\{\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right\}$ -free is not 46 but 44.

Make an exhaustive study: Exactly which of those pairs $\{\Phi, \Psi\}$ lead to pB numbers?

Problem 3. Some of the many automorphisms of parades were discussed in §6 above. Determine the number of equivalence classes (that is, the number of "essentially different" parades), under various important subgroups of those automorphisms.

Problem 4. The structure of several of the pB classes we've studied is strongly reminiscent of Young tableaux. (For example, lonesum matrices are the matrices obtainable from Ferrers diagrams under row and column permutation, and a Ferrers diagram has the shape of a Young tableau. Also, the matrices of min-closed relations are uniquely determined by the positions of the rightmost 1s in each row and the bottommost 1s in each column.) Is there a nice way to represent every parade as some sort of a Young tableau?

Problem 5. The probability distribution $P_{m,n}(r)$ in (17.4) is not unimodal. (For example, it seems to be true that $P_{m,n}(2) < P_{m,n}(3) > P_{m,n}(4)$ for $8 \leq n \leq 3m/2$; at least, one can readily check this relation for all $m \leq 50$.)

In fact, when m and n are large, the distribution appears to be almost uniform: The nonuniformity ratio $\max_{0 \leq r \leq n} P_{m,n}(r) / \min_{0 \leq r \leq n} P_{m,n}(r)$ is less than 2.867 when $m = 50$ and $n = 100$; it's less than 1.095 when $m = 200$ and $n = 100$!

Investigate the behavior of the nonuniformity ratio when m and n are large.

Problem 6. Matrices that are Γ -free in the sense of Lubiw [26], but not *strongly* Γ -free, were mentioned near the beginning of §11. Is there a good way to count the (weakly) Γ -free matrices of size $m \times n$? (Jeremy Spinrad [30] has derived asymptotic results.)

Problem 7. Consider parades in which there are infinitely many girls and infinitely many boys. (Only finitely many boys appear between any two girls; only finitely many girls appear between any two boys.) What do the bijections above tell us about infinite matrices of 0s and 1s?

Problem 8. One of the most natural ways to generalize the class of girl-and-boy parades is to consider parades where there are more than two kinds of marchers. For example, we can consider a parade that's formed with l clowns, m girls, and n boys. Let there be $B_{l,m,n}$ of them, with $B_{0,m,n} = B_{m,n}$.

It's not difficult to see that the bijection in §7 generalizes to this three-parameter case, with essentially the same argument as before: *The number of (l, m, n) -parades of clowns, girls, and boys is the number of acyclic orientations of the complete tripartite graph $K_{l,m,n}$.*

For example, when $(l, m, n) = (1, 2, 2)$, there are $B_{1,2,2} = 78$ such parades, of which $(14, 16, 18, 16, 14)$ have the clown in position $(1, 2, 3, 4, 5)$, respectively. Clearly $B_{1,1,n} = 2 \cdot 3^n$. Further small values are

$$\begin{aligned} B_{1,2,3} &= 330; B_{2,2,2} = 426; B_{2,2,3} = 2286; B_{2,2,4} = 12090; B_{2,2,5} = 63198; B_{2,2,6} = 327306; \\ B_{2,3,3} &= 15402; B_{2,3,4} = 101502; B_{3,3,3} = 122190. \end{aligned} \tag{19.2}$$

The triple exponential generating function is in fact a nice extension of (1.7) and (4.3):

$$\sum_{l,m,n \geq 0} B_{l,m,n} \frac{v^l}{l!} \frac{w^m}{m!} \frac{z^n}{n!} = \frac{e^{v+w+z}}{e^{w+z} + e^{z+v} + e^{v+w} - 2e^{v+w+z}} = \frac{1}{e^{-v} + e^{-w} + e^{-z} - 2}. \tag{19.3}$$

(We get (1.7) and (4.3) by setting $v = 0$.) It is a consequence of exercises 3.109 and 5.6 in [31].

There's an interesting way to derive (19.3) more directly. Let

$$X(g, b) = 1 + g + b + gb + bg + gbg + bgb + \dots \tag{19.4}$$

be the formal sum of all words (aka "strings") of g 's and b 's with no repeated letters. Clearly $X(g, b) = (1 + g)(1 + b)/(1 - gb)$ when $gb = bg$. Similarly, let

$$X(c, g, b) = 1 + c + g + b + cg + cb + gc + gb + bc + bg + cgb + cbg + \dots \tag{19.5}$$

be the formal sum of words without $\{cc, gg, bb\}$ when c 's are added into the mix. Since

$$X(c, g, b) = X(c, X(g, b) - 1), \quad (19.6)$$

we have

$$X(c, g, b) = \frac{(1+c)X(g, b)}{1-cX(g, b)+c} = \frac{(1+c)(1+g)(1+b)}{1-cg-cb-gb-2cgb}; \quad (19.7)$$

consequently $X(e^u - 1, e^w - 1, e^z - 1)$ is the generating function (19.3).

