A conjecture about noncrossing paths

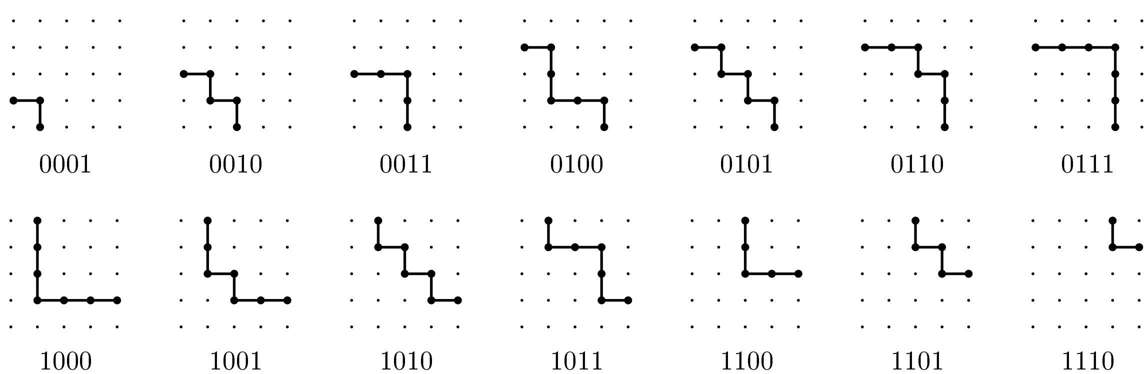
(Don Knuth, 6 February 2019)

For each of the $2^{n+1}-2$ sequences $x = x_0x_1\ldots x_n$ of 0s and 1s that are not constant, define a sequence of points as follows:

$$\text{Let } p_0 = (n+1,0); \qquad p_{j+1} = \begin{cases} p_j + (-1,0), & \text{if } x_j = 0; \\ p_j + (0,1), & \text{if } x_j = 1; \end{cases} \quad \text{for } 0 \le j \le n.$$

Notice that p_{n+1} always equals $(\nu x, \nu x)$, where $\nu x = x_0 + x_1 + \cdots + x_n$.

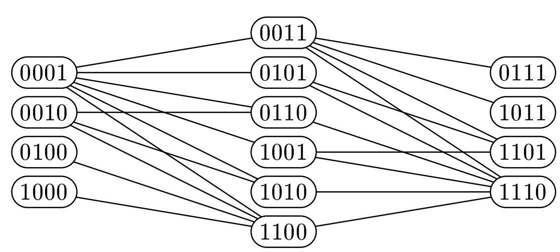
Let j be minimum with $x_j \neq x_0$, and let $P(x)$ be the path that runs from p_j to p_{j+1} to $\cdots$ to p_n to p_{n+1} to p'_n to $\cdots$ to p'_{j+1} to p'_j, where $(a,b)' = (b,a)$. For example, here are the 14 paths that arise when $n = 3$:



Notice that exactly $\binom{n+1}{k}$ paths cross the line $x = y$ at point (k,k), for $1 \le k \le n$.

We get a pleasant decomposition of an $(n+1) \times (n+1)$ square into polyominoes from every set of n paths $P(x)$ that do not cross each other. Such a set will of course contain one path $P(x^{(k)})$ that passes through (k,k), for each k. When $n = 3$ there are in fact 16 ways to choose noncrossing $x^{(1)}$, $x^{(2)}$, and $x^{(3)}$:

$x^{(1)}$	$x^{(2)}$	$x^{(3)}$	
0001	0011	0111	(a)
0001	0011	1011	(b)
0001	0011	1101	(c)
0001	0011	1110	(d)
0001	0101	1101	(e)
0001	0101	1110	(f)
0001	0110	1110	(g)
0001	1001	1101	(h)
0001	1001	1110	(g′)
0001	1010	1110	(f′)
0001	1100	1110	(d′)
0010	0110	1110	(h′)
0010	1010	1110	(e′)
0010	1100	1110	(c′)
0100	1100	1110	(b′)
1000	1100	1110	(a′)



These choices correspond to the set of all left-to-right paths in the tripartite graph shown.

The number of such noncrossing paths has the following values $T(n)$ for small n:

$$\begin{array}{rcccccccccc} n & = & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ T(n) & = & 1 & 2 & 5 & 16 & 66 & 352 & 2431 & 21760 & 252586 & 3803648 \end{array}$$

Of course the numbers that arise in a fairly natural problem such as this are likely to have arisen also in connection with other problems. And sure enough, these numbers are the opening values of sequence M1499 in the 1995 *Encyclopedia of Integer Sequences* by Sloane and Plouffe — now known as sequence A005157 in the wonderful *Online Encyclopedia of Integer Sequences* (OEIS), where it's called "The number of totally symmetric plane partitions that fit in an $n \times n \times n$ box."

