## 6.5 BERNOULLI NUMBERS

The next important sequence of numbers on our agenda is named after Jakob Bernoulli (1654–1705), who discovered curious relationships while working out the formulas for sums of $m$th powers [26]. Let's write

$$S_m(n) \;=\; 1^m + 2^m + \cdots + n^m \;=\; \sum_{k=1}^{n} k^m. \qquad (6.77)$$

(Thus we have $S_m(n) = H_n^{(-m)}$ in the notation of generalized harmonic numbers.) Bernoulli looked at the following sequence of formulas and spotted a pattern:

$$\begin{aligned}
S_0(n) &= n\\
S_1(n) &= \tfrac{1}{2}n^2 + \tfrac{1}{2}n\\
S_2(n) &= \tfrac{1}{3}n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{6}n\\
S_3(n) &= \tfrac{1}{4}n^4 + \tfrac{1}{2}n^3 + \tfrac{1}{4}n^2\\
S_4(n) &= \tfrac{1}{5}n^5 + \tfrac{1}{2}n^4 + \tfrac{1}{3}n^3 - \tfrac{1}{30}n\\
S_5(n) &= \tfrac{1}{6}n^6 + \tfrac{1}{2}n^5 + \tfrac{5}{12}n^4 - \tfrac{1}{12}n^2\\
S_6(n) &= \tfrac{1}{7}n^7 + \tfrac{1}{2}n^6 + \tfrac{1}{2}n^5 - \tfrac{1}{6}n^3 + \tfrac{1}{42}n\\
S_7(n) &= \tfrac{1}{8}n^8 + \tfrac{1}{2}n^7 + \tfrac{7}{12}n^6 - \tfrac{7}{24}n^4 + \tfrac{1}{12}n^2\\
S_8(n) &= \tfrac{1}{9}n^9 + \tfrac{1}{2}n^8 + \tfrac{2}{3}n^7 - \tfrac{7}{15}n^5 + \tfrac{2}{9}n^3 - \tfrac{1}{30}n\\
S_9(n) &= \tfrac{1}{10}n^{10} + \tfrac{1}{2}n^9 + \tfrac{3}{4}n^8 - \tfrac{7}{10}n^6 + \tfrac{1}{2}n^4 - \tfrac{3}{20}n^2\\
S_{10}(n) &= \tfrac{1}{11}n^{11} + \tfrac{1}{2}n^{10} + \tfrac{5}{6}n^9 - n^7 + n^5 - \tfrac{1}{2}n^3 + \tfrac{5}{66}n
\end{aligned}$$

Can you see it too? The coefficient of $n^{m+1}$ in $S_m(n)$ is always $1/(m+1)$. The coefficient of $n^m$ is always $1/2$. The coefficient of $n^{m-1}$ is always ... let's see ... $m/12$. The coefficient of $n^{m-2}$ is always zero. The coefficient of $n^{m-3}$ is always ... let's see ... hmmm ... yes, it's $-m(m-1)(m-2)/720$. The coefficient of $n^{m-4}$ is always zero. And it looks as if the pattern will continue, with the coefficient of $n^{m-k}$ always being some constant times $m^{\underline{k}}$.

That was Bernoulli's empirical discovery. In modern notation we write the coefficients in the form

$$\begin{aligned}
S_m(n) \;&=\; \frac{1}{m+1}\left(B_0\, n^{m+1} + \binom{m+1}{1} B_1\, n^m + \cdots + \binom{m+1}{m} B_m\, n\right)\\
&=\; \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k} B_k\, n^{m+1-k}. \qquad (6.78)
\end{aligned}$$

Since $S_m(1) = 1$, the numbers $B_k$ satisfy an implicit recurrence relation,

$$\sum_{k=0}^{m} \binom{m+1}{k} B_k \;=\; m+1, \qquad \text{for all } m \geqslant 0. \tag{6.79}$$

For example, $\binom{2}{0}B_0 + \binom{2}{1}B_1 = 2$. The first few values turn out to be

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $B_n$ | 1 | $\frac{1}{2}$ | $\frac{1}{6}$ | 0 | $\frac{-1}{30}$ | 0 | $\frac{1}{42}$ | 0 | $\frac{-1}{30}$ | 0 | $\frac{5}{66}$ | 0 | $\frac{-691}{2730}$ |

(All conjectures about a simple closed form for $B_n$ are wiped out by the appearance of the strange fraction $-691/2730$.)

We can prove Bernoulli's formula (6.78) by induction on $m$, using the perturbation method (one of the ways we found $S_2(n) = \Box_n$ in Chapter 2):

$$\begin{aligned} S_{m+1}(n) + (n{+}1)^{m+1} \;&=\; \sum_{k=0}^{n} (k+1)^{m+1} \;=\; \sum_{k=0}^{n} \sum_{j=0}^{m+1} \binom{m+1}{j} k^j \\ &=\; \sum_{j=0}^{m+1} \binom{m+1}{j} \bigl(0^j + S_j(n)\bigr) \\ &=\; 1 + \sum_{j=0}^{m+1} \binom{m+1}{j} S_j(n). \end{aligned} \tag{6.80}$$

*Warning: Different authors use different notations for Bernoulli's coefficients. For example, many 20th-century reference books such as [2] say that $B_1$ is $-1/2$, not $+1/2$. Indeed, the authors of the present book followed that lead, before 2021. But we've now adopted Bernoulli's original choice, because it actually fits best with 21st-century practice.*

Let $\widehat{S}_m(n)$ be the right-hand side of (6.78); we wish to show that $S_m(n) = \widehat{S}_m(n)$, assuming that $S_j(n) = \widehat{S}_j(n)$ for $0 \leqslant j < m$. We begin as we did for $m = 2$ in Chapter 2, subtracting $S_{m+1}(n)$ from both sides of (6.80). Then we expand each $S_j(n)$ using (6.78), and regroup so that the coefficients of powers of $n$ on the right-hand side are brought together and simplified:

$$\begin{aligned} (n{+}1)^{m+1} - 1 \;&=\; \sum_{j=0}^{m} \binom{m+1}{j} S_j(n) \;=\; \sum_{j=0}^{m} \binom{m+1}{j} \widehat{S}_j(n) + \binom{m+1}{m} \Delta \\ &=\; \sum_{j=0}^{m} \binom{m+1}{j} \frac{1}{j{+}1} \sum_{k=0}^{j} \binom{j+1}{k} B_k n^{j+1-k} + (m{+}1)\,\Delta \\ &=\; \sum_{0 \leqslant k \leqslant j \leqslant m} \binom{m+1}{j} \binom{j+1}{k} \frac{B_k}{j{+}1} n^{j+1-k} + (m{+}1)\,\Delta \\ &=\; \sum_{0 \leqslant k \leqslant j \leqslant m} \binom{m+1}{j} \binom{j+1}{j-k} \frac{B_{j-k}}{j{+}1} n^{k+1} + (m{+}1)\,\Delta \\ &=\; \sum_{0 \leqslant k \leqslant m} \frac{n^{k+1}}{k{+}1} \sum_{k \leqslant j \leqslant m} \binom{m+1}{j} \binom{j}{k} B_{j-k} + (m{+}1)\,\Delta \end{aligned}$$

$$= \sum_{0\leqslant k\leqslant m} \frac{n^{k+1}}{k+1}\binom{m+1}{k}\sum_{k\leqslant j\leqslant m}\binom{m+1-k}{j-k}B_{j-k} + (m+1)\,\Delta$$
$$= \sum_{0\leqslant k\leqslant m} \frac{n^{k+1}}{k+1}\binom{m+1}{k}\sum_{0\leqslant j\leqslant m-k}\binom{m+1-k}{j}B_{j} + (m+1)\,\Delta$$
$$= \sum_{0\leqslant k\leqslant m} \frac{n^{k+1}}{k+1}\binom{m+1}{k}(m+1-k) + (m+1)\,\Delta$$
$$= \sum_{0\leqslant k\leqslant m} n^{k+1}\binom{m+1}{k+1} + (m+1)\,\Delta$$
$$= (n+1)^{m+1} - 1 + (m+1)\,\Delta\,.$$

(Here $\Delta = S_m(n) - \widehat{S}_m(n)$. This derivation is a good review of the standard manipulations we learned in Chapter 5.) Thus $\Delta = 0$, QED.

*Here's some more neat stuff that you'll probably want to skim through the first time.*
*—Friendly TA*

In Chapter 7 we'll use generating functions to obtain a much simpler proof of (6.78). The key idea will be to show that the Bernoulli numbers are the coefficients of the power series

$$\frac{z}{1-e^{-z}} = \sum_{n\geqslant 0} B_n \frac{z^n}{n!}\,. \tag{6.81}$$

*Start Skimming*

Let's simply assume for now that equation (6.81) holds, so that we can derive some of its amazing consequences. If we subtract $\frac{1}{2}z$ from both sides, thereby cancelling the term $B_1 z^1/1! = \frac{1}{2}z$ from the right, we get

$$\frac{z}{1-e^{-z}} - \frac{z}{2} = \frac{z}{2}\,\frac{1+e^{-z}}{1-e^{-z}} = \frac{z}{2}\,\frac{e^{z/2}+e^{-z/2}}{e^{z/2}-e^{-z/2}} = \frac{z}{2}\coth\frac{z}{2}\,. \tag{6.82}$$

Here coth is the "hyperbolic cotangent" function, otherwise known in calculus books as $\cosh z/\sinh z$; we have

$$\sinh z = \frac{e^z - e^{-z}}{2}\,;\qquad \cosh z = \frac{e^z + e^{-z}}{2}\,. \tag{6.83}$$

Changing $z$ to $-z$ gives $\left(\frac{-z}{2}\right)\coth\left(\frac{-z}{2}\right) = \frac{z}{2}\coth\frac{z}{2}$; hence every odd-numbered coefficient of $\frac{z}{2}\coth\frac{z}{2}$ must be zero, and we have

$$B_3 = B_5 = B_7 = B_9 = B_{11} = B_{13} = \cdots = 0\,. \tag{6.84}$$

Furthermore (6.82) leads to a closed form for the coefficients of coth:

$$z\coth z = \frac{2z}{1-e^{-2z}} - \frac{2z}{2} = \sum_{n\geqslant 0} B_{2n}\frac{(2z)^{2n}}{(2n)!} = \sum_{n\geqslant 0} 4^n B_{2n}\frac{z^{2n}}{(2n)!}\,. \tag{6.85}$$

But there isn't much of a market for hyperbolic functions; people are more interested in the "real" functions of trigonometry. We can express ordinary trigonometric functions in terms of their hyperbolic cousins by using the rules

$$\sin z = -i \sinh iz\,, \qquad \cos z = \cosh iz\,; \tag{6.86}$$

the corresponding power series are

$$\sin z = \frac{z^1}{1!} - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots\,, \qquad \sinh z = \frac{z^1}{1!} + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\,;$$
$$\cos z = \frac{z^0}{0!} - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots\,, \qquad \cosh z = \frac{z^0}{0!} + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots\,.$$

Hence $\cot z = \cos z/\sin z = i\cosh iz/\sinh iz = i\coth iz$, and we have

*I see, we get "real" functions by using imaginary numbers.*

$$z\cot z = \sum_{n\geqslant 0} B_{2n}\frac{(2iz)^{2n}}{(2n)!} = \sum_{n\geqslant 0}(-4)^n B_{2n}\frac{z^{2n}}{(2n)!}\,. \tag{6.87}$$

Another remarkable formula for $z\cot z$ was found by Euler (exercise 73):

$$z\cot z = 1 - 2\sum_{k\geqslant 1}\frac{z^2}{k^2\pi^2 - z^2}\,. \tag{6.88}$$

We can expand Euler's formula in powers of $z^2$, obtaining

$$\begin{aligned} z\cot z &= 1 - 2\sum_{k\geqslant 1}\left(\frac{z^2}{k^2\pi^2} + \frac{z^4}{k^4\pi^4} + \frac{z^6}{k^6\pi^6} + \cdots\right) \\ &= 1 - 2\left(\frac{z^2}{\pi^2}H_\infty^{(2)} + \frac{z^4}{\pi^4}H_\infty^{(4)} + \frac{z^6}{\pi^6}H_\infty^{(6)} + \cdots\right). \end{aligned}$$

Equating coefficients of $z^{2n}$ with those in our other formula, (6.87), gives us an almost miraculous closed form for infinitely many infinite sums:

$$\zeta(2n) = H_\infty^{(2n)} = (-1)^{n-1}\frac{2^{2n-1}\pi^{2n}B_{2n}}{(2n)!}\,, \qquad \text{integer } n > 0. \tag{6.89}$$

For example,

$$\zeta(2) = H_\infty^{(2)} = 1 + \tfrac{1}{4} + \tfrac{1}{9} + \cdots = \pi^2 B_2 = \pi^2/6\,; \tag{6.90}$$
$$\zeta(4) = H_\infty^{(4)} = 1 + \tfrac{1}{16} + \tfrac{1}{81} + \cdots = -\pi^4 B_4/3 = \pi^4/90\,. \tag{6.91}$$

Formula (6.89) is not only a closed form for $H_\infty^{(2n)}$, it also tells us the approximate size of $B_{2n}$, since $H_\infty^{(2n)}$ is very near 1 when $n$ is large. And it tells us that $(-1)^{n-1}B_{2n} > 0$ for all $n > 0$; thus the Bernoulli numbers $B_2$, $B_4$, $B_6$, $B_8$, … are alternately positive and negative.

And that's not all. Bernoulli numbers also appear in the coefficients of the tangent function,

*Start Skipping*

$$\tan z \;=\; \frac{\sin z}{\cos z} \;=\; \sum_{n\geqslant 1}(-1)^{n-1}4^n(4^n-1)B_{2n}\frac{z^{2n-1}}{(2n)!}\,, \tag{6.92}$$

as well as other trigonometric functions (exercise 72). Formula (6.92) leads to another important fact about the Bernoulli numbers, namely that

$$T_{2n-1} \;=\; (-1)^{n-1}\frac{4^n(4^n-1)}{2n}\,B_{2n} \quad \text{is a positive integer.} \tag{6.93}$$

We have, for example:

| $n$ | 1 | 3 | 5 | 7 | 9 | 11 | 13 |
|---|---|---|---|---|---|---|---|
| $T_n$ | 1 | 2 | 16 | 272 | 7936 | 353792 | 22368256 |

(The T's are called *tangent numbers.*)

One way to prove (6.93), following an idea of B. F. Logan, is to consider the power series

$$\frac{\sin z + x\cos z}{\cos z - x\sin z} \;=\; x + (1{+}x^2)z + (2x^3{+}2x)\frac{z^2}{2} + (6x^4{+}8x^2{+}2)\frac{z^3}{6} + \cdots$$

$$= \sum_{n\geqslant 0} T_n(x)\frac{z^n}{n!}\,, \tag{6.94}$$

*When $x = \tan w$, this is $\tan(z+w)$. Hence, by Taylor's theorem, the $n$th derivative of $\tan w$ is $T_n(\tan w)$.*

where $T_n(x)$ is a polynomial in $x$; setting $x = 0$ gives $T_n(0) = T_n$, the $n$th tangent number. If we differentiate (6.94) with respect to $x$, we get

$$\frac{1}{(\cos z - x\sin z)^2} \;=\; \sum_{n\geqslant 0} T_n'(x)\frac{z^n}{n!}\,;$$

but if we differentiate with respect to $z$, we get

$$\frac{1+x^2}{(\cos z - x\sin z)^2} \;=\; \sum_{n\geqslant 1} T_n(x)\frac{z^{n-1}}{(n-1)!} \;=\; \sum_{n\geqslant 0} T_{n+1}(x)\frac{z^n}{n!}\,.$$

(Try it — the cancellation is very pretty.) Therefore we have

$$T_{n+1}(x) \;=\; (1+x^2)T_n'(x)\,, \qquad T_0(x) \;=\; x\,, \tag{6.95}$$

a simple recurrence from which it follows that the coefficients of $T_n(x)$ are nonnegative integers. Moreover, we can easily prove that $T_n(x)$ has degree $n+1$, and that its coefficients are alternately zero and positive. Therefore $T_{2n+1}(0) = T_{2n+1}$ is a positive integer, as claimed in (6.93).

Recurrence (6.95) gives us a simple way to calculate Bernoulli numbers, via tangent numbers, using only simple operations on integers; by contrast, the defining recurrence (6.79) involves difficult arithmetic with fractions.

If we want to compute the sum of $m$th powers from $a+1$ to $b$ instead of from 1 to $n$, the theory of Chapter 2 tells us that

$$\sum_{k=a+1}^{b} k^m \;=\; \sum\nolimits_a^b (x+1)^m \,\delta x \;=\; S_m(b) - S_m(a)\,. \tag{6.96}$$

This identity has interesting consequences when we consider negative values of $k$: We have

$$\sum_{k=-n+1}^{0} k^m \;=\; (-1)^m \sum_{k=1}^{n-1} k^m\,, \qquad \text{when } m>0,$$

hence

$$S_m(0) - S_m(-n) \;=\; (-1)^m S_m(n-1)\,.$$

But $S_m(0)=0$, so we have the identity

$$S_m(-n) \;=\; (-1)^{m+1} S_m(n-1)\,, \qquad m>0. \tag{6.97}$$

(Johann Faulhaber implicitly used (6.97) in 1631 [119] to find simple formulas for $S_m(n)$ as polynomials in $n(n+1)/2$ when $m \leqslant 17$; see [222].)

Therefore $S_m(-1)=0$. The polynomial $S_m(n)$ will always have the factors $n$ and $(n+1)$, because it has the roots 0 and $-1$.

