A proof for the even closed-form rule in *Claude's Cycles*

GPT-5.4 Pro

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Abstract

We prove that the closed-form C rule in `even_closed_form.c` defines, for every even integer $m \geq 8$, a decomposition of the arc set of the directed $m \times m \times m$ torus into three directed Hamiltonian cycles of length m^3. Following Knuth's note *Claude's Cycles*, we use the fiber decomposition $F_s = \{(i, j, k) \in \mathbb{Z}_m^3 : i + j + k = s\}$ and reduce the problem to three explicit return maps on F_0. The key point is that the rule is trivial on all fibers except F_{m-2} and F_{m-1}; those two fibers form a small splice gadget that turns the obvious row cycles into Hamiltonian cycles.

1 The problem and the rule

Fix an even integer $m \geq 8$, and write $h = m/2$. All arithmetic is in $\mathbb{Z}_m$ unless an interval such as $0, 1, \dots, h$ is explicitly being treated as a set of ordinary representatives.

We follow Knuth's reformulation of the problem. A rule is a map

$$\sigma : \mathbb{Z}_m^3 \to S_3,$$

where the three symbols $0, 1, 2$ mean "bump i", "bump j", and "bump k", respectively. If $\sigma(i, j, k) = a_0 a_1 a_2$, then the cth factor map

$$\phi_c : \mathbb{Z}_m^3 \to \mathbb{Z}_m^3$$

bumps the coordinate indicated by a_c.

Thus each factor is a functional digraph on m^3 vertices, and the goal is to prove that each of the three functional digraphs is a single cycle of length m^3.

The C file defines the rule in exactly this way. Let

$$s = i + j + k \in \mathbb{Z}_m.$$

When we compare s with $m-2$ or $m-1$, we mean the unique representative of s in $\{0, 1, \dots, m-1\}$. Then the rule is:

$$\sigma(i,j,k) = \texttt{012}, \qquad \text{if } s \leq m-3,$$

$$\sigma(i,j,k) = \begin{cases} \texttt{210}, & i \in \{h-1, h\}, \\ \texttt{120}, & \text{otherwise}, \end{cases} \qquad \text{if } s = m-1,$$

$$\sigma(i,j,k) = \tau(i,j), \qquad \text{if } s = m-2,$$

where the explicit case table for τ is exactly the function `d_layer_m2` in the C program; for completeness it is rewritten mathematically in Appendix A.

The important structural fact is immediate:

Remark 1. Only the last two fibers F_{m-2} and F_{m-1} are nontrivial. On every other fiber the rule is simply `012`. This is why the proof can be reduced to explicit two-dimensional return maps.

2 Fiber reduction and an excursion lemma

Following Knuth, let

$$F_s = \{(i, j, k) \in \mathbb{Z}_m^3 : i + j + k = s\}.$$

Every step of every factor increases $i + j + k$ by 1, so each ϕ_c maps F_s into F_{s+1}. In particular, every orbit meets the fibers in the cyclic order

$$F_0, F_1, \ldots, F_{m-1}, F_0, \ldots.$$

For each factor $c \in \{0, 1, 2\}$, define the return map

$$T_c = \phi_c^m\big|_{F_0} : F_0 \to F_0.$$

Lemma 2. *If T_c is a single cycle on F_0 of length m^2, then ϕ_c is a single cycle on $\mathbb{Z}_m^3$ of length m^3.*

Proof. Because ϕ_c advances the fiber index by 1, a point of F_0 can return to F_0 only after a multiple of m steps. If T_c has period m^2, then the corresponding ϕ_c-orbit has period $m \cdot m^2 = m^3$. Since $|\mathbb{Z}_m^3| = m^3$, that orbit already contains every vertex. □

So the problem is now two-dimensional: we must show that each T_c is a Hamiltonian cycle on the fiber F_0.

The next lemma is the general device that we shall use three times.

Lemma 3 (Excursion lemma)**.** *Let $T : X \to X$ be a map on a finite set, let $S \subseteq X$, and assume:*

(1) every orbit starting in S eventually returns to S;

(2) the first-return map $Q : S \to S$ is one cycle on S;

(3) the first-return excursions from points of S are pairwise disjoint away from S, and together they cover all of X.

Then T is one cycle on X.