Generalizing to t classes of marchers, let $X(a_1, \dots, a_t)$ be the words without double letters. Then

$$X(a_1) = 1 + a_1; \quad X(a_0, a_1, \dots, a_t) = X(a_0, X(a_1, \dots, a_t) - 1). \quad (19.8)$$

Setting $Y(a_1, \dots, a_t) = X(1/a_1 - 1, \dots, 1/a_t - 1)$, the recurrence becomes simpler:

$$Y(a_1) = 1/a_1; \quad Y(a_0, a_1, \dots, a_t) = Y(a_0, Y(a_1, \dots, a_t)). \quad (19.9)$$

Hence, by induction on t ,

$$Y(a_1, \dots, a_t) = a_1 + \dots + a_t - (t-1). \quad (19.10)$$

And the generating function for t -dimensional parades is $X(e^{z_1} - 1, \dots, e^{z_t} - 1) = Y(e^{-z_1}, \dots, e^{-z_t}) = 1/(e^{-z_1} + \dots + e^{-z_t} - (t-1))$, in agreement with the answer to exercise 5.6 in [31].

Surely there is a good generating function for the number of (l, m, n) -parades that begin with a clown, etc.

Problem 9. Parades of clowns, girls, and boys do *not* correspond to “lonesum tensors,” which are the $l \times m \times n$ tensors (t_{ijk}) of 0s and 1s that are uniquely determined by their $l + m + n$ “slice sums” $\sum_{j,k} t_{ijk}$, $\sum_{i,k} t_{ijk}$, and $\sum_{i,j} t_{ijk}$. If $L_{l,m,n}$ is the number of such tensors, we have $L_{1,m,n} = B_{m,n}$; also, for example,

$$\begin{aligned} L_{1,2,3} = 46; L_{2,2,2} = 104; L_{2,2,3} = 644; L_{2,2,4} = 3668; L_{2,2,5} = 19964; L_{2,2,6} = 105764; \\ L_{2,3,3} = 6712; L_{2,3,4} = 60968; L_{3,3,3} = 126164. \end{aligned} \quad (19.11)$$

Those counts have nothing in common with the numbers $B_{l,m,n}$ in (19.2).

The 104 lonesum tensors of size $2 \times 2 \times 2$ exclude not only the cases where one of the slices is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ or $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, but also tensors like $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, whose slice sums match those of $\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$, $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$, and $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$.

Tensors of size $2 \times m \times n$ that have no nonlonesum $2 \times 2 \times 2$ subtensors appear to be lonesum (at least in cases that are small enough to check by brute force). But that's not true of the tensor

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \quad (19.12)$$

All 27 of its $2 \times 2 \times 2$ subtensors are lonesum; yet it has the same slice sums $(2, 4, 7, 2, 4, 7, 2, 4, 7)$ after each of its layers has been transposed.

We might also consider “weakly lonesum” tensors, whose $lm + mn + nl$ sums with two coordinates fixed are enough to determine them completely. If there are $\widehat{L}_{l,m,n}$ of them, we have $\widehat{L}_{1,m,n} = 2^{mn}$ and

$$\begin{aligned} \widehat{L}_{1,2,3} = 64; \widehat{L}_{2,2,2} = 254; \widehat{L}_{2,2,3} = 4006; \widehat{L}_{2,2,4} = 62834; \widehat{L}_{2,2,5} = 980926; \widehat{L}_{2,2,6} = 15251714; \\ \widehat{L}_{2,3,3} = 246214; \widehat{L}_{2,3,4} = 14904586; \widehat{L}_{3,3,3} = 112589750. \end{aligned} \quad (19.13)$$

Is there any decent way to compute $L_{l,m,n}$ and/or $\widehat{L}_{l,m,n}$ for larger values of l , m , and n ?

