Cylindrical Algebraic Coverings for Quantifiers

Gereon Kremer, Jasper Nalbach

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SC^2 @ FLoC, Haifa

Disclaimer & Acknowledgements

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This works just as you would expect.

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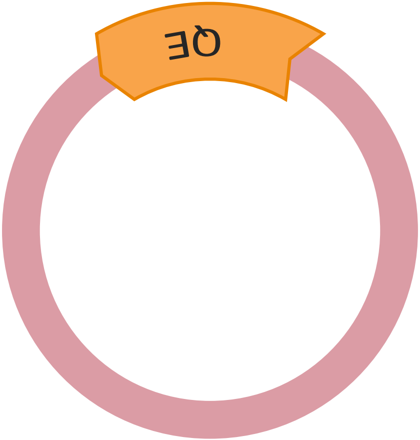
Thanks to Dagstuhl Seminar 22072

The Circle of Life – NRA edition



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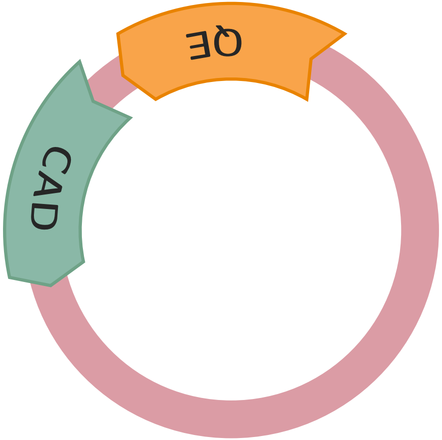

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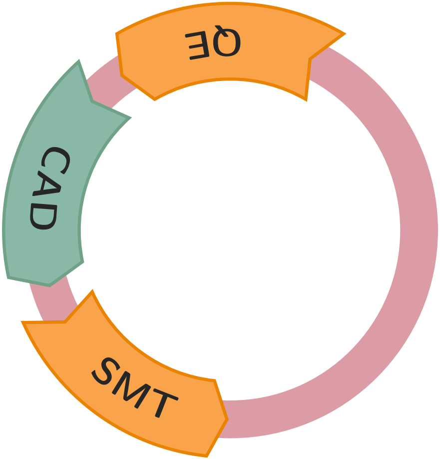

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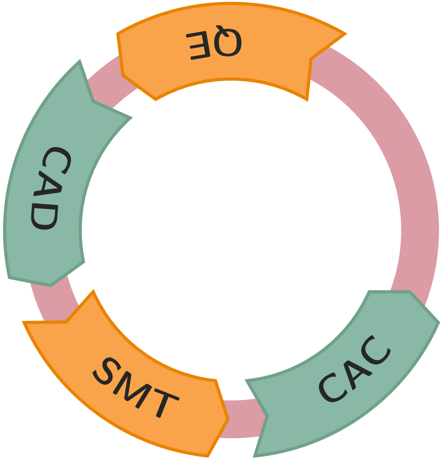

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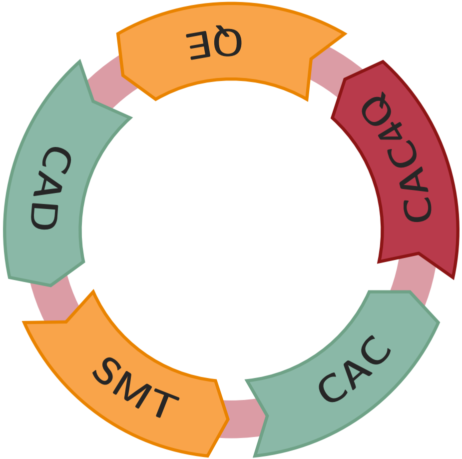

Cylindrical Algebraic Coverings

- Guess partial assignment

$$s_1 \times \cdots \times s_k \times s_{k+1}$$

[Ábrahám et al. 2021] [Kremer et al. 2021]

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- Refute partial assignment using intervals

$$s \notin s_1 \times \cdots \times s_k \times (a, b)$$

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- Lift covering to lower dimension

$$s_1 \times \cdots \times s_k \times \{(-\infty, a), [a, b], \ldots (z, \infty)\} \rightarrow s_1 \times \cdots \times s_{k-1} \times (\alpha, \beta)$$

