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CALVIN C. GOTLIEB, Editor-in-Chief
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Reply to: California Institute of Technology
Department of Mathematics
Pasadena, California

February 1, 1967

Dr. Marcel P. Schutzenberger
Institut Blaise Pascal
23 Rue de Maroc
Paris 19, France

Dear Marco:

Here are some detailed comments on your "Remark." (I have mixed trivial comments with nontrivial ones, but all are intended to improve the quality of the paper.)

1. Page 0, lines -4 and -3. What is an "ots governemental or private research sponsor"?

2. Page 1, line 5. This is maybe a nice joke, but the meaning of these symbols is "standard" only to a very small group of people. The notation N is especially ambiguous, as to whether it includes 0 or not. I didn't find any of these symbols in the papers of the automata theorists who ought to read this article. The first sentence should therefore be changed to something like "In what follows, Z denotes as usual the set of all integers, N denotes the non-negative integers, and Q denotes the rational numbers; we let X^*...". Actually you never use the symbol Q again, although I presume you meant to say ω is a polynomial over Q on page 5.

3. For the ACM Journal, change references to [1], [2], etc.

4. Why bother to say $|\alpha f| > 1$ for all $f \in XX^*$ when this follows trivially if it is true for all $f \in X$?

5. It is important to add a little mere motivation, so the readers will see more easily how your results generalize the Minsky-Papert and other theorems. Therefore I suggest adding the following comments just after the second paragraph:

"For example, if $X = \{0,1\}$, $\alpha 0 = \alpha 1 = 2$, and $\rho 0 = 0, \rho 1 = 1$, then ρf is the number whose binary representation is f."

Remark on acceptable sets of numbers
JACM 1968 300-303

6. Page 2. It seems inappropriate to mention Mike Harrison here since the theorem you quote is in my opinion the most well-known theorem about context-free languages. It certainly is not obscure. Perhaps you mean Mike suggested that the theorem might apply to this problem.

7. Page 2. In the statement of this theorem you must say $h' \neq e$ or else it is trivially true.

8. Page 3, it is more consistent to write "$\rho h' \cdot (1-\alpha h')^{-1} \cdot \alpha g$" on line 1, and $k \in N$ on line 9.

9. Page 4. I am displaying my ignorance: is the Hadamard product of two rational functions always a rational function? I'd like you to direct me to a proof of this, which I haven't seen yet. I always used roots of unity, which leaves the rational field. Anyway I see it is a rational function since it just amounts to a different choice of g'', h'', g', h', g, and h_0.

10. Page 4. I don't see how you apply which theorem of Polya.

The function $z^m\left(\frac{1}{1-qz^{2d}} + \frac{z^d}{1-q^2z^{2d}} + qz^{3d}\right)$ satisfies all the hypotheses you state for $\bar{\rho}'$, but not the conclusion.

As I understand Polya's result, all rational functions whose power series involve only finitely many primes can be obtained by the formula

$$\frac{u_1}{1-v_1z^m} + \frac{u_2z}{1-v_2z^m} + \cdots + \frac{u_mz^{m-1}}{1-v_mz^m} + p(z)$$

where $p(z)$ is a polynomial in z and the u's and v's are rational.

In our application we know $z^{-m}\bar{\rho}'(z^{1/d})$ has the form

$\frac{b_0'}{1-z} + \frac{b_1'}{1-c_1z} + \frac{b_2'}{1-c_2z}$. Since this vanishes for $z = \infty$, it must have the form above with $p(z) = 0$. So we can find another subsequence for which $\bar{\rho}''$ has the form you stated (namely every md-th term).

I recommend your using this form of proof since (a) it explains what you are doing and (b) uses only very elementary facts about power series besides Polya's theorem.

11. Page 5. There are at least two serious errors in your second example. In the first place you can't zero out the coefficient of t^{d-1} in $\omega(t)$ simply by replacing t by t - h when h is an integer. This affects the coefficient by an integer multiple of d. Hence you can't obviously assume π has such small degree. In the second place the "lim sup" estimate on the top of page 6 is wrong at least in the following case:

$d = 1$, $\pi = 0 = b_1$, $\zeta n = c'a^n + c''$. Then $\zeta(n+d) - a\zeta n = c''(1-a) \neq 0$.

In fact the second example is wrong if we take $\omega(t) = t$, $X = \{0,1\}$, $\alpha 0 = \alpha 1 = -2$, $\rho 0 = 0$, $\rho 1 = 1$. This is the fairly well known negative binary number system which is capable of representing all integers without signs.