In general, $S_m(n)$ is a polynomial of degree $m+1$ whose leading term is $\frac{1}{m+1}n^{m+1}$. Moreover, we can set $n=-\frac12$ in (6.97) to deduce that $S_m(-\frac12) = (-1)^{m+1}S_m(-\frac12)$; if $m$ is even, this makes $S_m(-\frac12)=0$, so $(n+\frac12)$ will be an additional factor. These observations explain why we found

$$S_2(n) \;=\; \tfrac13 n(n+\tfrac12)(n+1)$$

in Chapter 2; we could have used such reasoning to deduce the value of $S_2(n)$ without calculating it! Furthermore, (6.97) implies that the polynomial with the remaining factors, $\hat S_m(n) = S_m(n)/(n+\frac12)$, always satisfies

$$\hat S_m(-n) \;=\; \hat S_m(n-1)\,, \qquad m \text{ even}, \quad m>0.$$

It follows that $S_m(n)$ can always be written in the factored form

$$S_m(n) \;=\; \begin{cases} \dfrac{1}{m+1} \displaystyle\prod_{k=1}^{\lceil m/2\rceil} (n+\tfrac12-\alpha_k)(n+\tfrac12+\alpha_k)\,, & m \text{ odd;} \\[2ex] \dfrac{(n+\frac12)}{m+1} \displaystyle\prod_{k=1}^{m/2} (n+\tfrac12-\alpha_k)(n+\tfrac12+\alpha_k)\,, & m \text{ even.} \end{cases} \tag{6.98}$$

Here $\alpha_1 = \frac{1}{2}$, and $\alpha_2, \ldots, \alpha_{\lceil m/2 \rceil}$ are appropriate complex numbers whose values depend on $m$. For example,

$$\begin{aligned} S_3(n) &= n^2(n+1)^2/4\,; \\ S_4(n) &= n(n+\tfrac{1}{2})(n+1)(n+\tfrac{1}{2}+\sqrt{7/12}\,)(n+\tfrac{1}{2}-\sqrt{7/12}\,)/5\,; \\ S_5(n) &= n^2(n+1)^2(n+\tfrac{1}{2}+\sqrt{3/4}\,)(n+\tfrac{1}{2}-\sqrt{3/4}\,)/6\,; \\ S_6(n) &= n(n+\tfrac{1}{2})(n+1)(n+\tfrac{1}{2}+\alpha)(n+\tfrac{1}{2}-\alpha)(n+\tfrac{1}{2}+\overline{\alpha})(n+\tfrac{1}{2}-\overline{\alpha})/7\,, \\ &\qquad \text{where } \alpha = 2^{-3/2}\,3^{-1/4}\bigl(\sqrt{\sqrt{31}+\sqrt{27}}+i\sqrt{\sqrt{31}-\sqrt{27}}\,\bigr). \end{aligned}$$

*Stop Skipping*

If $m$ is odd and greater than 1, we have $B_m = 0$; hence $S_m(n)$ is divisible by $n^2$ (and by $(n+1)^2$). Otherwise the roots of $S_m(n)$ don't seem to obey a simple law.

Let's conclude our study of Bernoulli numbers by looking at how they relate to Stirling numbers. One way to compute $S_m(n)$ is to change ordinary powers to falling powers, since the falling powers have easy sums. After doing those easy sums we can convert back to ordinary powers:

$$\begin{aligned} S_m(n) - n^m + 0^m &= \sum_{k=0}^{n-1} k^m = \sum_{k=0}^{n-1}\sum_{j\geqslant 0} {m \brace j} k^{\underline{j}} = \sum_{j\geqslant 0}{m \brace j}\sum_{k=0}^{n-1} k^{\underline{j}} \\ &= \sum_{j\geqslant 0}{m \brace j}\frac{n^{\underline{j+1}}}{j+1} \\ &= \sum_{j\geqslant 0}{m \brace j}\frac{1}{j+1}\sum_{k\geqslant 0}(-1)^{j+1-k}{j+1 \brack k} n^k\,. \end{aligned}$$

Therefore, equating coefficients with those in (6.78), we must have the identity

$$\sum_{j\geqslant 0}{m \brace j}{j+1 \brack k}\frac{(-1)^{j+1-k}}{j+1} = \frac{1}{m+1}\binom{m+1}{k}B_{m+1-k}\,,\quad 0<k<m. \qquad (6.99)$$

It would be nice to prove this relation directly, thereby discovering Bernoulli numbers in a new way. Do the identities in Tables 264 or 265 offer any help?

Yes, the first one is worth a try: We can replace ${m \brace j}$ by ${m+1 \brace j+1} - (j+1){m \brace j+1}$. The $(j+1)$ nicely cancels with the awkward denominator, and the left-hand side becomes

$$\sum_{j\geqslant 0}{m+1 \brace j+1}{j+1 \brack k}\frac{(-1)^{j+1-k}}{j+1} \;-\; \sum_{j\geqslant 0}{m \brace j+1}{j+1 \brack k}(-1)^{j+1-k}\,.$$

The second sum is zero by (6.31), since $k < m$. That leaves us with the first sum, which cries out for a change in notation; let's rename all variables so that the index of summation is k, and so that the other parameters are m and n. Then identity (6.99) is equivalent to

$$\sum_k \left\{ n \atop k \right\} \left[ k \atop m \right] \frac{(-1)^{k-m}}{k} = \frac{1}{n}\binom{n}{m} B_{n-m}\,, \quad m > 0, \tag{6.100}$$

because this formula holds when $n = m+1$, when $n = m$, and when $n < m$. Good, we have something that looks more pleasant — although Tables 264 and 265 don't suggest any obvious next step.

The convolution formulas in Table 272 now come to the rescue. We can use (6.49) and (6.48) to rewrite the summand in terms of Stirling polynomials:

$$\left\{ n \atop k \right\} \left[ k \atop m \right] = (-1)^{n-k+1} \frac{n!}{(k-1)!} \sigma_{n-k}(-k) \cdot \frac{k!}{(m-1)!} \sigma_{k-m}(k)\,;$$

$$\left\{ n \atop k \right\} \left[ k \atop m \right] \frac{(-1)^{k-m}}{k} = (-1)^{n+1-m} \frac{n!}{(m-1)!}\, \sigma_{n-k}(-k)\, \sigma_{k-m}(k)\,.$$

Things are looking up; the convolution in (6.46), with $t = 1$, yields

$$\begin{aligned} \sum_{k=m}^{n} \sigma_{n-k}(-k)\, \sigma_{k-m}(k) &= \sum_{k=0}^{n-m} \sigma_{n-m-k}\big(-n + (n-m-k)\big)\, \sigma_k(m+k) \\ &= \frac{m-n}{(m)(-n)} \sigma_{n-m}\big(m - n + (n-m)\big)\,. \end{aligned}$$

Furthermore exercise 18, with $x = 0$, tells us that $\sigma_n(1) = -n\sigma_n(0) + [n=1]$. Thus the sum on the left of (6.100), when $n > m+1$, is equal to

$$\frac{(-1)^{n-m}}{n} \frac{n!}{m!} \sigma_{n-m}(1) = \frac{(-1)^{n-m}}{n} \binom{n}{m} (n-m)!\sigma_{n-m}(1)\,.$$

And finally we have $\sigma_{n-m}(1) = B_{n-m}/(n-m)!$, by (6.50). Thus formula (6.100) is indeed verified, because $(-1)^{n-m}B_{n-m} = B_{n-m}$ when $n > m+1$. (Whew.) We've also proved, incidentally, that Bernoulli numbers are related to the constant terms in the Stirling polynomials:

*Stop Skimming*

$$\frac{B_m}{m!} = -m\sigma_m(0) + [m=1]\,. \tag{6.101}$$

## 6.6 FIBONACCI NUMBERS

Now we come to a special sequence of numbers that is perhaps the most pleasant of all, the Fibonacci sequence $\langle F_n \rangle$:

| $n$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $F_n$ | 0 | 1 | 1 | 2 | 3 | 5 | 8 | 13 | 21 | 34 | 55 | 89 | 144 | 233 | 377 |

**16** What is the general solution of the double recurrence

$$A_{n,0} = a_n\,[n \geqslant 0]\,; \qquad A_{0,k} = 0\,, \qquad \text{if } k > 0;$$
$$A_{n,k} = kA_{n-1,k} + A_{n-1,k-1}\,, \qquad \text{integers } k, n,$$

when $k$ and $n$ range over the set of *all* integers?

**17** Solve the following recurrences, assuming that $\left|{n \atop k}\right|$ is zero when $n < 0$ or $k < 0$:

**a** $$\left|{n \atop k}\right| = \left|{n-1 \atop k}\right| + n\left|{n-1 \atop k-1}\right| + [n=k=0]\,, \qquad \text{for } n, k \geqslant 0.$$

**b** $$\left|{n \atop k}\right| = (n-k)\left|{n-1 \atop k}\right| + \left|{n-1 \atop k-1}\right| + [n=k=0]\,, \qquad \text{for } n, k \geqslant 0.$$

**c** $$\left|{n \atop k}\right| = k\left|{n-1 \atop k}\right| + k\left|{n-1 \atop k-1}\right| + [n=k=0]\,, \qquad \text{for } n, k \geqslant 0.$$

**18** Prove that the Stirling polynomials satisfy

$$(x+1)\,\sigma_n(x+1) = (x-n)\,\sigma_n(x) + x\sigma_{n-1}(x)\,.$$

**19** Prove that the generalized Stirling numbers satisfy

$$\sum_{k=0}^{n} \left\{{x+k \atop x}\right\}\left[{x \atop x-n+k}\right](-1)^k \Big/ \binom{x+k}{n+1} = 0\,, \quad \text{integer } n > 0;$$
$$\sum_{k=0}^{n} \left[{x+k \atop x}\right]\left\{{x \atop x-n+k}\right\}(-1)^k \Big/ \binom{x+k}{n+1} = 0\,, \quad \text{integer } n > 0.$$

**20** Find a closed form for $\sum_{k=1}^{n} H_k^{(2)}$.

**21** Show that if $H_n = a_n/b_n$ where $a_n$ and $b_n$ are integers, the denominator $b_n$ is a multiple of $2^{\lfloor \lg n \rfloor}$. *Hint:* Consider the number $2^{\lfloor \lg n \rfloor - 1}H_n - \frac{1}{2}$.

**22** Prove that the infinite sum

$$\sum_{k\geqslant 1} \left(\frac{1}{k} - \frac{1}{k+z}\right)$$

converges for all complex numbers $z$, except when $z$ is a negative integer; and show that it equals $H_z$ when $z$ is a nonnegative integer. (Therefore we can use this formula to define harmonic numbers $H_z$ when $z$ is complex.)

**23** Equation (6.81) gives the coefficients of $z/(1-e^{-z})$, when expanded in powers of $z$. What are the coefficients of $z/(1+e^{-z})$? *Hint:* Consider the identity $(1+e^{-z})(1-e^{-z}) = 1 - e^{-2z}$.

**72** Prove that the tangent function has the power series (6.92), and find the corresponding series for $z/\sin z$ and $\ln\bigl((\tan z)/z\bigr)$.

**73** Prove that $z \cot z$ is equal to

$$\frac{z}{2^n}\cot\frac{z}{2^n} - \frac{z}{2^n}\tan\frac{z}{2^n} + \sum_{k=1}^{2^{n-1}-1}\frac{z}{2^n}\left(\cot\frac{z+k\pi}{2^n} + \cot\frac{z-k\pi}{2^n}\right),$$

for all integers $n \geqslant 1$, and show that the limit of the $k$th summand is $2z^2/(z^2 - k^2\pi^2)$ for fixed $k$ as $n \to \infty$.

**74** Find a relation between the numbers $T_n(1)$ and the coefficients of $1/\cos z$.

**75** Prove that the tangent numbers and the coefficients of $1/\cos z$ appear at the edges of the infinite triangle that begins as follows:

```
                  1
               0     1
            1     1     0
         0     1     2     2
      5     5     4     2     0
   0     5    10    14    16    16
61    61    56    46    32    16     0
```

Each row contains partial sums of the previous row, going alternately left-to-right and right-to-left. *Hint:* Consider the coefficients of the power series $(\sin z + \cos z)/\cos(w+z)$.

**76** Find a closed form for the sum

$$\sum_k (-1)^k \left\{ {n \atop k} \right\} 2^{n-k} k!\,,$$

and show that it is zero when $n > 0$ is even.

**77** When $m$ and $n$ are integers, $n \geqslant 0$, the value of $\sigma_n(m)$ is given by (6.48) if $m < 0$, by (6.49) if $m > n$, and by (6.101) if $m = 0$. Show that in the remaining cases we have

$$\sigma_n(m) = \frac{(-1)^{m-1}}{m!\,(n-m)!}\sum_{k=0}^{m-1}\left[{m \atop m-k}\right]\frac{(-1)^k B_{n-k}}{n-k}, \quad \text{integer } n \geqslant m > 0.$$

**78** Prove the following relation that connects Stirling numbers, Bernoulli numbers, and Catalan numbers:

$$\sum_{k=0}^{n}\left\{ {n+k \atop k} \right\}\binom{2n}{n+k}\frac{(-1)^k}{k+1} = B_n\binom{2n}{n}\frac{(-1)^n}{n+1}.$$

**79** Show that the four chessboard pieces of the $64 = 65$ paradox can also be reassembled to prove that $64 = 63$.

**Table 351** Generating functions for special numbers.

$$\frac{1}{(1-z)^{m+1}}\ln\frac{1}{1-z} = \sum_{n\geqslant 0}(H_{m+n}-H_m)\binom{m+n}{n}z^n \qquad (7.43)$$

$$\frac{z}{1-e^{-z}} = \sum_{n\geqslant 0} B_n\frac{z^n}{n!} \qquad (7.44)$$

$$\frac{F_m z}{1-(F_{m-1}+F_{m+1})z+(-1)^m z^2} = \sum_{n\geqslant 0} F_{mn}\, z^n \qquad (7.45)$$

$$\sum_k \left\{{m \atop k}\right\}\frac{k!\, z^k}{(1-z)^{k+1}} = \sum_{n\geqslant 0} n^m z^n \qquad (7.46)$$

$$\left(z^{-1}\right)^{\overline{-m}} = \frac{z^m}{(1-z)(1-2z)\ldots(1-mz)} = \sum_{n\geqslant 0}\left\{{n \atop m}\right\}z^n \qquad (7.47)$$

$$z^{\overline{m}} = z(z+1)\ldots(z+m-1) = \sum_{n\geqslant 0}\left[{m \atop n}\right]z^n \qquad (7.48)$$

$$\left(e^z-1\right)^m = m!\sum_{n\geqslant 0}\left\{{n \atop m}\right\}\frac{z^n}{n!} \qquad (7.49)$$

$$\left(\ln\frac{1}{1-z}\right)^m = m!\sum_{n\geqslant 0}\left[{n \atop m}\right]\frac{z^n}{n!} \qquad (7.50)$$

$$\left(\frac{z}{\ln(1+z)}\right)^m = \sum_{n\geqslant 0}\frac{z^n}{n!}\left\{{m \atop m-n}\right\}\Big/\binom{m-1}{n} \qquad (7.51)$$

$$\left(\frac{z}{1-e^{-z}}\right)^m = \sum_{n\geqslant 0}\frac{z^n}{n!}\left[{m \atop m-n}\right]\Big/\binom{m-1}{n} \qquad (7.52)$$

$$e^{z+wz} = \sum_{m,n\geqslant 0}\binom{n}{m}w^m\frac{z^n}{n!} \qquad (7.53)$$

$$e^{w(e^z-1)} = \sum_{m,n\geqslant 0}\left\{{n \atop m}\right\}w^m\frac{z^n}{n!} \qquad (7.54)$$

$$\frac{1}{(1-z)^w} = \sum_{m,n\geqslant 0}\left[{n \atop m}\right]w^m\frac{z^n}{n!} \qquad (7.55)$$

$$\frac{1-w}{e^{(w-1)z}-w} = \sum_{m,n\geqslant 0}\left\langle{n \atop m}\right\rangle w^m\frac{z^n}{n!} \qquad (7.56)$$

this is the egf of $\langle g_1, g_2, \ldots \rangle$. Thus differentiation on egf's corresponds to the left-shift operation $\bigl(G(z) - g_0\bigr)/z$ on ordinary gf's. (We used this left-shift property of egf's when we studied hypergeometric series, (5.106).) Integration of an egf gives

$$\int_0^z \sum_{n\geqslant 0} g_n \frac{t^n}{n!}\,dt \;=\; \sum_{n\geqslant 0} g_n \frac{z^{n+1}}{(n+1)!} \;=\; \sum_{n\geqslant 1} g_{n-1}\frac{z^n}{n!}\,; \tag{7.74}$$

this is a right shift, the egf of $\langle 0, g_0, g_1, \ldots \rangle$.

The most interesting operation on egf's, as on ordinary gf's, is multiplication. If $\widehat{F}(z)$ and $\widehat{G}(z)$ are egf's for $\langle f_n \rangle$ and $\langle g_n \rangle$, then $\widehat{F}(z)\widehat{G}(z) = \widehat{H}(z)$ is the egf for a sequence $\langle h_n \rangle$ called the *binomial convolution* of $\langle f_n \rangle$ and $\langle g_n \rangle$:

$$h_n \;=\; \sum_k \binom{n}{k} f_k\, g_{n-k}\,. \tag{7.75}$$

Binomial coefficients appear here because $\binom{n}{k} = n!/k!\,(n-k)!$, hence

$$\frac{h_n}{n!} \;=\; \sum_{k=0}^{n} \frac{f_k}{k!}\,\frac{g_{n-k}}{(n-k)!}\,;$$

in other words, $\langle h_n/n! \rangle$ is the ordinary convolution of $\langle f_n/n! \rangle$ and $\langle g_n/n! \rangle$.