Problem 10. Max-closed *ternary* relations (see [15]), between variables (X, Y, Z) whose respective domains have sizes (l, m, n) , are equivalent to $l \times m \times n$ tensors (t_{ijk}) of another kind. In this case $t_{ijk} = 1$ means that the relation holds when $X = x_i$, $Y = y_j$, and $Z = z_k$. Therefore the tensor is max-closed if and only if $t_{ijk} = t_{i'j'k'} = 1$ implies $t_{\max\{i,i'\}\max\{j,j'\}\max\{k,k'\}} = 1$, for all $i, j, k, i', j',$ and k' .

Let $M_{l,m,n}$ be the number of $l \times m \times n$ max-closed tensors, and note that $M_{1,m,n} = B_{m,n}$. Also, for example,

$$\begin{aligned} M_{1,2,3} &= 46; M_{2,2,2} = 122; M_{2,2,3} = 898; M_{2,2,4} = 6086; M_{2,2,5} = 39394; M_{2,2,6} = 248102; \\ M_{2,3,3} &= 13094; M_{2,3,4} = 165534; M_{3,3,3} = 468732. \end{aligned} \quad (19.14)$$

So they too differ from what we've seen before.

In this case the tensors for $l, m, n > 1$ can be defined by forbidding them to contain any excluded $2 \times 2 \times 2$ subtensors, of which there are nine species:

$$\begin{pmatrix} *1 \\ 10 \end{pmatrix} \begin{pmatrix} ** \\ ** \end{pmatrix}; \begin{pmatrix} ** \\ ** \end{pmatrix} \begin{pmatrix} *1 \\ 10 \end{pmatrix}; \begin{pmatrix} *1 \\ ** \end{pmatrix} \begin{pmatrix} 10 \\ ** \end{pmatrix}; \begin{pmatrix} ** \\ *1 \end{pmatrix} \begin{pmatrix} ** \\ 10 \end{pmatrix}; \begin{pmatrix} ** \\ 1* \end{pmatrix} \begin{pmatrix} 1* \\ 0* \end{pmatrix}; \begin{pmatrix} ** \\ *1 \end{pmatrix} \begin{pmatrix} *1 \\ *0 \end{pmatrix}; \begin{pmatrix} *1 \\ ** \end{pmatrix} \begin{pmatrix} ** \\ 10 \end{pmatrix}; \begin{pmatrix} ** \\ 1* \end{pmatrix} \begin{pmatrix} *1 \\ *0 \end{pmatrix}; \begin{pmatrix} ** \\ *1 \end{pmatrix} \begin{pmatrix} 1* \\ *0 \end{pmatrix}. \quad (19.15)$$

Exactly 122 of the $2 \times 2 \times 2$ tensors contain none of these patterns.

The fact that max-closed relations can be defined by excluded subtensors means that BDD methods [21, §7.1.4] can be used to enumerate a few cases that go beyond the numbers in (19.14):

$$\begin{aligned} M_{2,3,5} &= 1930146; M_{2,3,6} = 21415862; M_{2,4,4} = 3648460; M_{2,4,5} = 71217786; \\ M_{3,3,4} &= 13432658; M_{3,3,5} = 339809648; M_{3,4,4} = 798255356; M_{4,4,4} = 114446643198. \end{aligned} \quad (19.16)$$

But BDD methods are limited because they chew up lots of space.

Wanted: A way to compute (say) $M_{9,9,9}$.

Problem 11. Develop a series of tutorial videos, aimed at college and/or advanced high school math students, in order to teach the fascinating theory of parades and their connections to combinatorial classes and special numbers. There are great opportunities here for animation, costumes, and music!

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Appendix. The fundamental recurrence $\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$ leads to an easy way to find the r th sample $1 \leq x_0 < \dots < x_{k-1} \leq n$ from the interval $[1..n]$, given $0 \leq k \leq n$ and $0 \leq r < \binom{n}{k}$:

$$\begin{aligned} &\text{“While } k > 0, \text{ test if } r < \binom{n-1}{k}; \text{ if not, set } r \leftarrow r - \binom{n-1}{k}, \\ &k \leftarrow k - 1, \text{ and } x_k \leftarrow n; \text{ then in either case set } n \leftarrow n - 1.” \end{aligned} \quad (\text{A.1})$$

The samples are obtained in colexicographic order: $b_1 \dots b_{k-1} b_k, b_1 \dots b_{k-2} b_{k-1} b_{k+1}, b_1 \dots b_{k-2} b_k b_{k+1}, \dots, b_{n-k+1} b_{n-k+2} \dots b_n$. Conversely,

$$\text{rank}(x_0 x_1 \dots x_{k-1}) = \binom{x_0 - 1}{1} + \binom{x_1 - 1}{2} + \dots + \binom{x_{k-1} - 1}{k}. \quad (\text{A.2})$$

(Theorem 7.2.1.3L in [21] traces this formula to Ernesto Pascal in 1887.)