12. Furthermore you must explain how to get the asymptotic estimate $\zeta n = c'a^{n/d} + c'' + o(1)$. I can't see how to get the second term to be c'' here.

13. In fact the whole discussion on page 6 needs a great deal of amplification. It is unclear how you obtain almost all the assertions on this page. There is no harm in showing more of the calculation, since some fairly delicate maneuvering is evidently being used.

14. On page 7, give a reference to Kleene's generalization of Jungen's theorem.

In summary, I believe you should (a) correct the errors, (b) supply more details for the proof of the second example (under suitably weakened hypotheses) so that I can convince myself there are no more errors in it, and (c) add a little bit more expository material as suggested above, before ACM publishes this "Remark."

The first duty of the ACM Journal is to its readers, and we want to teach them something, not merely record for posterity a set of assertions for which the method of calculation is mysteriously left unrevealed by the author. Your paper needs a few more concessions to the intended audience; everyone who reads it will appreciate it much more if it is clear what you are doing.

I'm looking forward to visiting you in May. What is Nivat's address?

Sincerely,

Donald E. Knuth
Programming Languages editor

DEK:ld

May 1974

Dear Don,

It is a pleasure to help such a paragon of scholarship as you are! But here I am afraid we are in very complicated situation:

Your lemma 13 originally due to

Fine and Wilf (Proc. Am M. Soc 16 1965 p 109 – 114)

It comes as an Exo in the beautifull theory of

A. Lentin. Equations dans les monoides libres. Gauthier-Villars Paris 1972 Mouton La Haye (a hard to read thing)

Relevant material is included in several thesis around us, also in

A. Lentin & MPS. Combi Math & Applic. Conf. At Chapell Hill 1967 and in a recent (feb or march?) Note aux Comptes Rendus of mine.

Let me apologize for self quotation.

Let me say that word w is ~~is~~ a <u>ρ-sesquipower</u> of word v iff w is an initial segment of length $\rho|v|$ ($|v|$ = length of v) of some power of v. (e.g. for alphabet $\{a, b\}$, word $w = abbabba$ is a $\frac{7}{3}$-sesquipower of $v = abb$). Then we have that if w is a ρ-sesquipower of v and a σ-sesquipower of u, a n. a. s. c for u and v to be powers of ~~the~~ common word is $1 \geq \frac{1}{\rho} + \frac{1}{\sigma}$.

This is just a reformulation

of Fine Wilf (– and weaker by forgetting the g.c.d term) but I sort of like it because it well applies to real functions (Cf Fine Wilf, again).

Incidentally, the problem of classifying intelligently strings which are (non trivially) ρ and σ-sesquipowers with $\rho, \sigma \geqslant 1 < \rho^{-1}+\sigma^{-1}$ is still unsolved (and maybe impossible).

I will <u>try</u> to get mailed to you the above mentionned thesis (but they have <u>no</u> connection ~~w~~with your paper)

Best regards

Marco

tel 64~~7~~.67-02.
<u>Note</u>: new address
97 RUE du RANELAGH. PARIS XVI

1975.02.05

STANFORD UNIVERSITY
STANFORD, CALIFORNIA 94305

COMPUTER SCIENCE DEPARTMENT

Telephone:
415-321-2300

February 5, 1975

Prof. Marco P. Schützenberger
97 Rue du Ranelagh
Paris 16, France

Dear Marco:

Thanks for your recent letter, but it was missing an important piece of data that I need!

To the best of my recollection, you once proved the following result (probably in more generality?): "Partition all words of X^+ into p disjoint subsets $X_1 \cup \cdots \cup X_p$. Then all words of length n^p or more have the form $tu_1 \ldots u_n v$, where each u_i belongs to the same X_j for some j." Your proof was very nice; this is the way I recall it: "Let $x_1 \ldots x_m$ be an arbitrary word of length $m \geq n^p$. For $0 \leq k \leq m$ let $c_k = (r_1^{(k)}, \ldots, r_p^{(k)})$ be the vector defined by the rule $r_j^{(k)} = \max\{r \mid x_1 \ldots x_k$ ends with r elements of $X_j\}$. Then $c_k \neq c_{k'}$ for $k < k'$ since $x_{k+1} \ldots x_{k'}$ is in some X_j and $r_j^{(k')} > r_j^{(k)}$. Consequently the distinct vectors $c_0, \ldots, c_m$ must include some c_k with a component $r_j^{(k)} \geq n$."

What I want to know is: Where did you publish this? I just spent two fruitless hours with Math. Reviews and couldn't find the reference.