Proof. Pick any $x_0 \in S$ and follow its forward orbit under T. By hypothesis (1), that orbit decomposes into the successive first-return excursions

$$x_0 \rightsquigarrow Q(x_0) \rightsquigarrow Q^2(x_0) \rightsquigarrow \cdots.$$

Hypothesis (2) says that the starting points in S appear in one cycle, and hypothesis (3) says that the corresponding excursions are pairwise disjoint away from S and together cover all of X. Hence the orbit of x_0 visits every point of X exactly once before returning to x_0. Therefore T is one cycle on X. □

3 Factor $c = 1$

We begin with the easiest factor.

Coordinates

Write a point of F_0 as

$$(i, j, k) = (y, -x - y, x),$$

so that $(x, y) = (k, i) \in \mathbb{Z}_m^2$ are coordinates on F_0. In these coordinates a row means a set of the form

$$R_y = \{(x, y) : x \in \mathbb{Z}_m\}.$$

We take the section

$$S_1 = R_0 = \{(x, 0) : x \in \mathbb{Z}_m\}.$$

The return map T_1

Starting from (x, y), after the first $m - 2$ trivial fibers the factor-1 orbit is at

$$(i, j, k) = (y, -x - y - 2, x) \in F_{m-2}.$$

Substituting this into the explicit case table for τ, and then applying the F_{m-1} rule, yields the following description of T_1.

Let $\varepsilon \in \{0, 1\}$ be the parity of h.

Proposition 4. *The return map T_1 acts row-by-row as follows.*

(a) For every row

$$1 \leq y \leq h - 3 \qquad \text{or} \qquad h + 2 \leq y \leq m - 1,$$

and also for row $y = 0$ away from the point $x = m - 1$, one has

$$T_1(x, y) = \begin{cases} (x + 1, y), & x \neq m - 1, \\ (0, y + 1), & x = m - 1. \end{cases}$$

Thus such a row is traversed consecutively from its entry point 0 to its exit point $m - 1$.

(b) In row $y = h - 2$, the long excursion enters at $x = 0$ and traverses the row in the order

$$0, 2, 4, \ldots, h + 2 + \varepsilon,\ h + 3 + \varepsilon, h + 4 + \varepsilon, \ldots, m - 1,\ 1, 3, 5, \ldots, h + 1 + \varepsilon,$$

then leaves the row from $x = h + 1 + \varepsilon$ into row $h - 1$.

(c) In row $y = h - 1$, the long excursion enters at $x = h + 1 + \varepsilon$, then moves consecutively through all m points of the row, and leaves from $x = h + \varepsilon$ into row h.

(d) In row $y = h$, the long excursion enters at $x = h + \varepsilon$, then moves consecutively through all m points of the row, and leaves from $x = h - 1 + \varepsilon$ into row $h + 1$.

(e) In row $y = h + 1$, the long excursion enters at $x = h + \varepsilon$ and traverses the row in the order

$$h + \varepsilon,\ h + 2 + \varepsilon,\ h + 4 + \varepsilon, \ldots, m - 2,\ 0, 1, 2, \ldots, h - 1 + \varepsilon,\ h + 1 + \varepsilon, h + 3 + \varepsilon, \ldots, m - 1,$$

then leaves from $x = m - 1$ into row $h + 2$ at the point $x = 0$.

Proof. This is a direct case translation of the C rule. For example, on row $y = h - 2$ the point on F_{m-2} has coordinates $(i, j, k) = (h - 2, h - x, x)$. The row $i = h - 2$ of τ in Appendix A now gives, according to the value of x, either a factor-1 bump of k, a bump of j, or a bump of i; the subsequent F_{m-1} rule contributes a second bump of k unless i has become $h - 1$ or h. Working through the finite list of cases yields exactly the five row descriptions above. The other rows are similar, and no other row type occurs. □

The section map

For $x \neq m-1$, the point $(x, 0)$ returns immediately to the section:

$$T_1(x, 0) = (x+1, 0).$$

Thus the only nontrivial first-return excursion starts from $(m-1, 0)$.

We now concatenate the row descriptions of Proposition 4. The point $(m-1, 0)$ maps to $(0, 1)$. Part (a) then forces complete consecutive traversal of the generic rows $1, 2, \ldots, h-3$, always entering at $x = 0$ and leaving at $x = m-1$. Parts (b)–(e) carry the orbit through the four special rows $h-2, h-1, h, h+1$, ending at $(0, h+2)$. Part (a) then resumes and forces complete consecutive traversal of the generic rows $h+2, h+3, \ldots, m-1$, again entering at $x = 0$ and leaving at $x = m-1$, and finally returns to $(0, 0)$.