Binomial convolutions occur frequently in applications. For example, we defined the Bernoulli numbers in (6.79) by the implicit recurrence

$$\sum_{j=0}^{m} \binom{m+1}{j} B_j \;=\; m+1\,, \qquad \text{for all } m \geqslant 0;$$

this can be rewritten as a binomial convolution, if we substitute $n$ for $m+1$ and add the term $B_n$ to both sides:

$$\sum_k \binom{n}{k} B_k \;=\; B_n + n\,, \qquad \text{for all } n \geqslant 0. \tag{7.76}$$

We can now relate this recurrence to power series (as promised in Chapter 6) by introducing the egf for Bernoulli numbers, $\widehat{B}(z) = \sum_{n\geqslant 0} B_n z^n/n!$. The left-hand side of (7.76) is the binomial convolution of $\langle B_n \rangle$ with the constant sequence $\langle 1, 1, 1, \ldots \rangle$; hence the egf of the left-hand side is $\widehat{B}(z)e^z$. The egf of the right-hand side is $\sum_{n\geqslant 0}(B_n + n)z^n/n! = \widehat{B}(z) + ze^z$. Therefore we must have $\widehat{B}(z) = ze^z/(e^z - 1) = z/(1 - e^{-z})$; we have proved equation (6.81), which appears also in Table 351 as equation (7.44).

Now let's look again at a sum that has been popping up frequently in this book,

$$S_m(n) \;=\; 1^m + 2^m + \cdots + n^m \;=\; \sum_{1\leqslant k\leqslant n} k^m\,.$$

This time we will try to analyze the problem with generating functions, in hopes that it will suddenly become simpler. We will consider $n$ to be fixed and $m$ variable; thus our goal is to understand the coefficients of the power series

$$S(z,n) \;=\; S_0(n) + S_1(n)\,z + S_2(n)\,z^2 + \cdots \;=\; \sum_{m\geqslant 0} S_m(n)\,z^m\,.$$

We know that the generating function for $\langle 1, k, k^2, \dots\rangle$ is

$$\frac{1}{1-kz} \;=\; \sum_{m\geqslant 0} k^m z^m\,;$$

hence

$$S(z,n) \;=\; \sum_{m\geqslant 0}\,\sum_{1\leqslant k\leqslant n} k^m z^m \;=\; \sum_{1\leqslant k\leqslant n} \frac{1}{1-kz}$$

by interchanging the order of summation. We can put this sum in closed form,

$$\begin{aligned} S(z,n) \;&=\; \frac{1}{z}\left(\frac{1}{z^{-1}-1} + \frac{1}{z^{-1}-2} + \cdots + \frac{1}{z^{-1}-n}\right) \\ &=\; \frac{1}{z}\bigl(H_{z^{-1}-1} - H_{z^{-1}-n-1}\bigr)\,; \end{aligned} \tag{7.77}$$

but we know nothing about expanding such a closed form in powers of $z$.

Exponential generating functions come to the rescue. The egf of our sequence $\langle S_0(n), S_1(n), S_2(n), \dots\rangle$ is

$$\widehat{S}(z,n) \;=\; S_0(n) + S_1(n)\,\frac{z}{1!} + S_2(n)\,\frac{z^2}{2!} + \cdots \;=\; \sum_{m\geqslant 0} S_m(n)\,\frac{z^m}{m!}\,.$$

To get these coefficients $S_m(n)$ we can use the egf for $\langle 1, k, k^2, \dots\rangle$, namely

$$e^{kz} \;=\; \sum_{m\geqslant 0} k^m\,\frac{z^m}{m!}\,,$$

and we have

$$\widehat{S}(z,n) \;=\; \sum_{m\geqslant 0}\,\sum_{1\leqslant k\leqslant n} k^m\,\frac{z^m}{m!} \;=\; \sum_{1\leqslant k\leqslant n} e^{kz}\,.$$

And the latter sum is a geometric progression, so there's a closed form

$$\widehat{S}(z,n) \;=\; \frac{e^{(n+1)z}-e^z}{e^z-1} \;=\; \frac{e^{nz}-1}{1-e^{-z}}\,. \qquad (7.78)$$

Eureka! All we need to do is figure out the coefficients of this relatively simple function, and we'll know $S_m(n)$, because $S_m(n) = m!\,[z^m]\,\widehat{S}(z,n)$.

Here's where Bernoulli numbers come into the picture. We observed a moment ago that the egf for Bernoulli numbers is

$$\widehat{B}(z) \;=\; \sum_{k\geqslant 0} B_k \frac{z^k}{k!} \;=\; \frac{z}{1-e^{-z}}\,;$$

hence we can write

$$\begin{aligned}\widehat{S}(z,n) \;&=\; \widehat{B}(z)\,\frac{e^{nz}-1}{z}\\ &=\; \Bigl(B_0\frac{z^0}{0!}+B_1\frac{z^1}{1!}+B_2\frac{z^2}{2!}+\cdots\Bigr)\Bigl(n\frac{z^0}{1!}+n^2\frac{z^1}{2!}+n^3\frac{z^2}{3!}+\cdots\Bigr)\,.\end{aligned}$$

The sum $S_m(n)$ is $m!$ times the coefficient of $z^m$ in this product. For example,

$$\begin{aligned}S_0(n) \;&=\; 0!\Bigl(B_0\frac{n}{1!\,0!}\Bigr) &&=\; n\,;\\ S_1(n) \;&=\; 1!\Bigl(B_0\frac{n^2}{2!\,0!}+B_1\frac{n}{1!\,1!}\Bigr) &&=\; \tfrac12 n^2+\tfrac12 n\,;\\ S_2(n) \;&=\; 2!\Bigl(B_0\frac{n^3}{3!\,0!}+B_1\frac{n^2}{2!\,1!}+B_2\frac{n}{1!\,2!}\Bigr) &&=\; \tfrac13 n^3+\tfrac12 n^2+\tfrac16 n\,.\end{aligned}$$

We have therefore derived the formula $\Box_n = S_2(n) = \frac13 n(n+\frac12)(n+1)$ for the umpteenth time, and this was the simplest derivation of all: In a few lines we have found the general behavior of $S_m(n)$ for all $m$.

The general formula can be written

$$S_{m-1}(n) \;=\; \frac{1}{m}\bigl(B_m(n+1)-B_m\bigr)\,, \qquad (7.79)$$

where $B_m(x)$ is the *Bernoulli polynomial* defined by

$$B_m(x) \;=\; \sum_k B_k\binom{m}{k}(x-1)^{m-k} \;=\; \sum_k \binom{m}{k}(-1)^k B_k\, x^{m-k}\,. \qquad (7.80)$$

Here's why: The Bernoulli polynomial is the binomial convolution of the sequence $\langle B_0, B_1, B_2, \ldots\rangle$ with $\langle 1, x-1, (x-1)^2, \ldots\rangle$; hence the exponential generating function for $\langle B_0(x), B_1(x), B_2(x), \ldots \rangle$ is the product of their egf's,

$$\widehat{B}(z,x) \;=\; \sum_{m\geqslant 0} B_m(x)\frac{z^m}{m!} \;=\; \frac{z}{1-e^{-z}}\sum_{m\geqslant 0}(x-1)^m\frac{z^m}{m!} \;=\; \frac{ze^{xz}}{e^z-1}\,. \qquad (7.81)$$

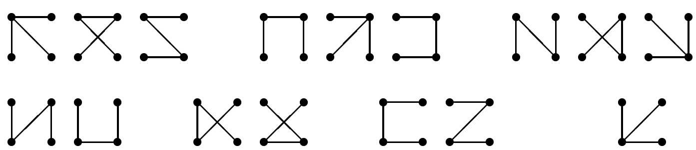
Equation (7.79) follows because the egf for $\langle 0, S_0(n), 2S_1(n), \ldots \rangle$ is

$$z\widehat{S}(z,n) \;=\; z\,\frac{e^{(n+1)z}-e^z}{e^z-1} \;=\; \widehat{B}(z,n{+}1) - \widehat{B}(z,1) \;=\; \widehat{B}(z,n{+}1) - \widehat{B}(z)\,,$$

by (7.78) and (7.44).

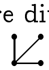
Let's turn now to another problem for which egf's are just the thing: How many spanning trees are possible in the *complete graph* on $n$ vertices $\{1,2,\ldots,n\}$? Let's call this number $t_n$. The complete graph has $\frac{1}{2}n(n-1)$ edges, one edge joining each pair of distinct vertices; so we're essentially looking for the total number of ways to connect up $n$ given things by drawing $n-1$ lines between them.

We have $t_1 = t_2 = 1$. Also $t_3 = 3$, because a complete graph on three vertices is a fan of order 2; we know that $f_2 = 3$. And there are sixteen spanning trees when $n = 4$:



(7.82)

Hence $t_4 = 16$.

Our experience with the analogous problem for fans suggests that the best way to tackle this problem is to single out one vertex, and to look at the blocks or components that the spanning tree joins together when we ignore all edges that touch the special vertex. If the non-special vertices form $m$ components of sizes $k_1$, $k_2$, $\ldots$, $k_m$, then we can connect them to the special vertex in $k_1k_2\ldots k_m$ ways. For example, in the case $n = 4$, we can consider the lower left vertex to be special. The top row of (7.82) shows $3t_3$ cases where the other three vertices are joined among themselves in $t_3$ ways and then connected to the lower left in 3 ways. The bottom row shows $2\cdot 1 \times t_2t_1 \times \binom{3}{2}$ solutions where the other three vertices are divided into components of sizes 2 and 1 in $\binom{3}{2}$ ways; there's also the case  where the other three vertices are completely unconnected among themselves.

This line of reasoning leads to the recurrence

$$t_n \;= \sum_{m>0}\frac{1}{m!}\sum_{k_1+\cdots+k_m=n-1}\binom{n-1}{k_1,k_2,\ldots,k_m}k_1k_2\ldots k_m\,t_{k_1}t_{k_2}\ldots t_{k_m}$$

**38** What is the probability generating function for the number of times you need to roll a fair die until all six faces have turned up? Generalize to $m$-sided fair dice: Give closed forms for the mean and variance of the number of rolls needed to see $l$ of the $m$ faces. What is the probability that this number will be exactly $n$?

**39** A *Dirichlet probability generating function* has the form

$$P(z) = \sum_{n\geqslant 1} \frac{p_n}{n^z}.$$

Thus $P(0) = 1$. If $X$ is a random variable with $\Pr(X=n) = p_n$, express $E(X)$, $V(X)$, and $E(\ln X)$ in terms of $P(z)$ and its derivatives.

**40** The $m$th cumulant $\kappa_m$ of the binomial distribution (8.57) has the form $nf_m(p)$, where $f_m$ is a polynomial of degree $m$. (For example, $f_1(p) = p$ and $f_2(p) = p - p^2$, because the mean and variance are $np$ and $npq$.)

**a** Find a closed form for the coefficient of $p^k$ in $f_m(p)$.

**b** Prove that $f_m(\frac{1}{2}) = (2^m - 1)B_m/m$, where $B_m$ is the $m$th Bernoulli number.

**41** Let the random variable $X_n$ be the number of flips of a fair coin until heads have turned up a total of $n$ times. Show that $E(X_{n+1}^{-1}) = (-1)^n(\ln 2 + H_{\lfloor n/2\rfloor} - H_n)$. Use the methods of Chapter 9 to estimate this value with an absolute error of $O(n^{-3})$.

**42** A certain man has a problem finding work. If he is unemployed on any given morning, there's constant probability $p_h$ (independent of past history) that he will be hired before that evening; but if he's got a job when the day begins, there's constant probability $p_f$ that he'll be laid off by nightfall. Find the average number of evenings on which he will have a job lined up, assuming that he is initially employed and that this process goes on for $n$ days. (For example, if $n = 1$ the answer is $1 - p_f$.)

*Does TEX choose optimal line breaks?*

**43** Find a closed form for the pgf $G_n(z) = \sum_{k\geqslant 0} p_{k,n} z^k$, where $p_{k,n}$ is the probability that a random permutation of $n$ objects has exactly $k$ cycles. What are the mean and standard deviation of the number of cycles?

**44** The athletic department runs an intramural "knockout tournament" for $2^n$ tennis players as follows. In the first round, the players are paired off randomly, with each pairing equally likely, and $2^{n-1}$ matches are played. The winners advance to the second round, where the same process produces $2^{n-2}$ winners. And so on; the $k$th round has $2^{n-k}$ randomly chosen matches between the $2^{n-k+1}$ players who are still undefeated. The $n$th round produces the champion. Unbeknownst to the tournament organizers, there is actually an ordering among the players, so that $x_1$ is best, $x_2$

*"We may not be big, but we're small."*

Since $\lfloor \lg n \rfloor!$ grows faster than any power of $n$, this minuscule error is overwhelmed by $\Sigma_c(n) = O(n^{-3})$. The error that comes from the original tail,

$$\Sigma_a(n) \;=\; \sum_{k \geqslant \lfloor \lg n \rfloor} a_k(n) \;<\; \sum_{k \geqslant \lfloor \lg n \rfloor} \frac{k + \ln n}{k!}\,,$$

is smaller yet.

Finally, it's easy to sum $\sum_{k \geqslant 0} b_k(n)$ in closed form, and we have obtained the desired asymptotic formula:

$$\sum_{k \geqslant 0} \frac{\ln(n + 2^k)}{k!} \;=\; e \ln n + \frac{e^2}{n} - \frac{e^4}{2n^2} + O\Big(\frac{1}{n^3}\Big)\,. \tag{9.65}$$

The method we've used makes it clear that, in fact,

$$\sum_{k \geqslant 0} \frac{\ln(n + 2^k)}{k!} \;=\; e \ln n + \sum_{k=1}^{m-1} (-1)^{k+1} \frac{e^{2^k}}{kn^k} + O\Big(\frac{1}{n^m}\Big)\,, \tag{9.66}$$

for any fixed $m > 0$. (This is a truncation of a series that diverges for all fixed $n$ if we let $m \to \infty$.)

There's only one flaw in our solution: We were too cautious. We derived (9.64) on the assumption that $k < \lfloor \lg n \rfloor$, but exercise 53 proves that the stated estimate is actually valid for all values of $k$. If we had known the stronger general result, we wouldn't have had to use the two-tail trick; we could have gone directly to the final formula! But later we'll encounter problems where exchange of tails is the only decent approach available.

## 9.5 EULER'S SUMMATION FORMULA

And now for our next trick — which is, in fact, the last important technique that will be discussed in this book — we turn to a general method of approximating sums that was first published by Leonhard Euler [101] in 1732. (The idea is sometimes also associated with the name of Colin Maclaurin, a professor of mathematics at Edinburgh who discovered it independently a short time later [263, page 305].)

Here's the formula:

$$\sum_{a < k \leqslant b} f(k) \;=\; \int_a^b f(x)\,dx \;+\; \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(x)\bigg|_a^b \;+\; R_m\,, \tag{9.67}$$

$$\text{where } R_m \;=\; (-1)^{m+1} \int_a^b \frac{B_m\big(\{x\}\big)}{m!} f^{(m)}(x)\,dx\,, \quad \begin{array}{c}\text{integers } a \leqslant b;\\ \text{integer } m \geqslant 1.\end{array} \tag{9.68}$$

On the left is a typical sum that we might want to evaluate. On the right is another expression for that sum, involving integrals and derivatives. If $f(x)$ is a sufficiently "smooth" function, it will have $m$ derivatives $f'(x), \ldots, f^{(m)}(x)$, and this formula turns out to be an identity. The right-hand side is often an excellent approximation to the sum on the left, in the sense that the remainder $R_m$ is often small. For example, we'll see that Stirling's approximation for $n!$ is a consequence of Euler's summation formula; so is our asymptotic approximation for the harmonic number $H_n$.

The numbers $B_k$ in (9.67) are the Bernoulli numbers that we met in Chapter 6; the function $B_m(\{x\})$ in (9.68) is the Bernoulli polynomial that we met in Chapter 7. The notation $\{x\}$ stands for the fractional part $x - \lfloor x \rfloor$, as in Chapter 3. Euler's summation formula sort of brings everything together.

Let's recall the values of small Bernoulli numbers, since it's always handy to have them listed near Euler's general formula:

$$\begin{aligned}&B_0 = 1, \ B_1 = \tfrac{1}{2}, \ B_2 = \tfrac{1}{6}, \ B_4 = -\tfrac{1}{30}, \ B_6 = \tfrac{1}{42}, \ B_8 = -\tfrac{1}{30};\\ &B_3 = B_5 = B_7 = B_9 = B_{11} = \cdots = 0.\end{aligned}$$

Jakob Bernoulli discovered these numbers when studying the sums of powers of integers, and Euler's formula explains why: If we set $f(x) = x^{m-1}$, we have $f^{(m)}(x) = 0$; hence $R_m = 0$, and (9.67) reduces to

$$\begin{aligned}\sum_{a<k\leqslant b} k^{m-1} &= \frac{x^m}{m}\bigg|_a^b + \sum_{k=1}^{m} \frac{B_k}{k!}(m-1)^{\underline{k-1}} x^{m-k}\bigg|_a^b\\ &= \frac{1}{m}\sum_{k=0}^{m}\binom{m}{k} B_k \cdot (b^{m-k} - a^{m-k}).\end{aligned}$$

For example, when $m = 3$ we have our favorite example of summation:

$$\sum_{0<k\leqslant n} k^2 = \frac{1}{3}\left(\binom{3}{0}B_0 n^3 + \binom{3}{1}B_1 n^2 + \binom{3}{2}B_2 n\right) = \frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6}.$$

(This is the last time we shall derive that famous formula in this book.)