Best regards,

Donald E. Knuth
Professor

DEK/pw

The construction can be made prettier as follows:

aabaabaac aabaabaac aabaabaa   26 letters 3^3-1

$A_1 = a^* - \phi^*$

$A_2 = \{a,b\}^* - a^*$

$A_3 = \{a,b,c\}^* - \{a,b\}^*$

No substring of the form A_j^3.

Take ~~If~~ the complete directed graph on n vertices $x_1,\dots,x_n$ with $x_i \to x_j$ iff $i<j$. Assign a color to each arc from k colors. This argument shows that there is a monochromatic path of length $\sqrt[k]{n}$. (Best possible) *

I forgot where I saw this material of Schützenberger's so I wrote him.** He replied that it wasn't published in a journal, but in Cours professé à l'Institut de Programmation en 1966/67 rédigé par J. F. Perrot. pp 1.2 – 1.6.

* $x_i \to x_j$ color t if the interval $[i,j)$ from i to j contains a multiple of k^t but not k^{t+1}

** Letter of 75.02.05 in letters-a files.

M. Harrison

QUELQUES PROBLEMES COMBINATOIRES DE LA THEORIE DES AUTOMATES

par M.P. SCHUTZENBERGER

Cours professé à l'Institut de
Programmation en 1966/1967,
rédigé par J.F. PERROT, Assistant.

A. THEOREME DE RAMSEY

-=-=-=-=-=-=-=-=-=-=-

I. Le théorème suivant a rendu célèbre le nom du mathématicien anglais F.P. RAMSEY qui le démontra en 1928 à la Société mathématique de Londres :

Soient E un ensemble, $\underline{P}_m(E)$ l'ensemble des parties de E ayant m éléments, et Θ une partition de $\underline{P}_m(E)$ en k classes : il existe, quelque soit n , un entier $R(n,m,k)$ tel que Card $(E) \geq R(n,m,k)$ entraîne l'existence d'un sous-ensemble à n éléments F de E tel que $\underline{P}_m(F)$ soit contenu tout entier dans une classe modulo Θ.

(Exposé, démonstration et bibliographie dans RYSER [8] chap. 4 ; sur les rapports avec la théorie des graphes voir ORE [7] chap. 13, sec. 5 ; on trouvera une discussion approfondie dans la thèse de C. FRASNAY [2]).

Pour $m = 1$, on a $R(n,1,k) = 1 + k(n-1)$ en vertu du principe des tiroirs selon lequel :" si $1 + k(n-1)$ éléments sont répartis entre k tiroirs, alors l'un des tiroirs renferme au moins n éléments " [2] .

Pour $m > 1$ la fonction $R(n,m,k)$ n'est pas connue.

Pour $m = 2$, nous en déduisons le résultat suivant :

THEOREME : Soient X un ensemble non vide, X^* le monoïde libre engendré par X, et une partition de l'ensemble XX^* des mots de longueur positive

<u>de X^* en k classes $A_1, A_2, \ldots, A_k$: pour tout entier n, il existe un entier $r_k(n)$ tel que, pour tout mot $w \in X^*$ de longueur supérieure à $r_k(n)$ il existe une classe A_i telle que w admette n facteurs non vides consécutifs dans A_i, i.e. $w \in X^* A_i^n X^*$.</u>

On a certainement $r_k(n) \leq R(n+1,2,k)$. Soit $E = \{1, 2, \ldots l(w)\}$; à chaque sous-ensemble à deux éléments $\{a,b\}$ de E $(a < b)$ nous associons bijectivement le facteur $w_{a,b}$ de w commençant au rang a et se terminant au rang b : la partition induite sur l'ensemble des facteurs de w par celle de XX^* permet de définir une partition de $\underline{P}_2(E)$ en au plus k classes, dont l'une au moins contient tous les sous-ensembles à deux éléments de $F \subset E, F = \{p_0, p_1, \ldots, p_n\}$ avec $p_i < p_{i+1}$. On a donc $w_{p_0, p_1} w_{p_1, p_2} \ldots w_{p_{n-1}, p_n} \in A_i^n$ pour une certaine classe A_i.

II. Nous allons maintenant préciser ce résultat en montrant directement que $r_k(n) = n^k$.

a) Cas où $k = 2$, $A_1 = A$, $A_2 = B$.

<u>Exemple</u> :

On montre que $r_2(2) = 4$, en raisonnant par l'absurde ; supposons qu'il existe un mot w de longueur supérieure ou égale à 4 et qui ne soit ni dans $X^* A^2 X^*$ ni dans $X^* B^2 X^*$: il est clair que, si w a une longueur strictement supérieure à 4, tous ses facteurs de longueur 4 possèdent la même propriété ; on peut donc supposer que w est de longueur 4, soit