Here is an algorithm that can be used to transform (5.3) into (5.4). It finds the parade Π of $m+1$ girls and n boys that extends a given parade Π' of m and n' boys by a given subset $\{b_{x_0}, \dots, b_{x_{k-1}}\}$. Here $n' = n$ if $k = 0$, otherwise $n' = n + 1 - k$. The boy represented by x_{k-1} is called “Max” in the comments below. His former place in the parade will be replaced by g_{m+1} followed by the subset that he leads. The algorithm is conceptually simple; but, as usual, God (or the devil) is in the details.

The parades are represented by two digits strings, $s_0 \dots s_{m+1}$ for the girls and $t_0 \dots t_n$ for the boys, as discussed in §2. For example, the digit strings for the parade $\Pi_{4,7,999999}$ in (5.4) are $s_0 \dots s_4 = 02331$ and $t_0 \dots t_7 = 02110302$. (Initially only $s_0 \dots s_m, t_0 \dots t_{n'}$, and $x_0 \dots x_{k-1}$ are given. All operations take place within those arrays, without needing any auxiliary memory.)

We assume that partition Π' has order d . In other words, both of the ordered set partitions initially have $d+1$ nonempty blocks. Block 0 for the girls corresponds to set $S_0 \cup \{g_0\}$, but block 0 for the boys corresponds to set $T_{d+1} \cup \{b_0\}$, as in §2 and §3 above. The algorithm increases d by 1 if the output parade Π has higher order than Π' .

- X1.** [Empty case?] If $k = 0$, go to step X10.
- X2.** [Is Max alone?] Set $\mu \leftarrow x_{k-1} - (k - 1)$, $p \leftarrow t_\mu$, and $q \leftarrow -1$. Then for $1 \leq j \leq n'$, set $q \leftarrow q + 1$ if $t_j = p$. Go to step X4 if $q = 0$.
- X3.** [Split block p .] Set $q \leftarrow 1$ and $d \leftarrow d + 1$. If $p = 0$, set $t_j \leftarrow d$ for $1 \leq j \leq n'$.
- X4.** [Begin the loop.] Set $i \leftarrow n'$, $j \leftarrow n$, and $l \leftarrow k - 2$.
- X5.** [Loop done?] (At this point $i \leq j$.) Go to X8 if $i = 0$.
- X6.** [Bypass the subset, except Max.] While $l \geq 0$ and $j = x_l$, set $l \leftarrow l - 1$, $j \leftarrow j - 1$, and repeat this step.
- X7.** [Update t_j .] Set $p' \leftarrow t_i$. If $q = 1$ and $p' > p > 0$, set $t_j \leftarrow p' + 1$; otherwise set $t_j \leftarrow p'$. Then set $i \leftarrow i - 1$, $j \leftarrow j - 1$, and return to X5. (Boy b_j has been renamed.)
- X8.** [Update the subset.] For $0 \leq l < k$, set $t_{x_l} \leftarrow 0$ if $p = 0$, otherwise set $t_{x_l} \leftarrow p + q$.
- X9.** [Update the girls.] If $p = 0$, set $s_{m+1} \leftarrow d$. Otherwise, if $q = 0$, set $s_{m+1} \leftarrow p - 1$. Otherwise, set $s_{m+1} \leftarrow p$, and also set $s_j \leftarrow s_j + 1$ for all $j \in [1..m]$ with $s_j \geq p$. Terminate the algorithm.
- X10.** [Extend by \emptyset .] (We will change Π' by simply appending g_{m+1} at the end.) If t_1, \dots, t_n are all nonzero, go to X12. (Otherwise there was at least one boy at the end.)
- X11.** [Increase the order.] Set $d \leftarrow d + 1$. Then set $t_j \leftarrow d$ for all $j \in [1..n]$ with $t_j = 0$.
- X12.** [Append g_{m+1} .] Set $s_{m+1} \leftarrow d$ and terminate the algorithm. ■

An experimental implementation of this algorithm, together with another program that unrankes parades according to the nonrecursive scheme of §3, can be found online [22]. That website also contains programs for the corresponding ranking algorithms.

into column 17, since b_{15} was originally named b_{17} . And so forth. Here's the glorious final result:

$$\Gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

The author's program RANK-PARADE2 (see [22]) displays types with original boys' names as well as their current names, thereby making it easy to "read off" the rows of this matrix when fed the parade (2.2).