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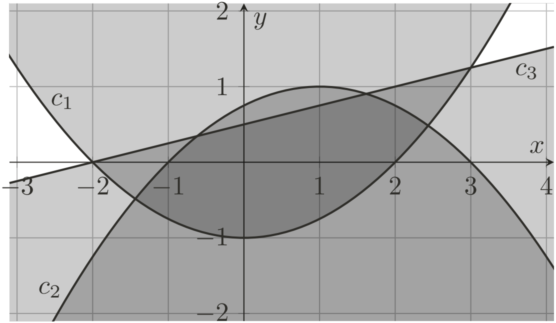
- Eventually get satisfying assignment or a covering in first dimension

$$s = s_1 \times \cdots \times s_n \qquad \text{or} \qquad s_1 \notin \{(-\infty, a), [a, b], \ldots (z, \infty)\}$$

[Ábrahám et al. 2021] [Kremer et al. 2021]

Cylindrical Algebraic Coverings – example

$$c_1 : 4 \cdot y < x^2 - 4 \qquad c_2 : 4 \cdot y > 4 - (x-1)^2 \qquad c_3 : 4 \cdot y > x + 2$$



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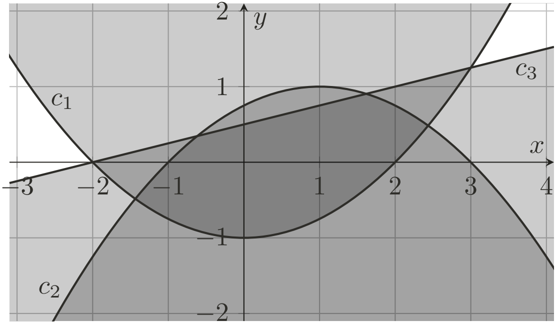

No constraint for x

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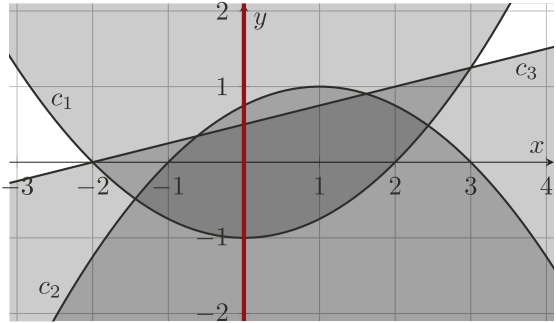

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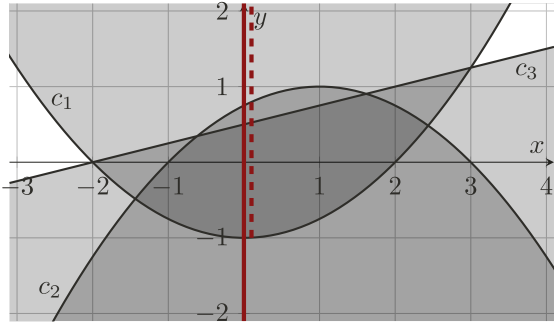

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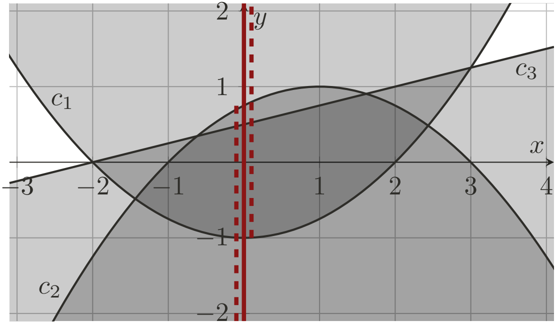

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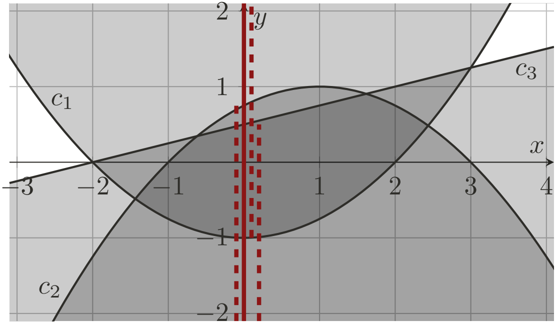

