An Alternative Hamiltonian Decomposition of the Three-Dimensional Torus Digraph

Let $m > 2$ be odd, and let G_m be the directed graph with vertex set

$$V = \mathbb{Z}_m^3 = \{(i,j,k) : 0 \le i,j,k < m\},$$

where arithmetic is modulo m, and where each vertex (i,j,k) has the three outgoing arcs

$$(i,j,k) \to (i+1,j,k), \qquad (i,j,k) \to (i,j+1,k), \qquad (i,j,k) \to (i,j,k+1).$$

Donald Knuth gave in *Claude's Cycles* an explicit decomposition of these arcs into three directed Hamiltonian cycles when m is odd. The purpose of this note is to record a different explicit decomposition. The construction is remarkably simple: the choice of direction depends only on the residue

$$s \equiv i+j+k \pmod m$$

and on whether $j = 0$ or $j = m-1$. The proof reduces each cycle to a return map on the slice $s = 0$.

For convenience, write

$$i^+ = (i+1,j,k), \qquad j^+ = (i,j+1,k), \qquad k^+ = (i,j,k+1)$$

for the three out-neighbors of (i,j,k).

The local permutation rule

At each vertex (i,j,k), let $\pi(i,j,k)$ be the ordered triple of directions assigned to the cycles C_0, C_1, C_2. Thus

$$\pi(i,j,k) = (d_0, d_1, d_2)$$

means that C_0 uses direction d_0, C_1 uses direction d_1, and C_2 uses direction d_2.

Define $s \equiv i+j+k \pmod m$. The rule is

$$\pi(i,j,k) = \begin{cases} (i,k,j), & s = 0,\ j \ne m-1, \\ (k,i,j), & s = 0,\ j = m-1, \\ (i,j,k), & 0 < s < m-1, \\ (j,i,k), & s = m-1,\ j = 0, \\ (j,k,i), & s = m-1,\ j \ne 0. \end{cases}$$

Equivalently, the three successor maps T_0, T_1, T_2 defining the cycles C_0, C_1, C_2 are as follows.

Cycle C_0.

$$T_0(i,j,k) = \begin{cases} (i,j,k+1), & s = 0,\ j = m-1, \\ (i,j+1,k), & s = m-1, \\ (i+1,j,k), & \text{otherwise}. \end{cases}$$

Cycle C_1.

$$T_1(i,j,k) = \begin{cases} (i+1,j,k), & s=0,\ j=m-1, \\ (i+1,j,k), & s=m-1,\ j=0, \\ (i,j+1,k), & 0<s<m-1, \\ (i,j,k+1), & \text{otherwise.} \end{cases}$$

Cycle C_2.

$$T_2(i,j,k) = \begin{cases} (i,j+1,k), & s=0, \\ (i+1,j,k), & s=m-1,\ j\neq 0, \\ (i,j,k+1), & \text{otherwise.} \end{cases}$$

At every vertex, the three directions chosen by T_0, T_1, T_2 are precisely i^+, j^+, k^+ in some order; hence the arcs of G_m are partitioned among C_0, C_1, C_2. It remains to prove that each C_r is a single directed cycle of length m^3.

A return-map lemma

Let

$$s(i,j,k) \equiv i+j+k \pmod{m},$$

and let

$$H = \{(i,j,k) \in \mathbb{Z}_m^3 : s(i,j,k) = 0\}.$$

Lemma. Suppose $T : V \to V$ is a permutation of V such that

$$s(T(v)) \equiv s(v) + 1 \pmod{m}$$

for every $v \in V$. Let $F : H \to H$ be the return map obtained by following T for exactly m steps:

$$F(v) = T^m(v) \qquad (v \in H).$$

If F is a single cycle of length $|H| = m^2$, then T is a single cycle of length $|V| = m^3$.