11.3. The solution to the recurrence $x_{1,n} = n$, $x_{m+1,n} = \max\{1 + x_{m,n}, 2 + x_{m,n-1}, \dots, n + x_{m,1}\}$ is $x_{m,n} = m + n - 1$. (This recurrence is derived from the construction of $\Gamma_{m,n,r}$.)

13.1.
$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

13.2. The text's bijection from Γ to Λ should be applied to the matrix Γ of answer 11.2 from bottom to top. That is, we start with the bottom row of Γ , getting our initial version of the bottom row of Λ ; then we use the next-to-last and last rows of Γ to get an initial version of the bottom two rows of Λ , namely $\begin{pmatrix} 00000000010000010001 \\ 00000000010000000001 \end{pmatrix}$; and so on. The result is

$$\Lambda = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

[The row sums (r_1, \dots, r_{16}) are $(15, 3, 5, 3, 4, 11, 15, 11, 7, 16, 3, 11, 18, 14, 18, 15)$ and the column sums (c_1, \dots, c_{20}) are $(16, 7, 11, 11, 2, 0, 7, 10, 10, 6, 13, 16, 0, 7, 3, 2, 12, 10, 16, 10)$. Of course they determine Λ uniquely, since Λ is lonesum.]

Since exercise 11.2 obtained the original matrix Γ from the parade (2.2), we might expect Λ to be precisely the lonesum matrix that corresponds to (2.2) according to the bijection of §10. In fact, however, (2.2)

corresponds to the top-to-bottom *reflection* of Λ , because the top row of Λ corresponds to the orientation of arcs from g_{16} , not g_1 . (The parade that does correspond to Λ is obtained from (2.2) by swapping $g_j \leftrightarrow g_{17-j}$.)

13.3. Let $p_0 = \infty$ and $p_{m+1} = 0$. If $p_{i-1} > p_i = \dots = p_j > p_{j+1}$ and $i \leq j$, put 1 into columns p_{j+1}, \dots, p_i of row i , and put 1 into column p_i (only), in rows $i+1, \dots, j$. (Notice that going from Λ to Γ in the inverse bijection requires us to zero out columns j_1, \dots, j_{i-1} of Γ' .)

13.4. If Λ' is nonzero and the top row of Γ is nonzero, we must have $t = 1$. Hence the matrices with $\Lambda = \Gamma$ are of three kinds: (i) at most one 1; (ii) more than one 1, all in a single row or a single column; (iii) more than one 1 in row i and more than one 1 in column j , with a 1 in cell (i, j) but no 1s in column j below row i . The number of possibilities is therefore the sum of (i) $mn + 1$; (ii) $m(2^n - n - 1) + (2^m - m - 1)n$; (iii) $\sum_{i=2}^m (2^{i-1} - 1) \sum_{k=2}^n \binom{n}{k} k = (2^m - m - 1)(2^{n-1} - 1)n$.

13.5. (a) True. (b) True. (c) True. It's a marvelously simple bijection.

14.1. Again we start from the matrix Γ in answer 11.2. This time we get, after rotating 180° ,

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

16.1. In a parade of order k , the $k! \binom{m+1}{k+1}$ ordered partitions of $\{g_0, \dots, g_m\}$ into nonempty blocks S_0, S_1, \dots, S_k include $(k+1)! \binom{m}{k+1}$ with $|S_0| > 1$ and $k! \binom{m}{k}$ with $|S_0| = 1$. (See (3.4).)

Similarly, the number of parades that begin with a girl and end with a boy is $\sum_{k \geq 0} (k+1)!^2 \binom{m}{k+1} \binom{n}{k+1}$. This agrees with $D_{m,n}$ in (16.5) when $k+1 \mapsto k$. (The term for $k = 0$ in (16.5) is omitted; but that's OK because it is zero—except in the peculiar case when $m = n = 0$, and we ponder what it means for a girl to come first and a boy to come last in the parade of 0 girls and 0 boys! For $D_{m,n}$, it's more accurate to say that “a boy doesn't come first and a girl doesn't come last.”)

17.1. It's the empty set, except when $m = n = 0$. ($\mathcal{P}_{m,n}^\times$ doesn't really mean “girl last”; it means “not boy last.”) So $|\mathcal{P}_{m,n}^\times| = [m=n=0]$.