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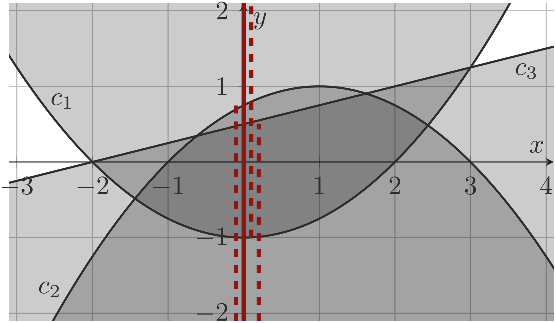

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Construct covering

$(-\infty, 0.5), (-1, \infty)$

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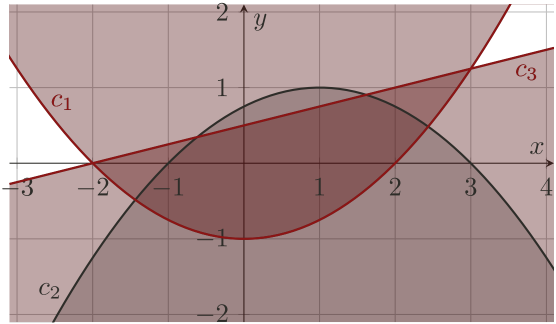

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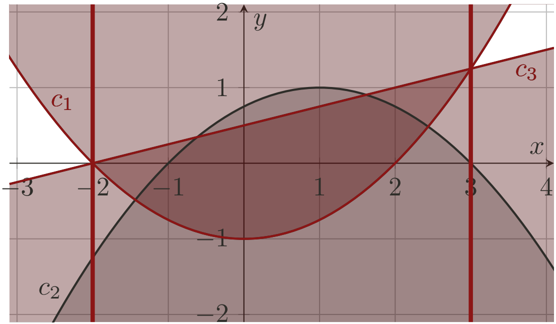

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Construct interval for x

$x \notin (-2, 3)$

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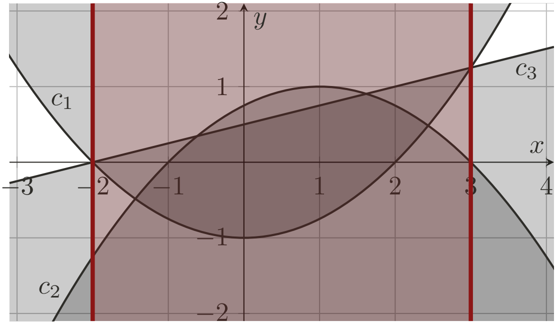

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Guess $x \mapsto 0$

$c_1 \to y \notin (-1, \infty)$

$c_2 \to y \notin (-\infty, 0.75)$

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Construct covering

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Construct interval for x

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New guess for x

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We want to characterize both true and false regions of quantified formulae.

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Core change

Instead of a model return satisfying interval with suitable characterization.

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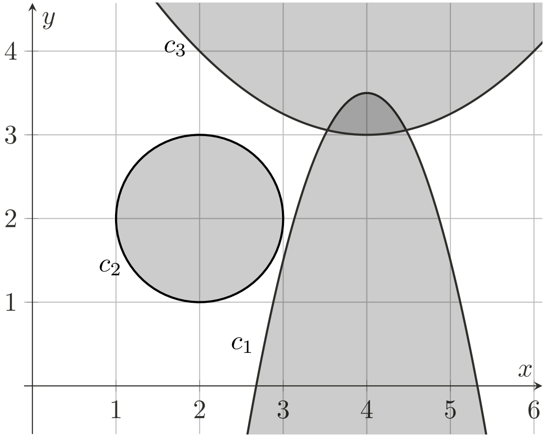
Challenges:

- Boolean structure?
  $\rightarrow$ consider constraints of (suitable) implicants
- Model construction?
  $\rightarrow$ reconstruct from characterization of true region.
- Interval in dimension zero?
  $\rightarrow$ just a technicality, use $\top, \bot$
- Termination guarantees?
  $\rightarrow$ still the same