Proof. Because s increases by 1 modulo m at each step, every T-orbit meets H exactly once every m steps. Hence the T-orbits are in bijection with the F-orbits on H: each F-orbit lifts to a T-orbit of length exactly m times as large. Therefore, if F has one orbit of size m^2, then T has one orbit of size $m \cdot m^2 = m^3$. □

For each $r \in \{0,1,2\}$, the map T_r increases exactly one of i, j, k by 1, so

$$s(T_r(i,j,k)) \equiv s(i,j,k) + 1 \pmod{m}.$$

Therefore the lemma applies to T_0, T_1, T_2.

To study the return maps, it is convenient to identify the slice H with $\mathbb{Z}_m^2$ by writing a point of H as (i,j), with k determined by

$$k \equiv -i-j \pmod{m}.$$

The cycle C_0

We compute the return map $F_0 = T_0^m$ on H.

Start from $(i, j) \in H$.

If $j \neq m-1$, then at $s = 0$ the rule for T_0 uses the i-direction, since the exceptional case $s = 0,\ j = m-1$ does not occur. At each intermediate level $s = 1, 2, \ldots, m-2$, the rule again uses the i-direction. Finally, at $s = m-1$, the rule uses the j-direction. Hence in one return from H to H,

$$i \text{ is incremented } m-1 \text{ times}, \qquad j \text{ is incremented once},$$

so

$$F_0(i, j) = (i-1,\ j+1) \qquad (j \neq m-1).$$

If $j = m-1$, then at $s = 0$ the exceptional case occurs and T_0 uses the k-direction. At the intermediate levels $s = 1, \ldots, m-2$, it uses the i-direction, and at $s = m-1$ it uses the j-direction. Hence

$$i \text{ is incremented } m-2 \text{ times}, \qquad j \text{ is incremented once},$$

so

$$F_0(i, m-1) = (i-2,\ 0).$$

Thus

$$F_0(i, j) = \begin{cases} (i-1,\ j+1), & j \neq m-1, \\ (i-2,\ 0), & j = m-1. \end{cases}$$

Now introduce new coordinates on H:

$$x = i + j, \qquad y = j.$$

In these coordinates,

$$F_0(x, y) = \begin{cases} (x,\ y+1), & y \neq m-1, \\ (x-1,\ 0), & y = m-1. \end{cases}$$

Therefore y increases through the cycle

$$0, 1, 2, \ldots, m-1, 0, 1, \ldots,$$

and each time y wraps from $m-1$ to 0, the coordinate x decreases by 1. Hence F_0 is a single cycle of length m^2 on H. By the lemma, T_0 is a single cycle of length m^3. Thus C_0 is Hamiltonian.

The cycle C_2

We compute the return map $F_2 = T_2^m$ on H.

Start from $(i, j) \in H$.

At $s = 0$, the rule for T_2 always uses the j-direction; thus j is incremented once immediately. At each intermediate level $s = 1, \ldots, m-2$, the rule uses the k-direction. At the final level $s = m-1$, the rule uses the i-direction if and only if the current value of j is nonzero.

If the initial $j \neq m-1$, then after the first step j becomes $j+1 \neq 0$, so the last step uses the i-direction. Hence

$$F_2(i,j) = (i+1,\ j+1) \qquad (j \neq m-1).$$

If the initial $j = m-1$, then after the first step j becomes 0, so the last step does *not* use the i-direction; it uses the k-direction instead. Hence

$$F_2(i, m-1) = (i,\ 0).$$

Thus

$$F_2(i,j) = \begin{cases} (i+1,\ j+1), & j \neq m-1, \\ (i,\ 0), & j = m-1. \end{cases}$$

Now introduce coordinates

$$x = i - j, \qquad y = j.$$

Then

$$F_2(x,y) = \begin{cases} (x,\ y+1), & y \neq m-1, \\ (x-1,\ 0), & y = m-1. \end{cases}$$

This is the same counter map as for C_0; hence F_2 is a single cycle of length m^2. By the lemma, T_2 is a single cycle of length m^3. Thus C_2 is Hamiltonian.

The cycle C_1

The proof for C_1 requires more care.

Start from $(i,j) \in H$, and follow T_1 until the next return to H.