17.2. Such orientations correspond to parades whose first and last blocks have size 1. There are exactly $m(m-1)(D_{m-2,n-1} + D_{m-1,n-1}) + mnD_{m-1,n-1} + nmD_{m-1,n-1} + n(n-1)(D_{m-1,n-2} + D_{m-1,n-1})$ such.

17.3. $|\mathcal{L}_{m,n}^\times| = D_{m-1,n} + D_{m-1,n+1}$.

17.4 There are d distinct row sums, plus 1 if a girl comes first (namely, one for each block of girls). There are d distinct column sums, plus 1 if a boy comes last (namely, one for each block of boys).

17.5. Mark the topmost 1 in every column. When row i of Γ is constructed (by removing g_{m+1-i} from the parade, as in the bijection of §11), we can prove by induction on i that all of the 1s in that row have been marked. Suppose the rightmost 1 in row i appears in column j ; mark the uppermost unmarked 1 in that column. (The other columns have nothing but 0s below row i .) Such an unmarked 1 will exist if and only if g_{m+1-i} is not a left-to-right minimum. (The left-to-right minima in parade (2.2) are g_2 and g_1 . Boys b_6 and b_{13} at the beginning of that parade have caused columns 6 and 13 of the matrix in exercise 11.2 to be zero.)

18.1 By modifying (9.2) and taking the permanent, we get (a) $C_{m-1,n}$; (b) $D_{m,n}$. (Too bad these permutations aren't likely to arise in practical problems.)

Progress on the “open problems.” Some of the unsolved problems mentioned above were resolved shortly after this note was first posted online. At the other extreme, several of them may be beyond reach; who knows? The author won’t be able to keep track of the current status of each one, but he will try to update this note periodically in order to mention pertinent things that he has learned.

One interesting surprise, *not* related to the stated problems, was Richard Stanley’s conjecture—proved by Ira Gessel—that the determinant of the $N \times N$ matrix formed from the numbers $B_{m,n}$ for $m, n < N$ is exactly $1!^2 2!^2 \dots (N-1)!^2$. (Stanley knows that number well, as the discriminant of $x(x+1) \dots (x+N-1)$.) The determinant of the corresponding matrix of D numbers is the same; and the determinant of the corresponding matrix of C numbers is almost the same, but multiplied by $N!$. (To which formula one is tempted to add ‘!’.)

Filip Stappers [https://archive.org/details/parades_problems] has found a nice solution to Problem 1, interpreting the left side of (19.1) as the number of parades of $m+1$ girls and n boys that begin with a girl, and obtaining the right side by considering the position of g_1 . He also exploited similar ideas to prove further convolution formulas.

In the same posting, Stappers also resolved Problem 2 completely, showing that exactly 25 of the 120 possibilities lead to pB numbers. All of those cases were already studied above, or equivalent by rotation or by reflection or by swapping $0 \leftrightarrow 1$.

Similar results on Problem 2 were reported by Ho Boon Suan. Beáta Bényi reported that she too had come to those conclusions some years ago, in joint (unpublished) work with Chad Brewbaker.

Beáta mentioned that the analog of (16.12), with $C_{k,n-k}$ in place of $B_{k,n-k}$, leads to the Genocchi numbers(!). See her paper with Matthieu Josuat-Vergès, “Combinatorial proof of an identity on Genocchi numbers,” *Integers* **24** (2021) #A21.7.6; that paper cites [18] as the source of the identity. In a sequence of papers with other coauthors, she has discovered interesting connections between pB numbers and many other branches of combinatorial mathematics.

Ho Boon Suan has made tantalizing progress on Problem 6, enumerating the (non-strong) Γ -free matrices. Denoting their number by $G_{m,n}$, he found empirically that the doubly exponential generating function has the form $\sum_{m \geq 0} \frac{w^m}{m!} e^{(m+1)z} p_m(z)$, where $p_m(z)$ is a polynomial of degree $\binom{m}{2}$ with unimodal positive coefficients, at least for $m \leq 4$. A symmetrical generating function presumably also exists, since $G_{m,n} = G_{n,m}$.

Progress on Problem 10 was first reported by Filip Stappers in August 2024, who found the generating function

$$\sum_{n \geq 0} M_{2,2,n} z^n = \frac{(1-z)(1-8z^2)}{(1-6z)(1-4z)(1-3z)(1-2z)}.$$

(Again see https://archive.org/details/parades_problems.)