The evidence is indeed overwhelming that $T(n)$ is given by OEIS A005157. So that is my conjecture. Ideally I'd like to see a bijection between every noncrossing set as described above and a suitable totally symmetric plane partition. Failing that, somebody should at least be able to prove that $T(n)$ has the same known value, which is

$$T(n) \;=\; \prod_{1\le i\le j\le k\le n} \frac{i+j+k-1}{i+j+k-2}.$$

[See R. L. Stanley, *Enumerative Combinatorics* **2** (1999), exercise 7.103a.]

The graph illustrated for $n = 3$ is easily characterized for general n: If x and y are binary sequences with $\nu y = \nu x + 1$, paths $P(x)$ and $P(y)$ will be disjoint if and only if we have

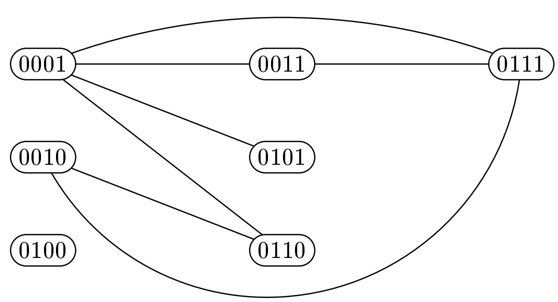
$$s_j = x_0 + \cdots + x_j \le y_0 + \cdots + y_j = t_j \text{ for } 0 \le j \le n; \quad \text{and } s_j = t_j \text{ implies } s_j \in \{0, j+1\}.$$

We can use this fact to generate the graphs easily, and to compute the number of routes from left to right, with essentially no difficulty when $n \le 30$, say.

Even better would be to find a bijection between the noncrossing paths above and the very similar noncrossing paths found by John R. Stembridge ["The enumeration of totally symmetric plane partitions," *Advances in Mathematics* **111** (1995), 227–243, Lemma 1.2], because he did establish a fairly simple bijection between *his* paths and the plane partitions in question.

In fact, Stembridge used just the first $2^n - 1$ of the paths above — the ones that begin with 0. Let's redraw the graph above, restricting it to only those seven vertices, and explicitly showing edges between every pair of vertices whose paths don't cross:

independent sets	
$\emptyset$	(A)
$\{0001\}$	(B)
$\{0001, 0011\}$	(C)
$\{0001, 0011, 0111\}$	(D)
$\{0001, 0101\}$	(E)
$\{0001, 0110\}$	(F)
$\{0001, 0111\}$	(G)
$\{0010\}$	(H)
$\{0010, 0110\}$	(I)
$\{0010, 0111\}$	(J)
$\{0011\}$	(K)
$\{0011, 0111\}$	(L)
$\{0100\}$	(M)
$\{0101\}$	(N)
$\{0110\}$	(O)
$\{0111\}$	(P)



All one has to do is find a (generalizable) one-to-one correspondence between the left-to-right routes in the former graph and the independent sets in this new graph. My guess is that independent set (A) should correspond to route (a′), and (D) should correspond to (a); but after that I'm stuck.

Another idea is to use Theorem 3.1 of Stembridge's seminal paper about "Nonintersecting paths, Pfaffians, and plane partitions" [*Advances in Mathematics* **83** (1990), 96–131], which uses Pfaffians to express the number of noncrossing paths in a fairly general setting. For example, when $n = 3$, that theorem will tell us the number of noncrossing paths to the vertices $\{u_1, u_2, u_3, u_4\} = \{(1,1), (2,2), (3,3), (4,4)\}$ from *any* of the vertices in the set $I = \{(1,0), (2,0), (3,0), (4,1), (4,2), (4,3), (4,4)\}$; it tells us that this number is

$$\operatorname{pf}\begin{pmatrix} 11 & 12 & 4 \\ & 11 & 6 \\ & & 4 \end{pmatrix} = 16.$$

(The entries '11', '12', ..., '4' in this Pfaffian are easily obtained as sums of 2×2 determinants of binomial coefficients.) Maybe linear algebra will deduce the value of $T(n)$, as Stembridge did in his paper of 1995.

But at the moment I'm stumped. I cannot believe that the conjecture might be false, nor do I expect that its eventual proof will turn out to involve any "deep thought."

(Maybe this new characterization of $T(n)$ will lead to a solution to the famous open problem about a q-generating function for the totally symmetric plane partitions. Who knows?!)

Addendum (8 February 2019)

Good news: Nikolai Beluhov established my conjecture almost immediately, by constructing a beautiful bijection. I hope he will publish it soon. Very roughly, one can regard his bijection as making three copies of the noncrossing paths, cutting them into triangles along the line $x = y$, and then assembling the six pieces kaleidoscope-fashion, so that one can essentially "see" a totally symmetric plane partition!

When $n = 3$, his construction yields the following bijection with Stembridge's independent sets:

a	b	c	d	e	f	g	h	a′	b′	c′	d′	e′	f′	g′	h′
D	L	J	I	G	F	N	P	A	B	H	M	E	K	O	C