Cylindrical Algebraic Coverings for Quantifiers

$\forall x.\exists y. \quad y > 3.5 - 2(x-4)^2 \quad (x-2)^2 + (y-2)^2 - 1 > 0 \quad y < 3 + 0.25(x-4)^2$



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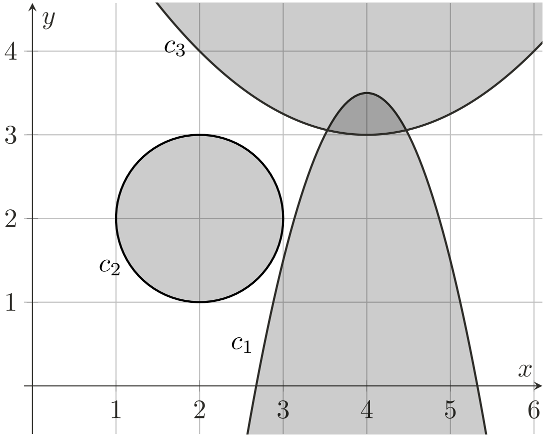

$\texttt{forall}(s = ())$

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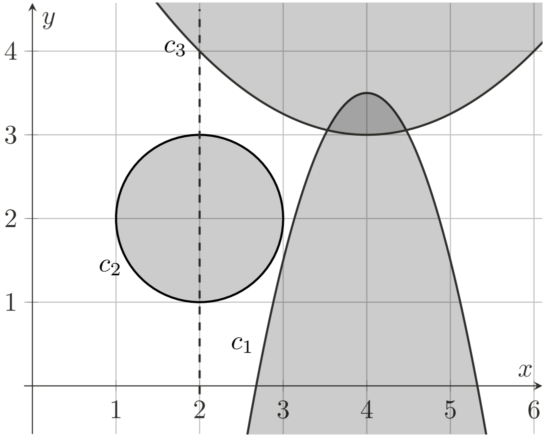


```
forall(s = ())
  exists(s = (2))
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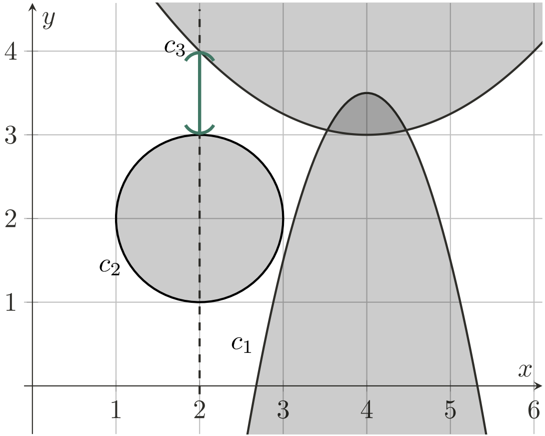

forall($s = ()$)
  exists($s = (2)$)
    SAT @ $s = (2, 3.5)$

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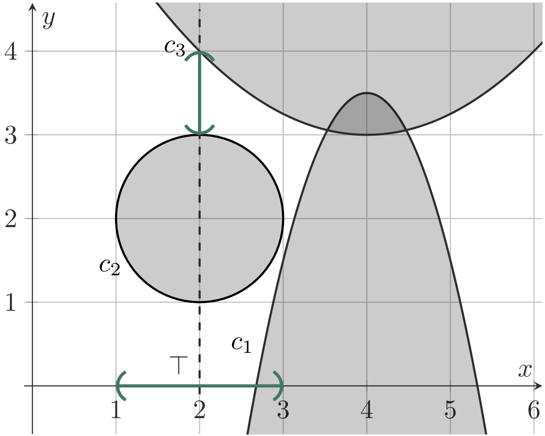

forall($s = ()$)
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    SAT @ $s = (2, 3.5)$
  SAT @ $s \times (1, 3)$

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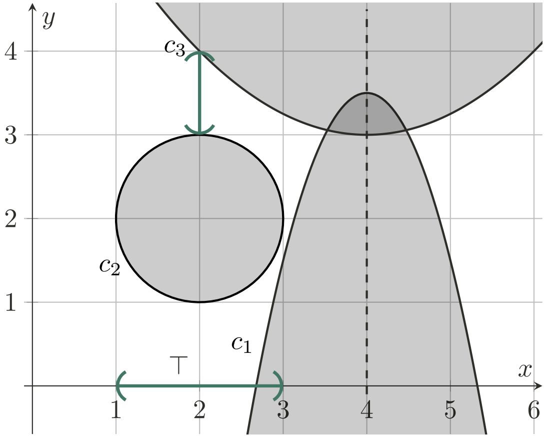


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forall(s = ())
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  SAT @ s × (1, 3)
  exists(s = (4))
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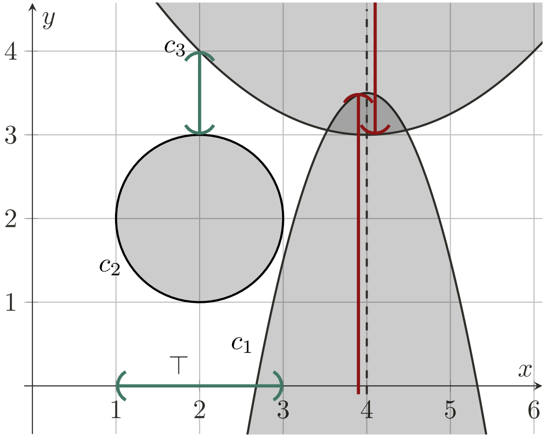


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  SAT @ s × (1, 3)
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    UNSAT @ s × (3, ∞)
    UNSAT @ s × (−∞, 3.5)
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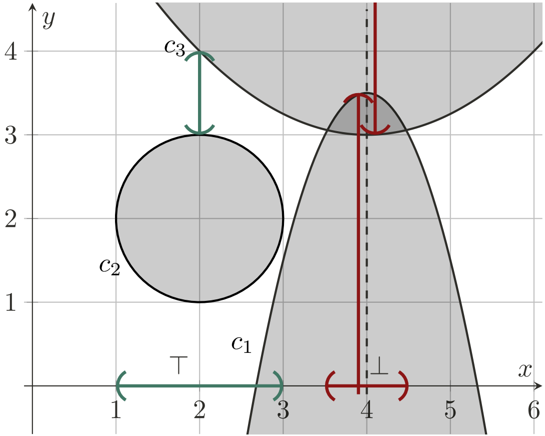