Therefore that one long excursion consists of the single section point $(m-1, 0)$ together with every point of every non-section row, each such row being traversed exactly once. It is disjoint from the immediate returns and covers all of $F_0 \setminus S_1$.

The first-return map on the section is therefore simply

$$Q_1(x) = x + 1 \qquad (x \in \mathbb{Z}_m),$$

which is one m-cycle.

By Lemma 3, T_1 is one cycle on F_0 of length m^2. By Lemma 2, factor $c = 1$ is one directed cycle of length m^3.

4 Factor $c = 2$

Coordinates

Write a point of F_0 as

$$(i, j, k) = (u + v, u, -2u - v),$$

so that $(u, v) = (j, i - j) \in \mathbb{Z}_m^2$ are coordinates on F_0. Let

$$R_u = \{(u, v) : v \in \mathbb{Z}_m\}, \qquad S_2 = R_0.$$

In these coordinates, the last step on F_{m-1} always bumps i. Consequently the return map T_2 uses only the three local moves

$$H : (u, v) \mapsto (u, v+1), \qquad U : (u, v) \mapsto (u+1, v), \qquad S : (u, v) \mapsto (u, v+2).$$

Generic rows

Proposition 5.

(a) For each row $1 \leq u \leq h-2$, put $a = h - 2 - u$. Then row R_u has upward exits precisely at

$$v = a, \ \ a+1, \ \ a+2,$$

and is horizontal elsewhere. If the row is entered at the three points

$$a+1, \ \ a+2, \ \ a+3,$$

then it splits into the three disjoint directed paths

$$a+1 \to U, \qquad a+2 \to U, \qquad a+3 \to a+4 \to \cdots \to a \to U,$$

so it is exited at the three points

$$a,\ a+1,\ a+2$$

as a set.

(b) For each row $h+2 \le u \le m-3$, put $a = h-1-u$. Then row R_u has upward exits precisely at

$$v = a,\ a+1,\ a+2,$$

and is horizontal elsewhere. If the row is entered at

$$a+1,\ a+2,\ a+3,$$

then it again splits into the three disjoint paths

$$a+1 \to U, \qquad a+2 \to U, \qquad a+3 \to a+4 \to \cdots \to a \to U,$$

and is exited at the three points

$$a,\ a+1,\ a+2$$

as a set.

Proof. Substitute $(i,j) = (u+v, u)$ into the table for τ. On the rows covered by part (a), the only non-`012`/`102` outputs are exactly the three `021`/`201` cases listed above, hence exactly the three upward exits stated. The same substitution gives part (b). Since all other points are horizontal, the path decompositions are immediate. □

Special rows

The five exceptional rows are $u = h-1, h, h+1, m-2, m-1$. Their behavior is finite and explicit; the following table is read directly from the same substitution. Throughout this subsection, expressions such as $h+4$ are interpreted modulo m.

row	incoming points	row decomposition / outgoing points
$u = h-1$	$0, 1, 2$	$0 \to U$, $2 \to U$, and $1 \to 3 \to 4 \to \cdots \to m-1 \to U$; outgoing $0, 2, m-1$
$u = h$	$0, 2, m-1$	$0 \to U$, $m-1 \to S \to 1 \to U$, and $2 \to 3 \to \cdots \to m-2 \to U$; outgoing $0, 1, m-2$
$u = h+1$	$0, 1, m-2$	$0 \to U$, $m-2 \to U$, and $1 \to 2 \to \cdots \to m-3 \to S \to m-1 \to U$; outgoing $0, m-2, m-1$
$u = m-2$	$h+2, h+3, h+4$	if h is odd: $h+2 \mapsto h+2$, $h+3 \mapsto h+1$, $h+4 \mapsto 3$; if h is even: $h+2 \mapsto h+2$, $h+3 \mapsto 3$, $h+4 \mapsto h+1$
$u = m-1$	$3, h+1, h+2$	if h is odd: $3 \mapsto h$, $h+1 \mapsto h+1$, $h+2 \mapsto h-1$; if h is even: $3 \mapsto h-1$, $h+1 \mapsto h+1$, $h+2 \mapsto h$

The important point is that every special row is again partitioned into three disjoint directed paths joining its incoming triple to its outgoing triple.