*All good things must come to an end.*

Before we prove Euler's formula, let's look at a high-level reason (due to Lagrange [234]) why such a formula ought to exist. Chapter 2 defines the difference operator $\Delta$ and explains that $\sum$ is the inverse of $\Delta$, just as $\int$ is the inverse of the derivative operator $D$. Summation over $a < k \leqslant b$ is the operator $\sum E$, because $\sum_{a<k\leqslant b} f(x) = \sum_{a\leqslant k<b} f(x+1)$; so its inverse is $E^{-1}\Delta$. We can express $E^{-1}\Delta$ in terms of $D$ using Taylor's formula as follows:

$$f(x+\epsilon) = f(x) + \frac{f'(x)}{1!}\epsilon + \frac{f''(x)}{2!}\epsilon^2 + \frac{f'''(x)}{3!}\epsilon^3 + \cdots.$$

Setting $\epsilon = -1$ tells us that

$$\begin{aligned} E^{-1}\Delta f(x) &= f(x) - f(x-1) \\ &= f'(x)/1! - f''(x)/2! + f'''(x)/3! - \cdots \\ &= (D/1! - D^2/2! + D^3/3! - \cdots)\, f(x) \;=\; (1 - e^{-D})\, f(x)\,. \end{aligned} \tag{9.69}$$

Here $e^{-D}$ stands for the differential operation $1 - D/1! + D^2/2! - D^3/3! + \cdots$. Since $E^{-1}\Delta = 1 - e^{-D}$, the inverse operator $\sum E = E/\Delta$ should be $1/(1-e^{-D})$; and we know from Table 351 that $z/(1-e^{-z}) = \sum_{k\geqslant 0} B_k z^k/k!$ is a power series involving Bernoulli numbers. Thus

$$\sum E \;=\; \frac{B_0}{D} + \frac{B_1}{1!} + \frac{B_2}{2!}D + \frac{B_3}{3!}D^2 + \cdots \;=\; \int + \sum_{k\geqslant 1}\frac{B_k}{k!}D^{k-1}\,. \tag{9.70}$$

Applying this operator equation to $f(x)$ and attaching limits yields

$$\sum\nolimits_a^b f(x+1)\,\delta x \;=\; \int_a^b f(x)\,dx + \sum_{k\geqslant 1}\frac{B_k}{k!}f^{(k-1)}(x)\Big|_a^b\,, \tag{9.71}$$

which is exactly Euler's summation formula (9.67) without the remainder term. (Euler did not, in fact, consider the remainder, nor did anybody else until S. D. Poisson [295] published an important memoir about approximate summation in 1823. The remainder term is important, because the infinite sum $\sum_{k\geqslant 1}(B_k/k!)f^{(k-1)}(x)\big|_a^b$ often diverges. Our derivation of (9.71) has been purely formal, without regard to convergence.)

Now let's prove (9.67), with the remainder included. It suffices to prove the case $a = 0$ and $b = 1$, namely

$$f(1) \;=\; \int_0^1 f(x)\,dx + \sum_{k=1}^{m}\frac{B_k}{k!}\,f^{(k-1)}(x)\Big|_0^1 - (-1)^m\int_0^1 \frac{B_m(x)}{m!}f^{(m)}(x)\,dx\,,$$

because we can then replace $f(x)$ by $f(x+l)$ for any integer $l$, getting

$$f(l) \;=\; \int_{l-1}^{l} f(x)\,dx + \sum_{k=1}^{m}\frac{B_k}{k!}\,f^{(k-1)}(x)\Big|_{l-1}^{l} - (-1)^m\int_{l-1}^{l}\frac{B_m\big(\{x\}\big)}{m!}f^{(m)}(x)\,dx\,.$$

The general formula (9.67) is just the sum of this identity over the range $a < l \leqslant b$, because intermediate terms telescope nicely.

The proof when $a = 0$ and $b = 1$ is by induction on $m$, starting with $m = 1$:

$$f(1) \;=\; \int_0^1 f(x)\,dx + \frac{1}{2}\big(f(1) - f(0)\big) + \int_0^1 (x - \tfrac{1}{2})f'(x)\,dx\,.$$

(The Bernoulli polynomial $B_m(x)$ was defined in (7.80) by the equation

$$B_m(x) \;=\; \binom{m}{0}B_0x^m - \binom{m}{1}B_1x^{m-1} + \cdots + (-1)^m\binom{m}{m}B_mx^0 \qquad (9.72)$$

in general, hence $B_1(x) = x - \frac{1}{2}$ in particular.) In other words, we want to prove that

$$\frac{f(0)+f(1)}{2} \;=\; \int_0^1 f(x)\,dx + \int_0^1 (x-\tfrac{1}{2})f'(x)\,dx\,.$$

But this is just a special case of the formula

$$u(x)v(x)\Big|_0^1 \;=\; \int_0^1 u(x)\,dv(x) \;+\; \int_0^1 v(x)\,du(x) \qquad (9.73)$$

for integration by parts, with $u(x) = f(x)$ and $v(x) = x - \frac{1}{2}$. Hence the case $m = 1$ is easy.

To pass from $m-1$ to $m$ and complete the induction when $m > 1$, we need to show that $R_{m-1} = (B_m/m!)f^{(m-1)}(x)\big|_0^1 + R_m$, namely that

$$(-1)^m\int_0^1 \frac{B_{m-1}(x)}{(m-1)!}\,f^{(m-1)}(x)\,dx$$
$$= \frac{B_m}{m!}\,f^{(m-1)}(x)\Big|_0^1 - (-1)^m\int_0^1 \frac{B_m(x)}{m!}\,f^{(m)}(x)\,dx\,.$$

This reduces to the equation

$$(-1)^mB_m\,f^{(m-1)}(x)\Big|_0^1 = m\int_0^1 B_{m-1}(x)f^{(m-1)}(x)\,dx + \int_0^1 B_m(x)f^{(m)}(x)\,dx.$$

Once again (9.73) applies to these two integrals, with $u(x) = f^{(m-1)}(x)$ and $v(x) = B_m(x)$, because the derivative of the Bernoulli polynomial (9.72) is

*Will the authors never get serious?*

$$\begin{aligned}\frac{d}{dx}\sum_k \binom{m}{k}(-1)^kB_kx^{m-k} \;&=\; \sum_k \binom{m}{k}(-1)^k(m-k)B_kx^{m-k-1}\\ &=\; m\sum_k \binom{m-1}{k}(-1)^kB_kx^{m-1-k}\\ &=\; mB_{m-1}(x). \qquad (9.74)\end{aligned}$$

(The absorption identity (5.7) was useful here.) Therefore the required formula will hold if and only if

$$(-1)^mB_mf^{(m-1)}(x)\Big|_0^1 \;=\; B_m(x)f^{(m-1)}(x)\Big|_0^1\,.$$

In other words, we need to have

$$(-1)^m B_m \;=\; B_m(1) \;=\; B_m(0)\,, \qquad \text{for } m > 1. \tag{9.75}$$

This is a bit embarrassing, because $B_m(1)$ is actually equal to $B_m$, not to $(-1)^m B_m$. But there's no problem really, because $m > 1$; we know that $B_m$ is zero when $m$ is odd and greater than 1. (Still, that was a close call.)

To complete the proof of Euler's summation formula we need to show that $B_m(1) = B_m(0)$, which is the same as saying that

$$\sum_k \binom{m}{k}(-1)^k B_k \;=\; (-1)^m B_m\,, \qquad \text{for } m > 1.$$

But this agrees with the definition of Bernoulli numbers, (6.79), so we're done.

The identity $B'_m(x) = mB_{m-1}(x)$ implies that

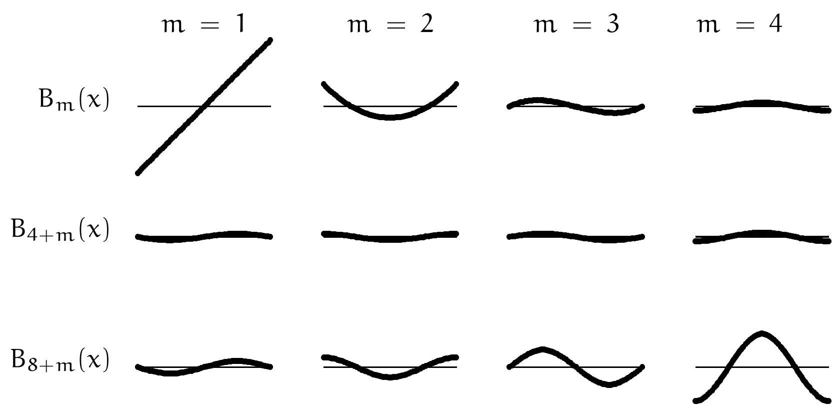
$$\int_0^1 B_m(x)\,dx \;=\; \frac{B_{m+1}(1) - B_{m+1}(0)}{m+1}\,,$$

and we know now that this integral is zero when $m \geqslant 1$. Hence the remainder term in Euler's formula,

$$R_m \;=\; \frac{(-1)^{m+1}}{m!}\int_a^b B_m\bigl(\{x\}\bigr)\,f^{(m)}(x)\,dx\,,$$

multiplies $f^{(m)}(x)$ by a function $B_m\bigl(\{x\}\bigr)$ whose average value is zero. This means that $R_m$ has a reasonable chance of being small.

Let's look more closely at $B_m(x)$ for $0 \leqslant x \leqslant 1$, since $B_m(x)$ governs the behavior of $R_m$. Here are the graphs for $B_m(x)$ for the first twelve values of $m$:




Although $B_3(x)$ through $B_9(x)$ are quite small, the Bernoulli polynomials and numbers ultimately get quite large. Fortunately $R_m$ has a compensating factor $1/m!$, which helps to calm things down.

The graph of $B_m(x)$ begins to look very much like a sine wave when $m \geqslant 3$; exercise 58 proves that $B_m(x)$ can in fact be well approximated by a negative multiple of $\cos(2\pi x - \frac{1}{2}\pi m)$, with error $O\bigl(2^{-m}\max_x B_m(\{x\})\bigr)$.

In general, $B_{4k+1}(x)$ is negative for $0 < x < \frac{1}{2}$ and positive for $\frac{1}{2} < x < 1$. Therefore its integral, $B_{4k+2}(x)/(4k+2)$, decreases for $0 < x < \frac{1}{2}$ and increases for $\frac{1}{2} < x < 1$. Moreover, we have

$$B_{4k+1}(1-x) \;=\; -B_{4k+1}(x)\,, \qquad \text{for } 0 \leqslant x \leqslant 1,$$

and it follows that

$$B_{4k+2}(1-x) \;=\; B_{4k+2}(x)\,, \qquad \text{for } 0 \leqslant x \leqslant 1.$$

The constant term $B_{4k+2}$ causes the integral $\int_0^1 B_{4k+2}(x)\,dx$ to be zero; hence $B_{4k+2} > 0$. The integral of $B_{4k+2}(x)$ is $B_{4k+3}(x)/(4k+3)$, which must therefore be positive when $0 < x < \frac{1}{2}$ and negative when $\frac{1}{2} < x < 1$; furthermore $B_{4k+3}(1-x) = -B_{4k+3}(x)$, so $B_{4k+3}(x)$ has the properties stated for $B_{4k+1}(x)$, but negated. Therefore $B_{4k+4}(x)$ has the properties stated for $B_{4k+2}(x)$, but negated. Therefore $B_{4k+5}(x)$ has the properties stated for $B_{4k+1}(x)$; we have completed a cycle that establishes the stated properties inductively for all k.

According to this analysis, the maximum value of $B_{2m}(x)$ must occur either at $x = 0$ or at $x = \frac{1}{2}$. Exercise 17 proves that

$$B_{2m}(\tfrac{1}{2}) \;=\; (2^{1-2m} - 1)B_{2m}\,; \tag{9.76}$$

hence we have

$$\bigl|B_{2m}(\{x\})\bigr| \;\leqslant\; |B_{2m}|\,. \tag{9.77}$$

This can be used to establish a useful upper bound on the remainder in Euler's summation formula, because we know from (6.89) that

$$\frac{|B_{2m}|}{(2m)!} \;=\; \frac{2}{(2\pi)^{2m}}\sum_{k\geqslant 1}\frac{1}{k^{2m}} \;=\; O\bigl((2\pi)^{-2m}\bigr)\,, \qquad \text{when } m > 0.$$

Therefore we can rewrite Euler's formula (9.67) as follows:

$$\sum_{a<k\leqslant b} f(k) \;=\; \int_a^b f(x)\,dx + \frac{1}{2}f(x)\Big|_a^b + \sum_{k=1}^{m}\frac{B_{2k}}{(2k)!}f^{(2k-1)}(x)\Big|_a^b \\ + O\bigl((2\pi)^{-2m}\bigr)\int_a^b \bigl|f^{(2m)}(x)\bigr|\,dx\,. \tag{9.78}$$

For example, if $f(x) = e^x$, all derivatives are the same and this formula tells us that $\sum_{a<k\leqslant b} e^k = (e^b - e^a)\bigl(1 + \frac{1}{2} + B_2/2! + B_4/4! + \cdots + B_{2m}/(2m)!\bigr) +$

$O\big((2\pi)^{-2m}\big)$. Of course, we know that this sum is actually a geometric series, equal to $(e^{b+1}-e^{a+1})/(e-1) = (e^b - e^a)\sum_{k\geqslant 0} B_k/k!$.

If $f^{(2m)}(x) \geqslant 0$ for $a \leqslant x \leqslant b$, the integral $\int_a^b |f^{(2m)}(x)|\,dx$ is just $f^{(2m-1)}(x)\big|_a^b$, so we have

$$|R_{2m}| \;\leqslant\; \left|\frac{B_{2m}}{(2m)!} f^{(2m-1)}(x)\Big|_a^b\right| ;$$

in other words, the remainder is bounded by the magnitude of the *final term* (the term just before the remainder), in this case. We can give an even better estimate if we know that

$$f^{(2m+2)}(x) \;\geqslant\; 0 \quad\text{and}\quad f^{(2m+4)}(x) \;\geqslant\; 0\,, \qquad \text{for } a \leqslant x \leqslant b. \tag{9.79}$$

For it turns out that this implies the relation

$$R_{2m} \;=\; \theta_m \frac{B_{2m+2}}{(2m+2)!} f^{(2m+1)}(x)\Big|_a^b\,, \qquad \text{for some } 0 \leqslant \theta_m \leqslant 1; \tag{9.80}$$

in other words, the remainder will then lie between 0 and the *first discarded term* in (9.78) — the term that would follow the final term if we increased $m$.

Here's the proof: Euler's summation formula is valid for all $m$, and $B_{2m+1} = 0$ when $m > 0$; hence $R_{2m} = R_{2m+1}$, and the first discarded term must be

$$R_{2m} - R_{2m+2}\,.$$

We therefore want to show that $R_{2m}$ lies between 0 and $R_{2m} - R_{2m+2}$; and this is true if and only if $R_{2m}$ and $R_{2m+2}$ have opposite signs. We claim that

$$f^{(2m+2)}(x) \;\geqslant\; 0 \quad \text{for } a \leqslant x \leqslant b \quad \text{implies} \quad (-1)^m R_{2m} \;\geqslant\; 0\,. \tag{9.81}$$

This, together with (9.79), will prove that $R_{2m}$ and $R_{2m+2}$ have opposite signs, so the proof of (9.80) will be complete.

It's not difficult to prove (9.81) if we recall the definition of $R_{2m+1}$ and the facts we proved about the graph of $B_{2m+1}(x)$. Namely, we have

$$R_{2m} \;=\; R_{2m+1} \;=\; \int_a^b \frac{B_{2m+1}\big(\{x\}\big)}{(2m+1)!}\, f^{(2m+1)}(x)\,dx\,,$$

and $f^{(2m+1)}(x)$ is increasing because its derivative $f^{(2m+2)}(x)$ is positive. (More precisely, $f^{(2m+1)}(x)$ is nondecreasing because its derivative is non-negative.) The graph of $B_{2m+1}\big(\{x\}\big)$ looks like $(-1)^{m+1}$ times a sine wave, so it is geometrically obvious that the second half of each sine wave is more influential than the first half when it is multiplied by an increasing function. This makes $(-1)^m R_{2m+1} \geqslant 0$, as desired. Exercise 16 proves the result formally.

## 9.6 FINAL SUMMATIONS

Now comes the summing up, as we prepare to conclude this book. We will apply Euler's summation formula to some interesting and important examples.

***Summation 1: This one is too easy.***

But first we will consider an interesting *unimportant* example, namely a sum that we already know how to do. Let's see what Euler's summation formula tells us if we apply it to the telescoping sum

$$S_n \;=\; \sum_{1<k\leqslant n} \frac{1}{k(k+1)} \;=\; \sum_{1<k\leqslant n} \Bigl(\frac{1}{k} - \frac{1}{k+1}\Bigr) \;=\; \frac{1}{2} - \frac{1}{n+1}\,.$$

It can't hurt to embark on our first serious application of Euler's formula with the asymptotic equivalent of training wheels.