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  UNSAT @ $s \times (3.5, 4.5)$

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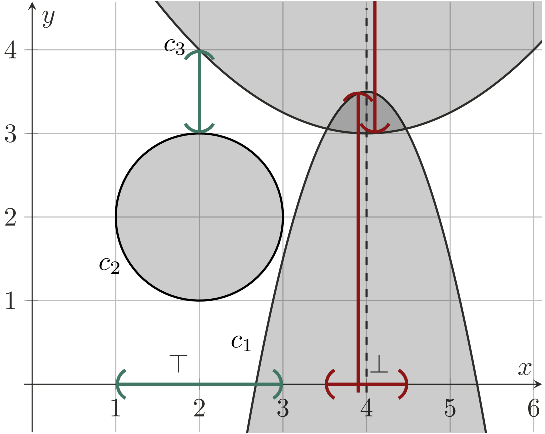

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  UNSAT @ $s \times (3.5, 4.5)$
UNSAT

Cylindrical Algebraic Coverings for QE

- Consider free variables first
- Use previous approach for bounded variables
- Collect all SAT regions for free variables
- Use solution formula construction from [Brown 1999]
- Return disjunction of all SAT regions

Conclusion

- Cylindrical Algebraic Coverings can be adapted to quantifiers and QE
- It mostly works as you would think it does
- A number of subtle challenges
- Provides for a few nice generalizations
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Any questions?

References

- Erika Ábrahám, James H. Davenport, Matthew England, and Gereon Kremer. "Deciding the Consistency of Non-Linear Real Arithmetic Constraints with a Conflict Driven Search Using Cylindrical Algebraic Coverings". In: Journal of Logical and Algebraic Methods in Programming 119 (2021), p. 100633. DOI: 10.1016/j.jlamp.2020.100633.
- Christopher W. Brown. "Solution Formula Construction for Truth Invariant CADs". PhD thesis. University of Delaware, 1999. URL: https://www.usna.edu/Users/cs/wcbrown/research/Thesis.html.
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