The section map

The section row $u = 0$ is almost trivial. The only section points with nontrivial return are

$$v = h - 2,\ h - 1,\ h.$$

All other section points return immediately, giving $Q_2(v) = v + 1$.

Proposition 6. *The first-return map $Q_2 : S_2 \to S_2$ is:*

$$Q_2(v) = v + 1 \qquad \text{for } v \notin \{h-2, h-1, h\},$$

$$Q_2(h-2) = \begin{cases} h-1, & h \text{ odd}, \\ h, & h \text{ even}, \end{cases}$$

$$Q_2(h-1) = \begin{cases} h, & h \text{ odd}, \\ h+1, & h \text{ even}, \end{cases}$$

$$Q_2(h) = \begin{cases} h+1, & h \text{ odd}, \\ h-1, & h \text{ even}. \end{cases}$$

Hence Q_2 is one m-cycle on S_2.

Proof. Let A, B, C denote the three exceptional excursions, starting at the section points

$$A : h - 2, \qquad B : h - 1, \qquad C : h.$$

For a generic row, write its ordered incoming triple as (L_1, L_2, L_3), where the positions are $(a + 1, a + 2, a + 3)$ in Proposition 5. The ordered outgoing triple on the next row, at the positions $(a, a + 1, a + 2)$, is then

$$\rho(L_1, L_2, L_3) := (L_3, L_1, L_2),$$

because the first two paths are one-point upward exits and the third path is the long horizontal path ending at a. Hence the early generic block $u = 1, \ldots, h - 2$ sends the initial ordered triple (A, B, C) to

$$\rho^{h-2}(A, B, C)$$

on row $u = h - 1$, ordered at the positions $(0, 1, 2)$.

Next consider the three middle rows. With the ordered incoming and outgoing triples

$$(0, 1, 2) \to (0, 2, m - 1), \qquad (0, 2, m - 1) \to (0, 1, m - 2), \qquad (0, 1, m - 2) \to (0, m - 2, m - 1),$$

each row acts on labels by

$$\tau(L_1, L_2, L_3) := (L_1, L_3, L_2).$$

Since $\tau^3 = \tau$, the ordered triple on row $u = h + 2$ at the positions $(0, m - 2, m - 1)$ is

$$\tau\rho^{h-2}(A, B, C).$$

Reordering this as the generic late-block order $(m - 2, m - 1, 0)$ contributes ρ^2, and each late generic row again contributes ρ. Therefore the ordered triple entering row $u = m - 2$ at the positions $(h + 2, h + 3, h + 4)$ is

$$\rho^{h-4}\rho^2\tau\rho^{h-2}(A, B, C) = \rho^{h-2}\tau\rho^{h-2}(A, B, C) = \tau(A, B, C) = (A, C, B),$$

because $\rho^3 = \text{id}$ and $\rho\tau\rho = \tau$.

We now apply the explicit correspondences for the last two rows. If h is odd, row $u = m-2$ sends (A, C, B) at $(h+2, h+3, h+4)$ to (B, C, A) at $(3, h+1, h+2)$, and row $u = m-1$ then sends that to (A, B, C) at $(h-1, h, h+1)$. Thus

$$Q_2(h-2) = h-1, \qquad Q_2(h-1) = h, \qquad Q_2(h) = h+1.$$

If h is even, row $u = m-2$ sends (A, C, B) to (C, B, A) at $(3, h+1, h+2)$, and row $u = m-1$ then sends that to (C, A, B) at $(h-1, h, h+1)$. Thus

$$Q_2(h-2) = h, \qquad Q_2(h-1) = h+1, \qquad Q_2(h) = h-1.$$

Together with the immediate returns outside $\{h-2, h-1, h\}$, this is exactly the stated formula. In the odd case it is simply $Q_2(v) = v+1$ for all v. In the even case the section cycle is

$$0 \to 1 \to \cdots \to h-3 \to h-2 \to h \to h-1 \to h+1 \to h+2 \to \cdots \to m-1 \to 0,$$

which is still a single m-cycle. □

Completion of the proof for $c = 2$

The three nontrivial section points $h-2, h-1, h$ launch three excursions. Their first exits from the section row are to row $u = 1$ at the incoming triple

$$h-2,\ h-1,\ h,$$

which is exactly the generic incoming triple of Proposition 5(a) for $u = 1$.