We might as well start by writing the function $f(x) = 1/\bigl(x(x+1)\bigr)$ in partial fraction form,

$$f(x) \;=\; \frac{1}{x} - \frac{1}{x+1}\,,$$

since this makes it easier to integrate and differentiate. Indeed, we have $f'(x) = -1/x^2 + 1/(x+1)^2$ and $f''(x) = 2/x^3 - 2/(x+1)^3$; in general

$$f^{(k)}(x) \;=\; (-1)^k k!\Bigl(\frac{1}{x^{k+1}} - \frac{1}{(x+1)^{k+1}}\Bigr)\,, \qquad \text{for } k \geqslant 0.$$

Furthermore

$$\int_1^n f(x)\,dx \;=\; \bigl(\ln x - \ln(x+1)\bigr)\Big|_1^n \;=\; \ln\frac{2n}{n+1}\,.$$

Plugging this into the summation formula (9.67) gives

$$S_n \;=\; \ln\frac{2n}{n{+}1} - \sum_{k=1}^{m} (-1)^k \frac{B_k}{k}\Bigl(\frac{1}{n^k} - \frac{1}{(n{+}1)^k} - 1 + \frac{1}{2^k}\Bigr) + R_m(n)\,,$$

$$\text{where } R_m(n) \;=\; -\int_1^n B_m\bigl(\{x\}\bigr)\Bigl(\frac{1}{x^{m+1}} - \frac{1}{(x+1)^{m+1}}\Bigr)\,dx\,.$$

For example, the right-hand side when $m = 4$ is

$$\ln\frac{2n}{n{+}1} + \frac{1}{2}\Bigl(\frac{1}{n} - \frac{1}{n+1} - \frac{1}{2}\Bigr) - \frac{1}{12}\Bigl(\frac{1}{n^2} - \frac{1}{(n+1)^2} - \frac{3}{4}\Bigr) + \frac{1}{120}\Bigl(\frac{1}{n^4} - \frac{1}{(n+1)^4} - \frac{15}{16}\Bigr) + R_4(n)\,.$$

This is kind of a mess; it certainly doesn't look like the real answer $1/2 - 1/(n+1)$. But let's keep going anyway, to see what we've got. We know how to expand the right-hand terms in negative powers of $n$ up to, say, $O(n^{-5})$:

$$\begin{aligned}
\ln\frac{n}{n+1} &= -n^{-1} + \tfrac{1}{2}n^{-2} - \tfrac{1}{3}n^{-3} + \tfrac{1}{4}n^{-4} + O(n^{-5})\,;\\
\frac{1}{n+1} &= n^{-1} - n^{-2} + n^{-3} - n^{-4} + O(n^{-5})\,;\\
\frac{1}{(n+1)^2} &= n^{-2} - 2n^{-3} + 3n^{-4} + O(n^{-5})\,;\\
\frac{1}{(n+1)^4} &= n^{-4} + O(n^{-5})\,.
\end{aligned}$$

Therefore the terms on the right of our approximation add up to

$$\begin{aligned}
&\ln 2 - \tfrac{1}{4} + \tfrac{1}{16} - \tfrac{1}{128} + \bigl(-1 + \tfrac{1}{2} - \tfrac{1}{2}\bigr)n^{-1} + \bigl(\tfrac{1}{2} + \tfrac{1}{2} - \tfrac{1}{12} + \tfrac{1}{12}\bigr)n^{-2}\\
&\quad + \bigl(-\tfrac{1}{3} - \tfrac{1}{2} - \tfrac{2}{12}\bigr)n^{-3} + \bigl(\tfrac{1}{4} + \tfrac{1}{2} + \tfrac{3}{12} + \tfrac{1}{120} - \tfrac{1}{120}\bigr)n^{-4} + R_4(n) + O(n^{-5})\\
&= \ln 2 - \tfrac{25}{128} - n^{-1} + n^{-2} - n^{-3} + n^{-4} + R_4(n) + O(n^{-5})\,.
\end{aligned}$$

The coefficients of $n^{-1}$, $n^{-2}$, $n^{-3}$, and $n^{-4}$ match those of $-1/(n+1)$.

If all were well with the world, we would be able to show that $R_4(n)$ is asymptotically small, maybe $O(n^{-5})$, and we would have an approximation to the sum. But we can't possibly show this, because we happen to know that the correct constant term is $1/2$, not $\ln 2 - \frac{25}{128}$ (which is approximately 0.4978). So $R_4(n)$ is actually equal to $\frac{89}{128} - \ln 2 + O(n^{-5})$, but Euler's summation formula doesn't tell us this.

In other words, we lose.

One way to try fixing things is to notice that the constant terms in the approximation form a pattern, if we let $m$ get larger and larger:

$$\ln 2 - \tfrac{1}{2}B_1 + \tfrac{1}{2}\cdot\tfrac{3}{4}B_2 - \tfrac{1}{3}\cdot\tfrac{7}{8}B_3 + \tfrac{1}{4}\cdot\tfrac{15}{16}B_4 - \tfrac{1}{5}\cdot\tfrac{31}{32}B_5 + \cdots\,.$$

Perhaps we can show that this series approaches $1/2$ as the number of terms becomes infinite? But no; the Bernoulli numbers get very large. For example, $B_{22} = \frac{854513}{138} > 6192$; therefore $\bigl|R_{22}(n)\bigr|$ will be much larger than $\bigl|R_4(n)\bigr|$. We lose totally.

There is a way out, however, and this escape route will turn out to be important in other applications of Euler's formula. The key is to notice that $R_4(n)$ approaches a definite limit as $n \to \infty$:

$$\lim_{n\to\infty} R_4(n) \;=\; -\int_1^\infty B_4\bigl(\{x\}\bigr)\Bigl(\frac{1}{x^5} - \frac{1}{(x+1)^5}\Bigr)\,dx \;=\; R_4(\infty)\,.$$

The integral $\int_1^\infty B_m(\{x\})\, f^{(m)}(x)\, dx$ will exist whenever $f^{(m)}(x) = O(x^{-2})$ as $x \to \infty$, and in this case $f^{(4)}(x)$ surely qualifies. Moreover, we have

$$R_4(n) \;=\; R_4(\infty) + \int_n^\infty B_4(\{x\})\Big(\frac{1}{x^5} - \frac{1}{(x+1)^5}\Big)\, dx$$
$$\;=\; R_4(\infty) + O\Big(\int_n^\infty x^{-6}\, dx\Big) \;=\; R_4(\infty) + O(n^{-5})\,.$$

Thus we have used Euler's summation formula to prove that

$$\sum_{1<k\leqslant n} \frac{1}{k(k+1)} \;=\; \ln 2 - \tfrac{25}{128} - (n+1)^{-1} + R_4(\infty) + O(n^{-5})$$
$$\;=\; C - (n+1)^{-1} + O(n^{-5})$$

for some constant $C$. We do not know what the constant is — some other method must be used to establish it — but Euler's summation formula is able to let us deduce that the constant exists.

Suppose we had chosen a much larger value of $m$. Then the same reasoning would tell us that

$$R_m(n) \;=\; R_m(\infty) + O(n^{-m-1})\,,$$

and we would have the formula

$$\sum_{1<k\leqslant n} \frac{1}{k(k+1)} \;=\; C + c_1 n^{-1} + c_2 n^{-2} + c_3 n^{-3} + \cdots + c_m n^{-m} + O(n^{-m-1})$$

for certain constants $c_1$, $c_2$, … . We know that $c_m$ happens be $(-1)^m$ in this case; but let's prove it, just to restore some of our confidence (in Euler's formula if not in ourselves). The term $\ln\frac{n}{n+1}$ contributes $(-1)^m/m$ to $c_m$; the term $(-1)^{m+1}(B_m/m)n^{-m}$ contributes $(-1)^{m+1}B_m/m$; and the term $(-1)^k(B_k/k)(n+1)^{-k}$ contributes $(-1)^m\binom{m-1}{k-1}B_k/k$. Therefore

$$(-1)^m c_m \;=\; \frac{1}{m} - \frac{B_m}{m} + \sum_{k=1}^{m}\binom{m-1}{k-1}\frac{B_k}{k}$$
$$\;=\; \frac{1}{m} - \frac{B_m}{m} + \frac{1}{m}\sum_{k=1}^{m}\binom{m}{k}B_k \;=\; \frac{1}{m}\big(1 - B_m + B_m(2) - B_0\big)\,.$$

(See (7.79).) Sure enough, it's 1, when $m \geqslant 1$. We have proved that

$$\sum_{1<k\leqslant n} \frac{1}{k(k+1)} \;=\; C - (n+1)^{-1} + O(n^{-m-1})\,, \quad \text{for all } m \geqslant 1. \qquad (9.82)$$

This is not enough to prove that the sum is exactly equal to $C - (n+1)^{-1}$; the actual value might be $C - (n+1)^{-1} + 2^{-n}$ or something. But Euler's summation formula does give us the error bound $O(n^{-m-1})$ for arbitrarily large $m$, even though we haven't evaluated any remainders explicitly.

***Summation 1, again: Recapitulation and generalization.***

Before we leave our training wheels, let's review what we just did from a somewhat higher perspective. We began with a sum

$$S_n = \sum_{1<k\leqslant n} f(k)$$

and we used Euler's summation formula to write

$$S_n = F(n) - F(1) + \sum_{k=1}^{m} \bigl(T_k(n) - T_k(1)\bigr) + R_m(n)\,, \tag{9.83}$$

where $F(x)$ was $\int f(x)\,dx$ and where $T_k(x)$ was a certain term involving $B_k$ and $f^{(k-1)}(x)$. We also noticed that there was a constant $c$ such that

$$f^{(m)}(x) = O(x^{c-m}) \quad \text{as } x \to \infty, \qquad \text{for all large } m.$$

Namely, $f(k)$ was $1/\bigl(k(k+1)\bigr)$; $F(x)$ was $\ln\bigl(x/(x+1)\bigr)$; $c$ was $-2$; and $T_k(x)$ was $(-1)^{k+1}(B_k/k)\bigl(x^{-k} - (x+1)^{-k}\bigr)$. For all large enough values of $m$, this implied that the remainders had a small tail,

$$\begin{aligned} R'_m(n) &= R_m(\infty) - R_m(n) \\ &= (-1)^{m+1}\int_n^\infty \frac{B_m\bigl(\{x\}\bigr)}{m!} f^{(m)}(x)\,dx = O(n^{c+1-m})\,. \end{aligned} \tag{9.84}$$

Therefore we were able to conclude that there exists a constant $C$ such that

$$S_n = F(n) + C + \sum_{k=1}^{m} T_k(n) - R'_m(n)\,. \tag{9.85}$$

(Notice that $C$ nicely absorbed the $T_k(1)$ terms, which were a nuisance.)

We can save ourselves unnecessary work in future problems by simply asserting the existence of $C$ whenever $R_m(\infty)$ exists.

Now let's suppose that $f^{(2m+2)}(x) \geqslant 0$ and $f^{(2m+4)}(x) \geqslant 0$ for $1 \leqslant x \leqslant n$. We have proved that this implies a simple bound (9.80) on the remainder,

$$R_{2m}(n) = \theta_{m,n}\bigl(T_{2m+2}(n) - T_{2m+2}(1)\bigr)\,,$$

where $\theta_{m,n}$ lies somewhere between 0 and 1. But we don't really want bounds that involve $R_{2m}(n)$ and $T_{2m+2}(1)$; after all, we got rid of $T_k(1)$ when we introduced the constant $C$. What we really want is a bound like

$$-R'_{2m}(n) = \phi_{m,n} T_{2m+2}(n)\,,$$

where $0 < \phi_{m,n} < 1$; this will allow us to conclude from (9.85) that

$$S_n \;=\; F(n) + C + T_1(n) + \sum_{k=1}^{m} T_{2k}(n) + \phi_{m,n} T_{2m+2}(n)\,, \qquad (9.86)$$

hence the remainder will truly be between zero and the first discarded term.

A slight modification of our previous argument will patch things up perfectly. Let us assume that

$$f^{(2m+2)}(x) \;\geqslant\; 0 \quad\text{and}\quad f^{(2m+4)}(x) \;\geqslant\; 0\,, \qquad \text{as } x \to \infty. \qquad (9.87)$$

The right-hand side of (9.85) is just like the negative of the right-hand side of Euler's summation formula (9.67) with $a = n$ and $b = \infty$, as far as remainder terms are concerned, and successive remainders are generated by induction on $m$. Therefore our previous argument can be applied.

***Summation 2: Harmonic numbers harmonized.***

Now that we've learned so much from a trivial (but safe) example, we can readily do a nontrivial one. Let us use Euler's summation formula to derive the approximation for $H_n$ that we have been claiming for some time.

In this case, $f(x) = 1/x$. We already know about the integral and derivatives of f, because of Summation 1; also $f^{(m)}(x) = O(x^{-m-1})$ as $x \to \infty$. Therefore we can immediately plug into formula (9.85):

$$\sum_{1<k\leqslant n} \frac{1}{k} \;=\; \ln n \,+\, C \,+\, B_1 n^{-1} \,-\, \sum_{k=1}^{m} \frac{B_{2k}}{2kn^{2k}} \,-\, R'_{2m}(n)\,,$$

for some constant C. The sum on the left is $H_n - 1$, not $H_n$; but of course we can add 1 to both sides. Let us call the constant $\gamma$ instead of $C + 1$, since Euler's constant $\gamma$ is, in fact, defined to be $\lim_{n\to\infty}(H_n - \ln n)$.

The remainder term can be estimated nicely by the theory we developed a minute ago, because $f^{(2m)}(x) = (2m)!/x^{2m+1} \geqslant 0$ for all $x > 0$. Therefore (9.86) tells us that

$$H_n \;=\; \ln n + \gamma + \frac{1}{2n} - \sum_{k=1}^{m} \frac{B_{2k}}{2kn^{2k}} - \theta_{m,n} \frac{B_{2m+2}}{(2m+2)n^{2m+2}}\,, \qquad (9.88)$$

where $\theta_{m,n}$ is some fraction between 0 and 1. This is the general formula whose first few terms are listed in Table 452. For example, when $m = 2$ we get

$$H_n \;=\; \ln n + \gamma + \frac{1}{2n} - \frac{1}{12n^2} + \frac{1}{120n^4} - \frac{\theta_{2,n}}{252n^6}\,. \qquad (9.89)$$

This equation, incidentally, gives us a good approximation to $\gamma$ even when $n = 2$:

$$\gamma \;=\; H_2 - \ln 2 - \tfrac{1}{4} + \tfrac{1}{48} - \tfrac{1}{1920} + \epsilon \;=\; 0.577165\ldots + \epsilon\,,$$

where $\epsilon$ is between zero and $\frac{1}{16128}$. If we take $n = 10^4$ and $m = 250$, we get the value of $\gamma$ correct to 1271 decimal places, beginning thus:

$$\gamma \;=\; 0.57721\,56649\,01532\,86060\,65120\,90082\,40243\ldots\,. \tag{9.90}$$

But Euler's constant appears also in other formulas that allow it to be evaluated even more efficiently [345].

***Summation 3: Stirling's approximation.***

If $f(x) = \ln x$, we have $f'(x) = 1/x$, so we can evaluate the sum of logarithms using almost the same calculations as we did when summing reciprocals. Euler's summation formula yields

$$\sum_{1<k\leqslant n} \ln k \;=\; n\ln n - n + \sigma + \frac{\ln n}{2} + \sum_{k=1}^{m} \frac{B_{2k}}{2k(2k-1)n^{2k-1}} + \varphi_{m,n}\frac{B_{2m+2}}{(2m+2)(2m+1)n^{2m+1}}$$

where $\sigma$ is a certain constant, "Stirling's constant," and $0 < \varphi_{m,n} < 1$. (In this case $f^{(2m)}(x)$ is negative, not positive; but we can still say that the remainder is governed by the first discarded term, because we could have started with $f(x) = -\ln x$ instead of $f(x) = \ln x$.)

Thus, for example,

$$\ln n! \;=\; n\ln n - n + \frac{\ln n}{2} + \sigma + \frac{1}{12n} - \frac{1}{360n^3} + \frac{\varphi_{2,n}}{1260n^5} \tag{9.91}$$

when $m = 2$. And we can get the approximation in Table 452 by taking 'exp' of both sides. (Stirling's original formula was actually a bit different; (9.91) is de Moivre's modification [76]. Stirling [343, p. 137] also stated without proof that $e^\sigma = \sqrt{2\pi}$. We'll soon be ready to prove that remarkable fact.)

If $m$ is fixed and $n \to \infty$, the general formula gives a better and better approximation to $\ln n!$ in the sense of absolute error, hence it gives a better and better approximation to $n!$ in the sense of relative error. But if $n$ is fixed and $m$ increases, the error bound $|B_{2m+2}|/(2m+2)(2m+1)n^{2m+1}$ decreases to a certain point and then begins to increase. Therefore the approximation reaches a point beyond which a sort of uncertainty principle limits the amount by which $n!$ can be approximated via Euler's formula. (See exercise 26.)