Step $s = 0$. At level $s = 0$, the rule for T_1 uses the i-direction if and only if $j = m-1$; otherwise it uses the k-direction. Hence the contribution at this step is

$$\Delta i = \mathbf{1}_{\{j=m-1\}}, \qquad \Delta j = 0.$$

Intermediate steps $s = 1, 2, \ldots, m-2$. At every such level, T_1 uses the j-direction. Since there are exactly $m-2$ intermediate levels, their total contribution is

$$\Delta i = 0, \qquad \Delta j = m - 2 \equiv -2 \pmod{m}.$$

Step $s = m-1$. At the last step before returning to H, the rule for T_1 uses the i-direction if and only if the *current* value of j equals 0. The current value of j at that moment is

$$j + (m-2) \equiv j - 2 \pmod{m}.$$

Therefore the last step uses the i-direction if and only if

$$j - 2 \equiv 0 \pmod{m},$$

that is, if and only if $j = 2$. Hence the contribution at the last step is

$$\Delta i = \mathbf{1}_{\{j=2\}}, \qquad \Delta j = 0.$$

Combining the three stages, we obtain the return map

$$F_1(i, j) = \big(i + \varepsilon(j),\ j - 2\big),$$

where

$$\varepsilon(j) = \mathbf{1}_{\{j=m-1\}} + \mathbf{1}_{\{j=2\}}.$$

This formula is valid for all odd $m > 2$; when $m = 3$, the two indicators coincide, so $\varepsilon(2) = 2$, as it should.

We now analyze the dynamics of F_1.

First, the second coordinate always changes by

$$j \mapsto j - 2.$$

Since m is odd, $\gcd(2, m) = 1$, so this permutation of $\mathbb{Z}_m$ is a single cycle of length m. Therefore, for fixed i, the successive values of j under iteration of F_1 run through all residues modulo m exactly once before repeating.

Second, over one full j-orbit of length m, the total increment of i is

$$\sum_{j \in \mathbb{Z}_m} \varepsilon(j) = \sum_{j \in \mathbb{Z}_m} \mathbf{1}_{\{j=m-1\}} + \sum_{j \in \mathbb{Z}_m} \mathbf{1}_{\{j=2\}} = 2.$$

Hence

$$F_1^m(i, j) = (i + 2,\ j)$$

for every $(i, j) \in H$.

Finally, since m is odd, we also have $\gcd(2, m) = 1$, so the translation

$$i \mapsto i + 2$$

on $\mathbb{Z}_m$ is a single cycle of length m. Therefore the map F_1^m has one orbit of length m on each horizontal row $j =$ constant, and the fact that $j \mapsto j - 2$ itself is a single m-cycle forces F_1 to be a single cycle of length m^2.

Equivalently, one may argue directly as follows. If $F_1^t(i, j) = (i, j)$, then the second coordinate shows that

$$-2t \equiv 0 \pmod{m}.$$

Because $\gcd(2, m) = 1$, this implies $m \mid t$, say $t = mu$. Then

$$F_1^t(i, j) = F_1^{mu}(i, j) = (i + 2u,\ j),$$

so $2u \equiv 0 \pmod{m}$. Again $\gcd(2, m) = 1$, hence $m \mid u$, and therefore $m^2 \mid t$. Thus every orbit of F_1 has length at least m^2; since $|H| = m^2$, F_1 is a single m^2-cycle.

By the lemma, T_1 is a single cycle of length m^3. Thus C_1 is Hamiltonian.

Conclusion

For every odd integer $m > 2$, the rule

$$\pi(i,j,k) = \begin{cases} (i,k,j), & s = 0,\ j \neq m-1, \\ (k,i,j), & s = 0,\ j = m-1, \\ (i,j,k), & 0 < s < m-1, \\ (j,i,k), & s = m-1,\ j = 0, \\ (j,k,i), & s = m-1,\ j \neq 0, \end{cases} \qquad s \equiv i+j+k \pmod{m},$$

partitions the arcs of G_m into three directed Hamiltonian cycles C_0, C_1, C_2. This gives an explicit Hamiltonian decomposition of the three-dimensional torus digraph, different from the one exhibited by Knuth in *Claude's Cycles*.