Now argue by induction on the rows. In the early generic block $1 \le u \le h-2$, Proposition 5(a) sends the incoming triple

$$a+1,\ a+2,\ a+3$$

to the outgoing triple

$$a,\ a+1,\ a+2$$

as a set; that is exactly the incoming triple for the next row. Hence every early generic row is partitioned by the three excursions, and the last early generic row produces the incoming set $\{0, 1, 2\}$ for row $u = h-1$.

The special rows then propagate the incoming sets

$$\{0, 1, 2\} \to \{0, 2, m-1\} \to \{0, 1, m-2\} \to \{0, m-2, m-1\},$$

namely from rows $h-1$, h, and $h+1$ to the next rows. At this point there are two cases.

If $m = 8$ (so $h = 4$), the late generic block is empty, and the outgoing set

$$\{0, m-2, m-1\} = \{h+4, h+2, h+3\}$$

is already the incoming set for row $u = m-2$.

If $m > 8$, then $\{0, m-2, m-1\}$ is exactly the generic incoming triple of Proposition 5(b) for $u = h+2$. In the late generic block $h+2 \le u \le m-3$, the same induction applies: each row sends

$$a+1,\ a+2,\ a+3$$

to

$$a,\ a+1,\ a+2$$

as a set, so the three excursions continue disjointly and cover every late generic row. The final late generic row produces the incoming set $\{h+2,h+3,h+4\}$ for row $u=m-2$.

Finally the last two rows carry

$$\{h+2,h+3,h+4\} \to \{3,h+1,h+2\} \to \{h-1,h,h+1\},$$

again by disjoint path decompositions that cover the entire row. Thus the three excursions are pairwise disjoint away from the section and their union is all of $F_0 \setminus S_2$.

By Proposition 6, the first-return map on the section is a single m-cycle. Hence Lemma 3 shows that T_2 is a single m^2-cycle on F_0. By Lemma 2, factor $c=2$ is a single directed cycle of length m^3.

5 Factor $c=0$

Coordinates

Write a point of F_0 as

$$(i,j,k) = (-x-y, x, y),$$

so that $(x,y)=(j,k) \in \mathbb{Z}_m^2$ are coordinates on F_0. Again let

$$R_y = \{(x,y) : x \in \mathbb{Z}_m\}, \qquad S_0 = R_0.$$

In these coordinates the return map T_0 uses five local moves:

$$H:(x,y)\mapsto(x+1,y), \quad V:(x,y)\mapsto(x,y+1), \quad D:(x,y)\mapsto(x+1,y+1),$$

$$S:(x,y)\mapsto(x+2,y), \qquad W:(x,y)\mapsto(x,y+2).$$

Generic early rows

Proposition 7. *For each row $0 \le y \le h-2$, put $a=h-y$. Then the row R_y has the following local rules:*

- *vertical exits at $x=a-1$ and $x=a$;*
- *a diagonal exit at $x=m-2$;*
- *horizontal motion at every other point.*

Consequently:

(a) on the section row $y=0$, the exceptional points $x=h-1,h,m-2$ go to row 1 at $x=h-1,h,m-1$;

(b) for each row $1 \le y \le h-2$, if the incoming triple is

$$a,\ a+1,\ m-1,$$

then the row splits into the three disjoint paths

$$a \to V,$$

$$a+1 \to a+2 \to \cdots \to m-2 \to D,$$
$$m-1 \to 0 \to 1 \to \cdots \to a-1 \to V,$$

and therefore the outgoing triple is

$$a-1,\ a,\ m-1$$

as a set.

Proof. Substitute $(i,j) = (-x-y-2, x)$ into the table for τ. On the rows $0 \le y \le h-2$, the only non-`012`/`021` cases occur exactly at $x = a-1$, $x = a$, and $x = m-2$, producing the moves V, V, and D. The path decompositions then follow immediately. ☐

Middle rows: the parity split

The middle rows are the only place where the proof depends on the parity of h.

Proposition 8.

(a) If h is even, then:

- *row $y = h-1$ receives $1, 2, m-1$ and sends them to row h at $0, 1$ and to row $h+1$ at $m-1$;*
- *row $y = h$ receives $0, 1$ and sends them to row $h+1$ at 0 and to row $h+2$ at $m-1$;*
- *row $y = h+1$ receives $0, m-1$ and sends them to row $h+2$ at $0, m-2$.*

Hence the late generic block starts at row $y = h+2$ with incoming triple $0, m-2, m-1$.