*Heisenberg may have been here.*

In Chapter 5, equation (5.83), we generalized factorials to arbitrary real $\alpha$ by using a definition

$$\frac{1}{\alpha!} = \lim_{n\to\infty} \binom{n+\alpha}{n} n^{-\alpha}$$

suggested by Euler. Suppose $\alpha$ is a large number; then

$$\ln \alpha! = \lim_{n\to\infty} \Bigl(\alpha \ln n + \ln n! - \sum_{k=1}^{n} \ln(\alpha+k)\Bigr),$$

and Euler's summation formula can be used with $f(x) = \ln(x+\alpha)$ to estimate this sum:

$$\begin{aligned}
\sum_{k=1}^{n} \ln(k+\alpha) &= F_m(\alpha,n) - F_m(\alpha,0) + R_{2m}(\alpha,n),\\
F_m(\alpha,x) &= (x+\alpha)\ln(x+\alpha) - x + \frac{\ln(x+\alpha)}{2}\\
&\qquad + \sum_{k=1}^{m} \frac{B_{2k}}{2k(2k-1)(x+\alpha)^{2k-1}},\\
R_{2m}(\alpha,n) &= \int_0^n \frac{B_{2m}\bigl(\{x\}\bigr)}{2m} \frac{dx}{(x+\alpha)^{2m}}.
\end{aligned}$$

If we subtract this approximation for $\sum_{k=1}^{n} \ln(k+\alpha)$ from Stirling's approximation for $\ln n!$, then add $\alpha \ln n$ and take the limit as $n \to \infty$, we get

$$\begin{aligned}
\ln \alpha! &= \alpha \ln \alpha - \alpha + \frac{\ln \alpha}{2} + \sigma\\
&\qquad + \sum_{k=1}^{m} \frac{B_{2k}}{(2k)(2k-1)\alpha^{2k-1}} - \int_0^\infty \frac{B_{2m}\bigl(\{x\}\bigr)}{2m} \frac{dx}{(x+\alpha)^{2m}},
\end{aligned}$$

because $\alpha \ln n + n \ln n - n + \frac{1}{2}\ln n - (n+\alpha)\ln(n+\alpha) + n - \frac{1}{2}\ln(n+\alpha) \to -\alpha$ and the other terms not shown here tend to zero. Thus Stirling's approximation behaves for generalized factorials (and for the Gamma function $\Gamma(\alpha+1) = \alpha!$) exactly as for ordinary factorials.

***Summation 4: A bell-shaped summand.***

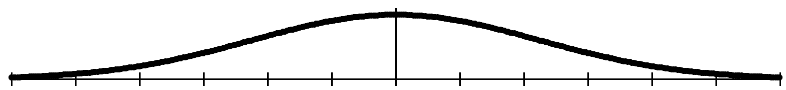
Let's turn now to a sum that has quite a different flavor:

$$\begin{aligned}
\Theta_n &= \sum_k e^{-k^2/n} &(9.92)\\
&= \cdots + e^{-9/n} + e^{-4/n} + e^{-1/n} + 1 + e^{-1/n} + e^{-4/n} + e^{-9/n} + \cdots.
\end{aligned}$$

This is a doubly infinite sum, whose terms reach their maximum value $e^0 = 1$ when $k = 0$. We call it $\Theta_n$ because it is a power series whose terms have the form $e^{p(k)}$, where $p(k)$ is a polynomial of degree 2; such power series are traditionally called "theta functions." If $n = 10^{100}$, we have

$$e^{-k^2/n} \;=\; \begin{cases} e^{-.01} \approx 0.99005, & \text{when } k = 10^{49}; \\ e^{-1} \;\;\approx 0.36788, & \text{when } k = 10^{50}; \\ e^{-100} < 10^{-43}, & \text{when } k = 10^{51}. \end{cases}$$

So the summand stays very near 1 until $k$ gets up to about $\sqrt{n}$, when it drops off and stays very near zero. We can guess that $\Theta_n$ will be proportional to $\sqrt{n}$. Here is a graph of $e^{-k^2/n}$ when $n = 10$:



Larger values of $n$ just stretch the graph horizontally by a factor of $\sqrt{n}$.

We can estimate $\Theta_n$ by letting $f(x) = e^{-x^2/n}$ and taking $a = -\infty$, $b = +\infty$ in Euler's summation formula. (If infinities seem too scary, let $a = -A$ and $b = +B$, then take limits as $A, B \to \infty$.) The integral of $f(x)$ is

$$\int_{-\infty}^{+\infty} e^{-x^2/n}\,dx \;=\; \sqrt{n}\int_{-\infty}^{+\infty} e^{-u^2}\,du \;=\; \sqrt{n}\,C\,,$$

if we replace $x$ by $u\sqrt{n}$. The value of $\int_{-\infty}^{+\infty} e^{-u^2}\,du$ is well known, but we'll call it $C$ for now and come back to it after we have finished plugging into Euler's summation formula.

The next thing we need to know is the sequence of derivatives $f'(x)$, $f''(x)$, ..., and for this purpose it's convenient to set

$$f(x) \;=\; g\bigl(x/\sqrt{n}\,\bigr)\,, \qquad g(x) \;=\; e^{-x^2}\,.$$

Then the chain rule of calculus says that

$$\frac{df(x)}{dx} \;=\; \frac{dg(y)}{dy}\frac{dy}{dx}\,, \qquad y \;=\; \frac{x}{\sqrt{n}}\,;$$

and this is the same as saying that

$$f'(x) \;=\; \frac{1}{\sqrt{n}}\,g'\bigl(x/\sqrt{n}\,\bigr)\,.$$

By induction we have

$$f^{(k)}(x) \;=\; n^{-k/2}g^{(k)}\bigl(x/\sqrt{n}\,\bigr)\,.$$

**24** Suppose $a_n = O\bigl(f(n)\bigr)$ and $b_n = O\bigl(f(n)\bigr)$. Prove or disprove that the convolution $\sum_{k=0}^{n} a_k b_{n-k}$ is also $O\bigl(f(n)\bigr)$, in the following cases:

**a** $f(n) = n^{-\alpha}$, $\alpha > 1$.

**b** $f(n) = \alpha^{-n}$, $\alpha > 1$.

**25** Prove (9.1) and (9.2), with which we opened this chapter.

**26** Equation (9.91) shows how to evaluate $\ln 10!$ with an absolute error $< \frac{1}{126000000}$. Therefore if we take exponentials, we get $10!$ with a relative error that is less than $e^{1/126000000} - 1 < 10^{-8}$. (In fact, the approximation gives 3628799.9714.) If we now round to the nearest integer, knowing that $10!$ is an integer, we get an exact result.

Is it always possible to calculate $n!$ in a similar way, if enough terms of Stirling's approximation are computed? Estimate the value of $m$ that gives the best approximation to $\ln n!$, when $n$ is a fixed (large) integer. Compare the absolute error in this approximation with $n!$ itself.

**27** Use Euler's summation formula to find the asymptotic value of $H_n^{(-\alpha)} = \sum_{k=1}^{n} k^{\alpha}$, where $\alpha$ is any fixed real number. (Your answer may involve a constant that you do not know in closed form.)

**28** Exercise 5.13 defines the hyperfactorial function $Q_n = 1^1 2^2 \ldots n^n$. Find the asymptotic value of $Q_n$ with relative error $O(n^{-1})$. (Your answer may involve a constant that you do not know in closed form.)

**29** Estimate the function $1^{1/1} 2^{1/2} \ldots n^{1/n}$ as in the previous exercise.

**30** Find the asymptotic value of $\sum_{k>0} k^l e^{-k^2/n}$ with absolute error $O(n^{-3})$, when $l$ is a fixed nonnegative integer.

**31** Evaluate $\sum_{k\geqslant 0} 1/(c^k + c^m)$ with absolute error $O(c^{-3m})$, when $c > 1$ and $m$ is a positive integer.

### *Exam problems*

**32** Evaluate $e^{H_n + H_n^{(2)}}$ with absolute error $O(n^{-1})$.

**33** Evaluate $\sum_{k\geqslant 0} \binom{n}{k} / n^{\overline{k}}$ with absolute error $O(n^{-3})$.

**34** Determine values $A$ through $F$ such that $(1 + 1/n)^{nH_n}$ is

$$An + B(\ln n)^2 + C\ln n + D + \frac{E(\ln n)^2}{n} + \frac{F\ln n}{n} + O(n^{-1})\,.$$

**35** Evaluate $\sum_{k=1}^{n} 1/kH_k$ with absolute error $O(1)$.

**36** Evaluate $S_n = \sum_{k=1}^{n} 1/(n^2 + k^2)$ with absolute error $O(n^{-5})$.

**37** Evaluate $\sum_{k=1}^{n} (n \bmod k)$ with absolute error $O(n \log n)$.

**38** Evaluate $\sum_{k\geqslant 0} k^k \binom{n}{k}$ with relative error $O(n^{-1})$.

**6.22** $|z/k(k+z)| \leqslant 2|z|/k^2$ when $k > 2|z|$, so the sum is well defined when the denominators are not zero. If $z = n$ we have $\sum_{k=1}^{m}\bigl(1/k - 1/(k+n)\bigr) = H_m - H_{m+n} + H_n$, which approaches $H_n$ as $m \to \infty$. (The quantity $H_{z-1} - \gamma$ is often called the psi function $\psi(z)$.)

**6.23** $z/(1+e^{-z}) = 2z/(1-e^{-2z}) - z/(1-e^{-z}) = \sum_{n\geqslant 0}(2^n - 1)B_n z^n/n!$.

**6.24** When $n$ is odd, $T_n(x)$ is a polynomial in $x^2$, hence its coefficients are multiplied by even numbers when we form the derivative and compute $T_{n+1}(x)$ by (6.95). (In fact we can prove more: The Bernoulli number $B_{2n}$ always has 2 to the first power in its denominator, by exercise 54; hence $2^{2n-k} \backslash\backslash T_{2n+1} \iff 2^k \backslash\backslash (n+1)$. The odd positive integers $(n+1)T_{2n+1}/2^{2n}$ are called Genocchi numbers $\langle 1, 1, 3, 17, 155, 2073, \ldots \rangle$, after Genocchi [145].)

*(Of course Euler knew the Genocchi numbers long before Genocchi was born; see [110], Volume 2, Chapter 7, §181.)*

**6.25** $100n - nH_n < 100(n-1) - (n-1)H_{n-1} \iff H_{n-1} > 99$. (The least such $n$ is approximately $e^{99-\gamma}$, while he finishes at $N \approx e^{100-\gamma}$, about $e$ times as long. So he is getting closer during the final 63% of his journey.)

**6.26** Let $u(k) = H_{k-1}$ and $\Delta v(k) = 1/k$, so that $u(k) = v(k)$. Then we have $S_n - H_n^{(2)} = \sum_{k=1}^{n} H_{k-1}/k = H_{k-1}^2\big|_1^{n+1} - S_n = H_n^2 - S_n$.

**6.27** Observe that when $m > n$ we have $\gcd(F_m, F_n) = \gcd(F_{m-n}, F_n)$ by (6.108). This yields a proof by induction.

**6.28** (a) $Q_n = \alpha(L_n - F_n)/2 + \beta F_n$. (The solution can also be written $Q_n = \alpha F_{n-1} + \beta F_n$.) (b) $L_n = \phi^n + \hat{\phi}^n$.

**6.29** When $k = 0$ the identity is (6.133). When $k = 1$ it is, essentially,

$$\begin{aligned} K(x_1, \ldots, x_n)x_m \;=\; & K(x_1, \ldots, x_m)\, K(x_m, \ldots, x_n) \\ & - K(x_1, \ldots, x_{m-2})\, K(x_{m+2}, \ldots, x_n)\,; \end{aligned}$$

in Morse code terms, the second product on the right subtracts out the cases where the first product has intersecting dashes. When $k > 1$, an induction on $k$ suffices, using both (6.127) and (6.132). (The identity is also true when one or more of the subscripts on $K$ become $-1$, if we adopt the convention that $K_{-1} = 0$. When multiplication is not commutative, Euler's identity remains valid for $k = n - 1$ if we write it in the form

$$\begin{aligned} & K_{m+n}(x_1, \ldots, x_{m+n})\, K_{n-1}(x_{m+n-1}, \ldots, x_{m+1}) \\ &\quad = K_{m+n-1}(x_1, \ldots, x_{m+n-1})\, K_n(x_{m+n}, \ldots, x_{m+1}) \\ &\qquad - (-1)^n K_{m-1}(x_1, \ldots, x_{m-1})\,. \end{aligned}$$

For example, we obtain the somewhat surprising noncommutative factorizations

$$(abc + a + c)(1 + ba) \;=\; (ab + 1)(cba + a + c)$$

from the case $m = 0$, $n = 3$.)

by 7 only when $n = 6$, 42, 48, 295, 299, 337, 341, 2096, 2390, 14675, 16731, 16735, and 102728. See the answer to exercise 92.)

**6.53** Summation by parts yields

$$\frac{n+1}{(n+2)^2}\left(\frac{(-1)^m}{\binom{n+1}{m+1}}\big((n+2)H_{m+1}-1\big)-1\right).$$

**6.54** (a) If $m \geqslant p$ we have $S_m(p) \equiv S_{m-(p-1)}(p) \pmod p$, since $k^{p-1} \equiv 1$ when $1 \leqslant k < p$. Also $S_{p-1}(p) \equiv p-1 \equiv -1$. If $0 < m < p-1$, we can write

$$S_m(p) \;\equiv\; \sum_{k=0}^{p-1}\sum_{j=0}^{m}\left\{ {m \atop j} \right\} k^{\underline{j}} \;=\; \sum_{j=0}^{m}\left\{ {m \atop j} \right\}\frac{p^{\underline{j+1}}}{j+1} \;\equiv\; 0.$$

(b) The condition in the hint implies that the denominator of $I_{2n}$ is not divisible by any prime $p$; hence $I_{2n}$ must be an integer. To prove the hint, we may assume that $n>1$. Then

*(The numerators of Bernoulli numbers played an important role in early studies of Fermat's Last Theorem; see Ribenboim [308].)*

$$B_{2n} + \frac{\big[(p-1)\backslash(2n)\big]}{p} + \sum_{k=0}^{2n-2}\binom{2n+1}{k}B_k\frac{p^{2n-k}}{2n+1}$$

is an integer, by (6.78), (6.84), and part (a). So we want to verify that none of the fractions $\binom{2n+1}{k}B_k p^{2n-k}/(2n+1) = \binom{2n}{k}B_k p^{2n-k}/(2n-k+1)$ has a denominator divisible by $p$. The denominator of $\binom{2n}{k}B_k p$ isn't divisible by $p$, since $B_k$ has no $p^2$ in its denominator (by induction); and the denominator of $p^{2n-k-1}/(2n-k+1)$ isn't divisible by $p$, since $2n-k+1 < p^{2n-k}$ when $k \leqslant 2n-2$; QED. (The numbers $I_{2n}$ are tabulated in [224]. Hermite calculated them through $I_{18}$ in 1875 [184]. It turns out that $I_2 = I_4 = I_6 = I_8 = I_{10} = I_{12} = 1$; hence there *is* actually a "simple" pattern to the Bernoulli numbers displayed in the text, including $\frac{-691}{2730}$(!). But the numbers $I_{2n}$ don't seem to have any memorable features when $2n > 12$. For example, $B_{24} = -86579 - \frac12 - \frac13 - \frac15 - \frac17 - \frac1{13}$, and 86579 is prime.)

(c) The numbers $2-1$ and $3-1$ always divide $2n$. If $n$ is prime, the only divisors of $2n$ are 1, 2, $n$, and $2n$, so the denominator of $B_{2n}$ for prime $n > 2$ will be 6 unless $2n+1$ is also prime. In the latter case we can try $4n+3$, $8n+7$, …, until we eventually hit a nonprime (since $n$ divides $2^{n-1}n + 2^{n-1} - 1$). (This proof does not need the more difficult, but true, theorem that there are infinitely many primes of the form $6k+1$.) The denominator of $B_{2n}$ can be 6 also when $n$ has nonprime values, such as 49.

**6.55** The stated sum is $\frac{m+1}{x+m+1}\binom{x+n}{n}\binom{n}{m+1}$, by Vandermonde's convolution. To get (6.70), differentiate and set $x = 0$.

**6.74** Since $\tan 2z + \sec 2z = (\sin z + \cos z)/(\cos z - \sin z)$, setting $x = 1$ in (6.94) gives $T_n(1) = 2^n E_n$, where $1/\cos z = \sum_{n\geqslant 0} E_{2n} z^{2n}/(2n)!$. (The coefficients $E_n$ are called *Euler numbers* in combinatorics, not to be confused with the Eulerian numbers $\left\langle {n \atop k} \right\rangle$. We have $\langle E_0, E_1, E_2, \ldots \rangle = \langle 1, 1, 1, 2, 5, 16, 61, 272, 1385, 7936, 50521, \ldots \rangle$. Numerical analysts define Euler numbers differently: Their $E_n$ is $(-1)^{n/2} E_n [n \text{ even}]$ in the notation above.)

**6.75** Let $G(w, z) = \sin z / \cos(w+z)$ and $H(w, z) = \cos z / \cos(w+z)$, and let $G(w, z) + H(w, z) = \sum_{m,n} E_{m,n} w^m z^n / m!\, n!$. Then the equations $G(w, 0) = 0$ and $(\frac{\partial}{\partial z} - \frac{\partial}{\partial w}) G(w, z) = H(w, z)$ imply that $E_{m,0} = 0$ when $m$ is odd, $E_{m,n+1} = E_{m+1,n} + E_{m,n}$ when $m + n$ is even; the equations $H(0, z) = 1$ and $(\frac{\partial}{\partial w} - \frac{\partial}{\partial z}) H(w, z) = G(w, z)$ imply that $E_{0,n} = [n = 0]$ when $n$ is even, $E_{m+1,n} = E_{m,n+1} + E_{m,n}$ when $m+n$ is odd. Consequently the $n$th row below the apex of the triangle contains the numbers $E_{n,0}, E_{n-1,1}, \ldots, E_{0,n}$. At the left, $E_{n,0}$ is the secant number $E_n [n \text{ even}]$; at the right, $E_{0,n} = T_n + [n = 0]$.