(b) If h is odd, then:

- *row $y = h-1$ receives $1, 2, m-1$ and sends them to row h at $0, 1, m-1$;*
- *row $y = h$ receives $0, 1, m-1$ and sends them to row $h+1$ at $0, m-1$ and to row $h+2$ at $m-2$;*
- *row $y = h+1$ receives $0, m-1$ and sends them to row $h+2$ at $m-1$ and to row $h+3$ at $m-2$;*
- *row $y = h+2$ receives $m-2, m-1$ and sends them to row $h+3$ at $m-3, 0$.*

Hence the late generic block starts at row $y = h+3$ with incoming triple $0, m-3, m-2$.

Proof. Each statement is a direct translation of the row types produced by the explicit C cases. For instance, when h is even, row $h-1$ has local moves $V, V, H, \ldots, H, S, W$ at the points $0, 1, 2, \ldots, m-2, m-1$, so the incoming triple $1, 2, m-1$ yields the three paths

$$1 \to V,$$

$$2 \to 3 \to \cdots \to m-2 \to S \to 0 \to V,$$

$$m-1 \to W,$$

which gives the stated outputs. The other cases are similar finite checks. ☐

Generic late rows

Proposition 9. *Let*

$$y_0 = \begin{cases} h+2, & h \text{ even,} \\ h+3, & h \text{ odd.} \end{cases}$$

For each row $y_0 \le y \le m-2$, *put* $a = m+h-y$. *Then row* R_y *has:*

- *vertical exits at* $x = a-1$ *and* $x = a$;
- *a diagonal exit at* $x = m-1$;
- *horizontal motion elsewhere.*

If the incoming triple is

$$0,\ a,\ a+1,$$

then the row splits into the three disjoint paths

$$0 \to 1 \to \cdots \to a-1 \to V,$$

$$a \to V,$$

$$a+1 \to a+2 \to \cdots \to m-1 \to D,$$

and therefore the outgoing triple is

$$0,\ a-1,\ a$$

as a set.

Proof. Again this comes directly from substituting $(i,j) = (-x-y-2, x)$ into the table for τ on the indicated rows. Once the local moves are known, the path decomposition is immediate. □

The bottom row and the section map

The only section points with nontrivial return are

$$x = h-1,\ h,\ m-2.$$

All other section points return immediately, so $Q_0(x) = x+1$ away from those three points. The bottom row $y = m-1$ receives the triple

$$0,\ h+1,\ h+2.$$

Its local rule is especially simple: the diagonal exits occur exactly at

$$x = h-1,\ h,\ m-2,$$

and every other point moves by $S : (x, m-1) \mapsto (x+2, m-1)$.

Proposition 10. *The first-return map $Q_0 : S_0 \to S_0$ is:*

$$Q_0(x) = x + 1 \qquad\qquad \textit{for } x \notin \{h-1, h, m-2\},$$

$$Q_0(h-1) = \begin{cases} h, & h \textit{ even}, \\ h+1, & h \textit{ odd}, \end{cases}$$

$$Q_0(h) = \begin{cases} h+1, & h \textit{ even}, \\ m-1, & h \textit{ odd}, \end{cases}$$

$$Q_0(m-2) = \begin{cases} m-1, & h \textit{ even}, \\ h, & h \textit{ odd}. \end{cases}$$

Hence Q_0 is one m-cycle on S_0.

Proof. Let A, B, C denote the three exceptional excursions starting from

$$A : h-1, \qquad B : h, \qquad C : m-2.$$

For a generic early row, ordered at the positions $(a, a+1, m-1)$ as in Proposition 7(b), the ordered outgoing triple on the next row, at the positions $(a-1, a, m-1)$, is

$$\rho(L_1, L_2, L_3) := (L_3, L_1, L_2).$$

Therefore the early generic block sends the initial ordered triple (A, B, C) to

$$\rho^{h-2}(A, B, C)$$

on row $y = h-1$, ordered at the positions $(1, 2, m-1)$.

Suppose first that h is even. Tracing the ordered inputs $(1, 2, m-1)$ through the three middle rows of Proposition 8(a), one finds that the label at 1 reaches row $h+2$ at $m-1$, the label at 2 reaches row $h+2$ at $m-2$, and the label at $m-1$ reaches row $h+2$ at 0. Hence the ordered triple on row $y = h+2$ at the positions $(0, m-2, m-1)$ is

$$\nu(L_1, L_2, L_3) := (L_3, L_2, L_1).$$

On every generic late row, if the ordered incoming triple at positions $(0, a, a+1)$ is (L_1, L_2, L_3), then the ordered outgoing triple at positions $(0, a-1, a)$ is again

$$\rho(L_1, L_2, L_3) = (L_3, L_1, L_2).$$

Since there are $h-3$ late generic rows, the ordered triple entering the bottom row at the positions $(0, h+1, h+2)$ is

$$\rho^{h-3} \nu \rho^{h-2}(A, B, C) = (B, A, C).$$

Because $\rho^3 = \mathrm{id}$, this identity may be checked by reducing h modulo 3. Thus A, B, C enter the bottom row at $h+1, 0, h+2$, respectively.

The bottom-row paths are

$$0 \to 2 \to 4 \to \cdots \to h \to D,$$

$$h+1 \to h+3 \to \cdots \to m-1 \to 1 \to 3 \to \cdots \to h-1 \to D,$$

$$h+2 \to h+4 \to \cdots \to m-2 \to D.$$

Therefore the labels entering at $0, h+1, h+2$ return to the section at $h+1, h, m-1$, respectively. Since those entries carry the labels B, A, C, we obtain

$$Q_0(h-1) = h, \qquad Q_0(h) = h+1, \qquad Q_0(m-2) = m-1.$$

Suppose next that h is odd. Tracing the ordered inputs $(1, 2, m-1)$ through the four middle rows of Proposition 8(b), one finds that the label at 1 reaches row $h+3$ at $m-3$, the label at 2 reaches row $h+3$ at 0, and the label at $m-1$ reaches row $h+3$ at $m-2$. Hence the ordered triple on row $y = h+3$ at the positions $(0, m-3, m-2)$ is

$$\xi(L_1, L_2, L_3) := (L_2, L_1, L_3).$$

There are $h-4$ late generic rows, so the ordered triple entering the bottom row at the positions $(0, h+1, h+2)$ is

$$\rho^{h-4}\xi\rho^{h-2}(A, B, C) = (C, B, A).$$

Again one checks this using $\rho^3 = \mathrm{id}$. Thus A, B, C enter the bottom row at $h+2, h+1, 0$, respectively. The bottom-row paths are now

$$0 \to 2 \to 4 \to \cdots \to h-1 \to D,$$

$$h+1 \to h+3 \to \cdots \to m-2 \to D,$$

$$h+2 \to h+4 \to \cdots \to m-1 \to 1 \to 3 \to \cdots \to h \to D.$$

Therefore the labels entering at $0, h+1, h+2$ return to the section at $h, m-1, h+1$, respectively. Since those entries carry the labels C, B, A, we obtain

$$Q_0(h-1) = h+1, \qquad Q_0(h) = m-1, \qquad Q_0(m-2) = h.$$

In either parity, the values outside $\{h-1, h, m-2\}$ are the immediate returns $Q_0(x) = x+1$. The displayed formulas are therefore exactly the stated section map. In the even case that is again the full shift $x \mapsto x+1$; in the odd case the section cycle is

$$0 \to 1 \to \cdots \to h-2 \to h-1 \to h+1 \to h+2 \to \cdots \to m-2 \to h \to m-1 \to 0,$$

again a single m-cycle. □

Completion of the proof for $c = 0$

By Proposition 7(a), the three exceptional section points $h-1, h, m-2$ leave the section row and enter row $y = 1$ at the triple

$$h-1,\ h,\ m-1.$$

For $1 \le y \le h-2$, Proposition 7(b) sends the incoming triple

$$a,\ a+1,\ m-1$$

to the outgoing triple

$$a-1,\ a,\ m-1$$

as a set, which is exactly the incoming triple for the next row. Hence every early generic row is partitioned by the three excursions, and the last early row produces the incoming set $\{1, 2, m-1\}$ for row $y = h-1$.

The middle block now propagates the incoming sets exactly as described in Proposition 8. If h is even, the three middle rows combine to produce the incoming set

$$\{0, m-2, m-1\}$$

for row $y = h+2$. If h is odd, the four middle rows combine to produce the incoming set

$$\{0, m-3, m-2\}$$

for row $y = h+3$.