**6.76** Let $A_n$ denote the sum. Looking ahead to equation (7.49), we see that $\sum_n A_n z^n/n! = \sum_{n,k} (-1)^k \left\{ {n \atop k} \right\} 2^{n-k} k!\, z^n/n! = \sum_k (-1)^k 2^{-k} (e^{2z} - 1)^k = 2/(e^{2z} + 1) = 1 - \tanh z$. When $n > 0$, it follows by exercise 23 or 72 that

$$A_n = (2^{n+1} - 4^{n+1}) B_{n+1}/(n+1) = (-1)^{(n+1)/2} T_n .$$

**6.77** This follows by induction on $m$, using the recurrence in exercise 18. It can also be proved from (6.50), using the fact that

$$\begin{aligned} \frac{(-1)^{m-1}(m-1)!}{(e^z - 1)^m} &= (D+1)^{\overline{m-1}} \frac{1}{e^z - 1} \\ &= \sum_{k=0}^{m-1} \left[ {m \atop m-k} \right] \frac{d^{m-k-1}}{dz^{m-k-1}} \frac{1}{e^z - 1}, \quad \text{integer } m > 0. \end{aligned}$$

The latter equation, incidentally, is equivalent to

$$\frac{d^m}{dz^m} \frac{1}{e^z - 1} = (-1)^m \sum_k \left\{ {m+1 \atop k} \right\} \frac{(k-1)!}{(e^z - 1)^k}, \quad \text{integer } m \geqslant 0.$$

**6.78** If $p(x)$ is any polynomial of degree $\leqslant n$, we have

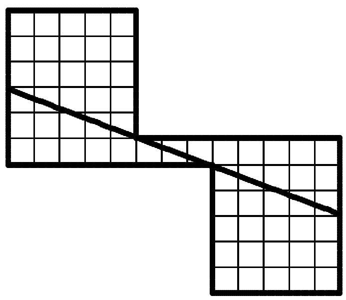
$$p(x) = \sum_k p(-k) \binom{-x}{k} \binom{x+n}{n-k},$$

because this equation holds for $x = 0, -1, \ldots, -n$. The stated identity is the special case where $p(x) = x\sigma_n(x)$ and $x = 1$. Incidentally, we obtain a simpler expression for Bernoulli numbers in terms of Stirling numbers by setting $k = 1$ in (6.99):

$$\sum_{k\geqslant 0} \left\{ m \atop k \right\} (-1)^{m-k} \frac{k!}{k+1} \;=\; B_m \,.$$

**6.79** Sam Loyd [256, pages 288 and 378] gave the construction

*He also published it in the Brooklyn Daily Eagle (28 August 1904), 39; (11 September 1904), 37.*



and claimed to have invented (but not published) the $64 = 65$ arrangement in 1858. (Similar paradoxes go back at least to the eighteenth century, but Loyd found better ways to present them.)

**6.80** We expect $A_m/A_{m-1} \approx \phi$, so we try $A_{m-1} = 618034 + r$ and $A_{m-2} = 381966 - r$. Then $A_{m-3} = 236068 + 2r$, etc., and we find $A_{m-18} = 144 - 2584r$, $A_{m-19} = 154 + 4181r$. Hence $r = 0$, $x = 154$, $y = 144$, $m = 20$.

**6.81** If $P(F_{n+1}, F_n) = 0$ for infinitely many *even* values of $n$, then $P(x, y)$ is divisible by $U(x, y) - 1$, where $U(x, y) = x^2 - xy - y^2$. For if $t$ is the total degree of $P$, we can write

$$P(x,y) \;=\; \sum_{k=0}^{t} q_k x^k y^{t-k} + \sum_{j+k<t} r_{j,k} x^j y^k \;=\; Q(x,y) + R(x,y)\,.$$

Then

$$\frac{P(F_{n+1}, F_n)}{F_n^t} \;=\; \sum_{k=0}^{t} q_k \left(\frac{F_{n+1}}{F_n}\right)^k + O(1/F_n)$$

and we have $\sum_{k=0}^{t} q_k \phi^k = 0$ by taking the limit as $n \to \infty$. Hence $Q(x, y)$ is a multiple of $U(x, y)$, say $A(x, y)U(x, y)$. But $U(F_{n+1}, F_n) = (-1)^n$ and $n$ is even, so $P_0(x, y) = P(x, y) - \big(U(x, y) - 1\big)A(x, y)$ is another polynomial such that $P_0(F_{n+1}, F_n) = 0$. The total degree of $P_0$ is less than $t$, so $P_0$ is a multiple of $U - 1$ by induction on $t$.

Similarly, $P(x, y)$ is divisible by $U(x, y) + 1$ if $P(F_{n+1}, F_n) = 0$ for infinitely many *odd* values of $n$. A combination of these two facts gives the desired necessary and sufficient condition: $P(x, y)$ is divisible by $U(x, y)^2 - 1$.

that $q_n(x) = i^n E_n(x)$, where $E_n(x)$ is called an Euler polynomial. We have $\sum(-1)^x x^n\,\delta x = \frac{1}{2}(-1)^{x+1}E_n(x)$, so Euler polynomials are analogous to Bernoulli polynomials, and they have factors analogous to those in (6.98). By exercise 6.23 we have $nE_{n-1}(x) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k B_k x^{n-k}(2-2^{k+1})$; this polynomial has integer coefficients by exercise 6.54. Hence $q_{2n}(x)$, whose coefficients have denominators that are powers of 2, must have integer coefficients. Hence $p_n(y)$ itself has integer coefficients. Finally, the relation $(4y-1)p_n''(y) + 2p_n'(y) = 2n(2n-1)p_{n-1}(y)$ shows that

$$2m(2m-1)\left|{n \atop m}\right| \;=\; m(m+1)\left|{n \atop m+1}\right| + 2n(2n-1)\left|{n-1 \atop m-1}\right|,$$

and it follows that the $\left|{n \atop m}\right|$'s are positive. (A similar proof shows that the related quantity $(-1)^n(2n+2)E_{2n+1}(x)/(2x-1)$ has positive integer coefficients, when expressed as an $n$th degree polynomial in $y$.) It can be shown that $\left|{n \atop 1}\right|$ is the Genocchi number $(-1)^{n-1}(2^{2n+1}-2)B_{2n}$ (see exercise 6.24), and that $\left|{n \atop n-1}\right| = \binom{n}{2}$, $\left|{n \atop n-2}\right| = 2\binom{n+1}{4} + 3\binom{n}{4}$, etc.

**7.53** It is $P_{(1+V_{4n+1}+V_{4n+3})/6}$. Thus, for example, $T_{20} = P_{12} = 210$; $T_{285} = P_{165} = 40755$.

**7.54** Let $E_k$ be the operation on power series that sets all coefficients to zero except those of $z^n$ where $n \bmod m = k$. The stated construction is equivalent to the operation

$$E_0\,S\,E_0\,S\,(E_0+E_1)\,S\,\ldots\,S\,(E_0+E_1+\cdots+E_{m-1})$$

applied to $1/(1-z)$, where S means "multiply by $1/(1-z)$." There are $m!$ terms

$$E_0\,S\,E_{k_1}\,S\,E_{k_2}\,S\,\ldots\,S\,E_{k_m}$$

where $0 \leqslant k_j < j$, and every such term evaluates to $z^{rm}/(1-z^m)^{m+1}$ if $r$ is the number of places where $k_j < k_{j+1}$. Exactly $\left\langle{m \atop r}\right\rangle$ terms have a given value of $r$, so the coefficient of $z^{mn}$ is $\sum_{r=0}^{m-1}\left\langle{m \atop r}\right\rangle\binom{n+m-r}{m} = (n+1)^m$ by (6.37). (The fact that operation $E_k$ can be expressed with complex roots of unity seems to be of no help in this problem.)

**7.55** Suppose that $P_0(z)F(z) + \cdots + P_m(z)F^{(m)}(z) = Q_0(z)G(z) + \cdots + Q_n(z)G^{(n)}(z) = 0$, where $P_m(z)$ and $Q_n(z)$ are nonzero. (a) Let $H(z) = F(z) + G(z)$. Then there are rational functions $R_{k,l}(z)$ for $0 \leqslant l < m+n$ such that $H^{(k)}(z) = R_{k,0}(z)F^{(0)}(z) + \cdots + R_{k,m-1}(z)F^{(m-1)}(z) + R_{k,m}(z)G^{(0)}(z) + \cdots + R_{k,m+n-1}(z)G^{(n-1)}(z)$. The $m+n+1$ vectors $\big(R_{k,0}(z), \ldots, R_{k,m+n-1}(z)\big)$ are linearly dependent in the $(m+n)$-dimensional vector space whose components are rational functions; hence there are rational functions $S_l(z)$, not

**8.37** The number of coin-toss sequences of length $n$ is $F_{n-1}$, for all $n > 0$, because of the relation between domino tilings and coin flips. Therefore the probability that exactly $n$ tosses are needed is $F_{n-1}/2^n$, when the coin is fair. Also $q_n = F_{n+1}/2^{n-1}$, since $\sum_{k\geqslant n} F_k z^k = (F_n z^n + F_{n-1}z^{n+1})/(1-z-z^2)$. (A systematic solution via generating functions is, of course, also possible.)

**8.38** When $k$ faces have been seen, the task of rolling a new one is equivalent to flipping coins with success probability $p_k = (m-k)/m$. Hence the pgf is $\prod_{k=0}^{l-1} p_k z/(1-q_k z) = \prod_{k=0}^{l-1}(m-k)z/(m-kz)$. The mean is $\sum_{k=0}^{l-1} p_k^{-1} = m(H_m - H_{m-l})$; the variance is $m^2\bigl(H_m^{(2)} - H_{m-l}^{(2)}\bigr) - m(H_m - H_{m-l})$; and equation (7.47) provides a closed form for the requested probability, namely $m^{-n}m!\left\{{n-1 \atop l-1}\right\}/(m-l)!$. (The problem discussed in this exercise is traditionally called "coupon collecting.")

**8.39** $E(X) = P(-1)$; $V(X) = P(-2) - P(-1)^2$; $E(\ln X) = -P'(0)$.

**8.40** (a) We have $\kappa_m = n\bigl(0!\left\{{m \atop 1}\right\}p - 1!\left\{{m \atop 2}\right\}p^2 + 2!\left\{{m \atop 3}\right\}p^3 - \cdots\bigr)$, by (7.49). Incidentally, the third cumulant is $npq(q-p)$ and the fourth is $npq(1-6pq)$. The identity $q+pe^t = (p+qe^{-t})e^t$ shows that $f_m(p) = (-1)^m f_m(q) + [m=1]$; hence we can write $f_m(p) = g_m(pq)(q-p)^{[m\ \text{odd}]}$, where $g_m$ is a polynomial of degree $\lfloor m/2 \rfloor$, whenever $m > 1$. (b) Let $p = \frac12$ and $F(t) = \ln(\frac12 + \frac12 e^t)$. Then $\sum_{m\geqslant 1} \kappa_m t^{m-1}/(m-1)! = F'(t) = 1/(1+e^{-t})$, and we can use exercise 6.23.

**8.41** If $G(z)$ is the pgf for a random variable $X$ that assumes only positive integer values, then $\int_0^1 G(z)\,dz/z = \sum_{k\geqslant 1} \Pr(X=k)/k = E(X^{-1})$. If $X$ is the distribution of the number of flips to obtain $n+1$ heads, we have $G(z) = \bigl(pz/(1-qz)\bigr)^{n+1}$ by (8.59), and the integral is

$$\int_0^1 \left(\frac{pz}{1-qz}\right)^{n+1} \frac{dz}{z} = \int_0^1 \frac{w^n\,dw}{1+(q/p)w}$$

if we substitute $w = pz/(1-qz)$. When $p = q$ the integrand can be written $(-1)^n\bigl((1+w)^{-1} - 1 + w - w^2 + \cdots + (-1)^n w^{n-1}\bigr)$, so the integral is $(-1)^n\bigl(\ln 2 - 1 + \frac12 - \frac13 + \cdots + (-1)^n/n\bigr)$. We have $H_{2n} - H_n = \ln 2 - \frac14 n^{-1} + \frac1{16}n^{-2} + O(n^{-4})$ by (9.28), and it follows that $E(X_{n+1}^{-1}) = \frac12 n^{-1} - \frac14 n^{-2} + O(n^{-4})$.

**8.42** Let $F_n(z)$ and $G_n(z)$ be pgf's for the number of employed evenings, if the man is initially unemployed or employed, respectively. Let $q_h = 1 - p_h$ and $q_f = 1 - p_f$. Then $F_0(z) = G_0(z) = 1$, and

$$\begin{aligned} F_n(z) &= p_h z G_{n-1}(z) + q_h F_{n-1}(z)\,; \\ G_n(z) &= p_f F_{n-1}(z) + q_f z G_{n-1}(z)\,. \end{aligned}$$

**9.3** Replacing $kn$ by $O(n)$ requires a different $C$ for each $k$; but each $O$ stands for a single $C$. In fact, the context of this $O$ requires it to stand for a set of functions of two variables $k$ and $n$. It would be correct to write $\sum_{k=1}^{n} kn = \sum_{k=1}^{n} O(n^2) = O(n^3)$.

**9.4** For example, $\lim_{n\to\infty} O(1/n) = 0$. On the left, $O(1/n)$ is the set of all functions $f(n)$ such that there are constants $C$ and $n_0$ with $\bigl|f(n)\bigr| \leqslant C/n$ for all $n \geqslant n_0$. The limit of all functions in that set is 0, so the left-hand side is the singleton set $\{0\}$. On the right, there are no variables; 0 represents $\{0\}$, the (singleton) set of all "functions of no variables, whose value is zero." (Can you see the inherent logic here? If not, come back to it next year; you probably can still manipulate O-notation even if you can't shape your intuitions into rigorous formalisms.)

**9.5** Let $f(n) = n^2$ and $g(n) = 1$; then $n$ is in the left set but not in the right, so the statement is false.

**9.6** $n \ln n + \gamma n + O(\sqrt{n} \ln n)$.

**9.7** $(1 - e^{-1/n})^{-1} = nB_0 + B_1 + B_2 n^{-1}/2! + \cdots = n + \frac{1}{2} + O(n^{-1})$.

**9.8** For example, let $f(n) = \lfloor n/2 \rfloor!^2 + n$, $g(n) = \bigl(\lceil n/2 \rceil - 1\bigr)!\,\lceil n/2 \rceil! + n$. These functions, incidentally, satisfy $f(n) = O\bigl(ng(n)\bigr)$ and $g(n) = O\bigl(nf(n)\bigr)$; more extreme examples are clearly possible.

**9.9** (For completeness, we assume that there is a side condition $n \to \infty$, so that two constants are implied by each $O$.) Every function on the left has the form $a(n) + b(n)$, where there exist constants $m_0$, $B$, $n_0$, $C$ such that $\bigl|a(n)\bigr| \leqslant B\bigl|f(n)\bigr|$ for $n \geqslant m_0$ and $\bigl|b(n)\bigr| \leqslant C\bigl|g(n)\bigr|$ for $n \geqslant n_0$. Therefore the left-hand function is at most $\max(B, C)\bigl(\bigl|f(n)\bigr| + \bigl|g(n)\bigr|\bigr)$, for $n \geqslant \max(m_0, n_0)$, so it is a member of the right side.

**9.10** If $g(x)$ belongs to the left, so that $g(x) = \cos y$ for some $y$, where $|y| \leqslant C|x|$ for some $C$, then $0 \leqslant 1 - g(x) = 2\sin^2(y/2) \leqslant \frac{1}{2}y^2 \leqslant \frac{1}{2}C^2x^2$; hence the set on the left is contained in the set on the right, and the formula is true.

**9.11** The proposition is true. For if, say, $|x| \leqslant |y|$, we have $(x+y)^2 \leqslant 4y^2$. Thus $(x+y)^2 = O(x^2) + O(y^2)$. Thus $O(x+y)^2 = O\bigl((x+y)^2\bigr) = O\bigl(O(x^2) + O(y^2)\bigr) = O\bigl(O(x^2)\bigr) + O\bigl(O(y^2)\bigr) = O(x^2) + O(y^2)$.

**9.12** $1 + 2/n + O(n^{-2}) = (1 + 2/n)\bigl(1 + O(n^{-2})/(1 + 2/n)\bigr)$ by (9.26), and $1/(1 + 2/n) = O(1)$; now use (9.26).

**9.13** $n^n\bigl(1 + 2n^{-1} + O(n^{-2})\bigr)^n = n^n \exp\bigl(n\bigl(2n^{-1} + O(n^{-2})\bigr)\bigr) = e^2 n^n + O(n^{n-1})$.

**9.14** It is $n^{n+\beta} \exp\bigl((n+\beta)\bigl(\alpha/n - \frac{1}{2}\alpha^2/n^2 + O(n^{-3})\bigr)\bigr)$.

*(It's interesting to compare this formula with the corresponding result for the middle binomial coefficient, exercise 9.60.)*

**9.15** $\ln\binom{3n}{n,n,n} = 3n\ln 3 - \ln n + \frac{1}{2}\ln 3 - \ln 2\pi + \left(\frac{1}{36} - \frac{1}{4}\right)n^{-1} + O(n^{-3})$, so the answer is

$$\frac{3^{3n+1/2}}{2\pi n}\bigl(1 - \tfrac{2}{9}n^{-1} + \tfrac{2}{81}n^{-2} + O(n^{-3})\bigr)\,.$$

**9.16** If $l$ is any integer in the range $a \leqslant l < b$ we have

$$\begin{aligned}\int_0^1 B(x)f(l+x)\,dx &= \int_{1/2}^1 B(x)f(l+x)\,dx - \int_0^{1/2} B(1-x)f(l+x)\,dx\\ &= \int_{1/2}^1 B(x)\bigl(f(l+x) - f(l+1-x)\bigr)\,dx\,.\end{aligned}$$

Since $l + x \geqslant l + 1 - x$ when $x \geqslant \frac{1}{2}$, this integral is positive when $f(x)$ is nondecreasing.