From that point on, Proposition 9 again gives an induction. On each late generic row the incoming triple

$$0,\ a,\ a+1$$

is sent to the outgoing triple

$$0,\ a-1,\ a$$

as a set, which is exactly the incoming triple for the next row. Thus the three excursions continue disjointly through every late generic row and cover it completely. The final late generic row yields the incoming set $\{0, h+1, h+2\}$ for the bottom row $y = m-1$.

Finally Proposition 10 analyzes the bottom row and shows how those three excursions return to the section. Therefore the three excursions are pairwise disjoint away from the section and their union is all of $F_0 \setminus S_0$.

By Proposition 10, the first-return map on the section is one m-cycle. Hence Lemma 3 gives that T_0 is a single cycle on F_0 of length m^2. Lemma 2 then implies that factor $c = 0$ is a single directed cycle of length m^3.

6 Conclusion

We have proved that, for every even $m \geq 8$, each of the three return maps T_0, T_1, T_2 is a single cycle on F_0 of length m^2. By Lemma 2, the three factor maps ϕ_0, ϕ_1, ϕ_2 are therefore single directed cycles of length m^3. In other words, the closed-form C rule in `even_closed_form.c` really does decompose the directed $m \times m \times m$ torus into three Hamiltonian cycles.

The proof also explains why the construction works. Because the rule is `012` on all fibers except F_{m-2} and F_{m-1}, the m-step return map on any factor is almost trivial: without the last two fibers one would simply have m disjoint row cycles. The two exceptional fibers act as a small splice gadget. In factor $c = 1$ they create one long excursion; in factors $c = 0$ and $c = 2$ they create three long excursions. Those excursions are exactly what stitches the row cycles together into Hamiltonian cycles.

A The layer $s = m - 2$

For completeness, here is the exact mathematical translation of the C function `d_layer_m2`. Recall that $h = m/2$. The value of $\tau(i, j)$ is one of the six strings `012`, `021`, `102`, `120`, `201`, `210`.

row condition	condition on j	$\tau(i,j)$
$i=0$	$j=m-2$	210
$i=0$	$j=m-1$	102
$i=0$	otherwise	012
$i=1$	$j=m-2$	201
$i=1$	$j=m-1$	210
$i=1$	otherwise	012
$2 \le i \le h-3$	$j=m-1-i$	102
$2 \le i \le h-3$	$j=m-1$	210
$2 \le i \le h-3$	otherwise	012
$i=h-2$	$j \le h$	021
$i=h-2$	$j=h+1$	120
$i=h-2$	$j=m-2,\ m \equiv 0 \pmod 4$	012
$i=h-2$	$j=m-2,\ m \equiv 2 \pmod 4$	102
$i=h-2$	$j=m-1,\ m \equiv 0 \pmod 4$	201
$i=h-2$	$j=m-1,\ m \equiv 2 \pmod 4$	021
$i=h-2$	otherwise	012
$i=h-1$	$j \le h-1$	021
$i=h-1$	$j=h$	120
$i=h-1$	$h+1 \le j \le m-3$	021
$i=h-1$	$j=m-2,\ m \equiv 0 \pmod 4$	021
$i=h-1$	$j=m-2,\ m \equiv 2 \pmod 4$	201
$i=h-1$	$j=m-1,\ m \equiv 0 \pmod 4$	201
$i=h-1$	$j=m-1,\ m \equiv 2 \pmod 4$	021
$i=h$	$j \le h-2$	021
$i=h$	$j=h-1$	120
$i=h$	$h \le j \le m-3$	021
$i=h$	$j=m-2,\ m \equiv 0 \pmod 4$	021
$i=h$	$j=m-2,\ m \equiv 2 \pmod 4$	201
$i=h$	$j=m-1,\ m \equiv 0 \pmod 4$	201
$i=h$	$j=m-1,\ m \equiv 2 \pmod 4$	021
$i=h+1$	$j \le h-3$	012
$i=h+1$	$j=h-2$	102
$i=h+1$	$h-1 \le j \le m-3$	021
$i=h+1$	$j=m-2,\ m \equiv 0 \pmod 4$	120
$i=h+1$	$j=m-2,\ m \equiv 2 \pmod 4$	210
$i=h+1$	$j=m-1$	012
$h+2 \le i \le m-1$	$j=m-1-i$	102
$h+2 \le i \le m-1$	$j=m-2$	210
$h+2 \le i \le m-1$	otherwise	012