**9.17** $\sum_{m\geqslant 0} B_m(\frac{1}{2})z^m/m! = ze^{z/2}/(e^z - 1) = z/(1 - e^{-z/2}) - z/(1 - e^{-z})$.

**9.18** The text's derivation for the case $\alpha = 1$ generalizes to give

$$b_k(n) \;=\; \frac{2^{(2n+1/2)\alpha}}{(2\pi n)^{\alpha/2}}e^{-k^2\alpha/n}\,,\quad c_k(n) \;=\; 2^{2n\alpha}\,n^{-(1+\alpha)/2+3\epsilon}e^{-k^2\alpha/n}\,;$$

the answer is $2^{2n\alpha}(\pi n)^{(1-\alpha)/2}\alpha^{-1/2}\bigl(1 + O(n^{-1/2+3\epsilon})\bigr)$.

**9.19** $H_{10} = 2.928968254{-} \approx 2.928968258$; $10! = 3628800 \approx 3628800.05$; $B_{10} = 0.0757575\ldots \approx 0.07575749$; $\pi(10) = 4 \approx 10.002$; $e^{0.1} = 1.1051709{+} \approx 1.1051708$; $\ln 1.1 = 0.095310{+} \approx 0.095308$; $1.1111111\ldots \approx 1.1111$; $1.1^{0.1} = 1.0095765{+} \approx 1.0095764$. (The approximation to $\pi(n)$ gives more significant figures when $n$ is larger; for example, $\pi(10^9) = 50847534 \approx 50840742$.)

**9.20** (a) Yes; the left side is $o(n)$ while the right side is equivalent to $O(n)$. (b) Yes; the left side is $e\cdot e^{O(1/n)}$. (c) No; the left side is about $\sqrt{n}$ times the bound on the right.

**9.21** We have $P_n = p = n\bigl(\ln p - 1 - 1/\ln p + O(1/\log n)^2\bigr)$, where

$$\begin{aligned}\ln p &= \ln n + \ln\ln p - 1/\ln n + \ln\ln n/(\ln n)^2 + O(1/\log n)^2\,;\\ \ln\ln p &= \ln\ln n + \frac{\ln\ln n}{\ln n} - \frac{(\ln\ln n)^2}{2(\ln n)^2} + \frac{\ln\ln n}{(\ln n)^2} + O(1/\log n)^2\,.\end{aligned}$$

It follows that

$$\begin{aligned}P_n \;=\; n\biggl(&\ln n + \ln\ln n - 1\\ &+ \frac{\ln\ln n - 2}{\ln n} - \frac{(\ln\ln n)^2/2 - 3\ln\ln n}{(\ln n)^2} + O(1/\log n)^2\biggr)\,.\end{aligned}$$

(A slightly better approximation replaces this $O(1/\log n)^2$ by the quantity $-5.5/(\ln n)^2 + O(\log\log n/\log n)^3$; then we estimate $P_{1000000} \approx 15480992.8$.)

*What does a drowning analytic number theorist say?*

*log log log log . . .*

**9.22** Replace $O(n^{-2k})$ by $-\frac{1}{12}n^{-2k} + O(n^{-4k})$ in the expansion of $H_{n^k}$; this replaces $O\bigl(\Sigma_3(n^2)\bigr)$ by $-\frac{1}{12}\Sigma_3(n^2) + O\bigl(\Sigma_3(n^4)\bigr)$ in (9.53). We have

$$\Sigma_3(n) \;=\; \tfrac{3}{4}n^{-1} + \tfrac{5}{36}n^{-2} + O(n^{-3}),$$

hence the term $O(n^{-2})$ in (9.54) can be replaced by $-\frac{19}{144}n^{-2} + O(n^{-3})$.

**9.23** $nh_n = \sum_{0\leqslant k<n} h_k/(n-k) + 2cH_n/(n+1)(n+2)$. Choose $c = e^{\pi^2/6} = \sum_{k\geqslant 0} g_k$ so that $\sum_{k\geqslant 0} h_k = 0$ and $h_n = O(\log n)/n^3$. The expansion of $\sum_{0\leqslant k<n} h_k/(n-k)$ as in (9.60) now yields $nh_n = 2cH_n/(n+1)(n+2) + O(n^{-2})$, hence

$$g_n \;=\; e^{\pi^2/6}\left(\frac{n + 2\ln n + O(1)}{n^3}\right).$$

**9.24** (a) If $\sum_{k\geqslant 0}\bigl|f(k)\bigr| < \infty$ and if $f(n-k) = O\bigl(f(n)\bigr)$ when $0 \leqslant k \leqslant n/2$, we have

$$\sum_{k=0}^{n} a_k b_{n-k} \;=\; \sum_{k=0}^{n/2} O\bigl(f(k)\bigr)O\bigl(f(n)\bigr) + \sum_{k=n/2}^{n} O\bigl(f(n)\bigr)O\bigl(f(n-k)\bigr),$$

which is $2O\bigl(f(n)\sum_{k\geqslant 0}|f(k)|\bigr)$, so this case is proved. (b) But in this case if $a_n = b_n = \alpha^{-n}$, the convolution $(n+1)\alpha^{-n}$ is not $O(\alpha^{-n})$.

**9.25** $S_n/\binom{3n}{n} = \sum_{k=0}^{n} n^{\underline{k}}/(2n+1)^{\overline{k}}$. We may restrict the range of summation to $0 \leqslant k \leqslant (\log n)^2$, say. In this range $n^{\underline{k}} = n^k\bigl(1 - \binom{k}{2}/n + O(k^4/n^2)\bigr)$ and $(2n+1)^{\overline{k}} = (2n)^k\bigl(1 + \binom{k+1}{2}/2n + O(k^4/n^2)\bigr)$, so the summand is

$$\frac{1}{2^k}\left(1 - \frac{3k^2 - k}{4n} + O\Bigl(\frac{k^4}{n^2}\Bigr)\right).$$

Hence the sum over k is $2 - 4/n + O(1/n^2)$. Stirling's approximation can now be applied to $\binom{3n}{n} = (3n)!/(2n)!\,n!$, proving (9.2).

**9.26** The minimum occurs at a term $|B_{2m}|/\bigl((2m)(2m-1)n^{2m-1}\bigr)$ where $2m \approx 2\pi n + \frac{3}{2}$, and this term is approximately equal to $1/(\pi e^{2\pi n}\sqrt{n}\,)$. The absolute error in $\ln n!$ is therefore too large to determine $n!$ exactly by rounding to an integer, when $n$ is greater than about $e^{2\pi+1}$.

**9.27** We may assume that $\alpha \neq -1$. Let $f(x) = x^\alpha$; the answer is

$$\sum_{k=1}^{n} k^\alpha \;=\; C_\alpha + \frac{n^{\alpha+1}}{\alpha+1} + \sum_{k=1}^{2m}\frac{B_k}{k}\binom{\alpha}{k-1}n^{\alpha-k+1} + O(n^{\alpha-2m-1}).$$

*In particular,* $\zeta(-n) = -B_{n+1}/(n+1)$ *for integer* $n \geqslant 0$.

(The constant $C_\alpha$ turns out to be $\zeta(-\alpha)$, which is in fact *defined* by this formula when $\alpha > -1$.)

**9.28** In general, suppose $f(x) = x^\alpha \ln x$ in Euler's summation formula, when $\alpha \neq -1$. Proceeding as in the previous exercise, we find

$$\sum_{k=1}^{n} k^\alpha \ln k \;=\; C'_\alpha + \frac{n^{\alpha+1}\ln n}{\alpha+1} - \frac{n^{\alpha+1}}{(\alpha+1)^2} + \sum_{k=1}^{2m} \frac{B_k}{k}\binom{\alpha}{k-1} n^{\alpha-k+1}(\ln n + H_\alpha - H_{\alpha-k+1}) + O(n^{\alpha-2m-1}\log n)\,;$$

the constant $C'_\alpha$ can be shown [74, §3.7] to be $-\zeta'(-\alpha)$. (The $\log n$ factor in the O term can be removed when $\alpha$ is a positive integer $\leqslant 2m$; in that case we also replace the kth term of the right sum by $B_k \alpha!\,(k-2-\alpha)! \times (-1)^\alpha n^{\alpha-k+1}/k!$ when $\alpha < k-1$.) To solve the stated problem, we let $\alpha = 1$ and $m = 1$, taking the exponential of both sides to get

$$Q_n \;=\; A \cdot n^{n^2/2+n/2+1/12} e^{-n^2/4}\bigl(1 + O(n^{-2})\bigr)\,,$$

where $A = e^{1/12-\zeta'(-1)} \approx 1.2824271291$ is "Glaisher's constant."

**9.29** Let $f(x) = x^{-1}\ln x$. A slight modification of the calculation in the previous exercise gives

$$\sum_{k=1}^{n} \frac{\ln k}{k} \;=\; \frac{(\ln n)^2}{2} + \gamma_1 + \sum_{k=1}^{2m} \frac{B_k}{k}(-1)^{k-1} n^{-k}(\ln n - H_{k-1}) + O(n^{-2m-2}\log n)\,,$$

where $\gamma_1 \approx -0.07281584548367672486$ is a "Stieltjes constant" (see the answer to 9.57). Taking exponentials gives

$$e^{\gamma_1}\sqrt{n^{\ln n}}\left(1 + \frac{\ln n}{2n} + O\Bigl(\frac{\log n}{n}\Bigr)^2\right).$$

**9.30** Let $g(x) = x^l e^{-x^2}$ and $f(x) = g(x/\sqrt{n}\,)$. Then $n^{-l/2}\sum_{k>0} k^l e^{-k^2/n}$ is

$$\int_0^\infty f(x)\,dx - \sum_{k=1}^{m} \frac{B_k}{k!} f^{(k-1)}(0) - (-1)^m \int_0^\infty \frac{B_m\bigl(\{x\}\bigr)}{m!} f^{(m)}(x)\,dx \;=\; n^{1/2}\int_0^\infty g(x)\,dx - \sum_{k=1}^{m} \frac{B_k}{k!} n^{(k-1)/2} g^{(k-1)}(0) + O(n^{-m/2})\,.$$

Since $g(x) = x^l - x^{2+l}/1! + x^{4+l}/2! - x^{6+l}/3! + \cdots$, the derivatives $g^{(m)}(x)$ obey a simple pattern, and the answer is

$$\frac{1}{2}n^{(l+1)/2}\Gamma\Big(\frac{l+1}{2}\Big) - \frac{B_{l+1}}{(l+1)!\,0!} + \frac{B_{l+3}n^{-1}}{(l+3)!\,1!} - \frac{B_{l+5}n^{-2}}{(l+5)!\,2!} + O(n^{-3})\,.$$

**9.31** The somewhat surprising identity $1/(c^{m-k}+c^m) + 1/(c^{m+k}+c^m) = 1/c^m$ makes the terms for $0 \leqslant k \leqslant 2m$ sum to $(m+\frac{1}{2})/c^m$. The remaining terms are

$$\begin{aligned}\sum_{k\geqslant 1}\frac{1}{c^{2m+k}+c^m} &= \sum_{k\geqslant 1}\Big(\frac{1}{c^{2m+k}} - \frac{1}{c^{3m+2k}} + \frac{1}{c^{4m+3k}} - \cdots\Big)\\ &= \frac{1}{c^{2m+1}-c^{2m}} - \frac{1}{c^{3m+2}-c^{3m}} + \cdots\,,\end{aligned}$$

and this series can be truncated at any desired point, with an error not exceeding the first omitted term.

**9.32** $H_n^{(2)} = \pi^2/6 - 1/n + O(n^{-2})$ by Euler's summation formula, since we know the constant; and $H_n$ is given by (9.89). So the answer is

$$ne^{\gamma+\pi^2/6}\big(1 - \tfrac{1}{2}n^{-1} + O(n^{-2})\big)\,.$$

*The world's top three constants, $(e, \pi, \gamma)$, all appear in this answer.*

**9.33** We have $n^{\underline{k}}/n^{\overline{k}} = 1 - k(k-1)n^{-1} + \frac{1}{2}k^2(k-1)^2n^{-2} + O(k^6n^{-3})$; dividing by $k!$ and summing over $k \geqslant 0$ yields $e - en^{-1} + \frac{7}{2}en^{-2} + O(n^{-3})$.

**9.34** $A = e^\gamma$; $B = 0$; $C = -\frac{1}{2}e^\gamma$; $D = \frac{1}{2}e^\gamma(1-\gamma)$; $E = \frac{1}{8}e^\gamma$; $F = \frac{1}{12}e^\gamma(3\gamma+1)$.

**9.35** Since $1/k\big(\ln k + O(1)\big) = 1/k\ln k + O\big(1/k(\log k)^2\big)$, the given sum is $\sum_{k=2}^{n} 1/k\ln k + O(1)$. The remaining sum is $\ln\ln n + O(1)$ by Euler's summation formula.

**9.36** This works out beautifully with Euler's summation formula:

$$S_n \;=\; \int_0^n \frac{dx}{n^2+x^2} + \frac{B_1}{1!}\frac{1}{n^2+x^2}\bigg|_0^n + \frac{B_2}{2!}\frac{-2x}{(n^2+x^2)^2}\bigg|_0^n + R_2\,,$$

where we have $R_2 = R_3$ and $R_3 = O(n^{3-8})$.

Hence $S_n = \frac{1}{4}\pi n^{-1} - \frac{1}{4}n^{-2} - \frac{1}{24}n^{-3} + O(n^{-5})$.

**9.37** This is

$$\sum_{k,q\geqslant 1}(n-qk)\big[n/(q+1) < k \leqslant n/q\big]$$

**9.63** Let $c = \phi^{2-\phi}$. The estimate $cn^{\phi-1}+o(n^{\phi-1})$ was proved by Fine [150]. Ilan Vardi observes that the sharper estimate stated can be deduced from the fact that the error term $e(n) = f(n) - cn^{\phi-1}$ satisfies the approximate recurrence $c^{\phi}n^{2-\phi}e(n) \approx -\sum_k e(k)[1 \leqslant k < cn^{\phi-1}]$. The function

$$\frac{n^{\phi-1}u(\ln\ln n/\ln\phi)}{\ln n}$$

*Additional progress on this problem has been made by Jean-Luc Rémy, Journal of Number Theory, vol. 66 (1997), 1–28.*

satisfies this recurrence asymptotically, if $u(x+1) = -u(x)$. (Vardi conjectures that

$$f(n) \;=\; n^{\phi-1}\Bigl(c + u\Bigl(\frac{\ln\ln n}{\ln\phi}\Bigr)(\ln n)^{-1} + O\bigl((\log n)^{-2}\bigr)\Bigr)$$

for some such function $u$.) Calculations for small $n$ show that $f(n)$ equals the nearest integer to $cn^{\phi-1}$ for $1 \leqslant n \leqslant 400$ except in one case: $f(273) = 39 > c\cdot 273^{\phi-1} \approx 38.4997$. But the small errors are eventually magnified, because of results like those in exercise 2.36. For example, $e(201636503) \approx 35.73$; $e(919986484788) \approx -1959.07$.

**9.64** (From this identity for $B_2(x)$ we can easily derive the identity of exercise 58 by induction on $m$.) If $0 < x < 1$, the integral $\int_x^{1/2} \sin N\pi t\, dt/\sin \pi t$ can be expressed as a sum of $N$ integrals that are each $O(N^{-2})$, so it is $O(N^{-1})$; the constant implied by this $O$ may depend on $x$. Integrating

$$\sum_{n=1}^{N} \cos 2n\pi t \;=\; \Re\Bigl(e^{2\pi i t}\frac{e^{2N\pi i t}-1}{e^{2\pi i t}-1}\Bigr) \;=\; -\frac{1}{2} + \frac{1}{2}\frac{\sin(2N+1)\pi t}{\sin \pi t}$$

and letting $N \to \infty$ now gives $\sum_{n\geqslant 1}(\sin 2n\pi x)/n = \frac{\pi}{2} - \pi x$, a relation that Euler knew ([107] and [110, part 2, §92]). Integrating again yields the desired formula. (This solution was suggested by E. M. E. Wermuth [367]; Euler's original derivation did not meet modern standards of rigor.)

**9.65** Since $a_0 + a_1 n^{-1} + a_2 n^{-2} + \cdots = 1 + (n-1)^{-1}(a_0 + a_1(n-1)^{-1} + a_2(n-1)^{-2} + \cdots)$, we obtain the recurrence $a_{m+1} = \sum_k \binom{m}{k} a_k$, which matches the recurrence for the Bell numbers. Hence $a_m = \varpi_m$.

A slightly longer but more informative proof can be based on the fact that $1/\bigl((n-1)\ldots(n-m)\bigr) = \sum_k \left\{ {k \atop m} \right\}/n^k$, by (7.47).

*"The paradox is now fully established that the utmost abstractions are the true weapons with which to control our thought of concrete fact."*
*— A. N. Whitehead [372]*

**9.66** The expected number of distinct elements in the sequence $1$, $f(1)$, $f(f(1))$, ..., when $f$ is a random mapping of $\{1, 2, \ldots, n\}$ into itself, is the function $Q(n)$ of exercise 56, whose value is $\frac{1}{2}\sqrt{2\pi n} + O(1)$; this might account somehow for the factor $\sqrt{2\pi n}$.

**9.67** It is known that $\ln \chi_n \sim \frac{3}{2}n^2 \ln \frac{4}{3}$; the constant $e^{-\pi/6}$ has been verified empirically to eight significant digits.

**9.68** This would fail if, for example, $e^{n-\gamma} = m + \frac{1}{2} + \epsilon/m$ for some integer $m$ and some $0 < \epsilon < \frac{1}{8}$; but no counterexamples are known.