Satisfiability Modulo Theories for Arithmetic Problems

... and a lot of references

Stanford University

Contains mostly other people's work!

Satisfiability Modulo Theories

$$\exists \overline{x}.\varphi(\overline{x})$$

Is an existential first-order formula satisfiable?

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Theories:

- uninterpreted functions
- arrays
- bit-vectors
- floating-point numbers
- arithmetic
- datatypes
- strings
- ...

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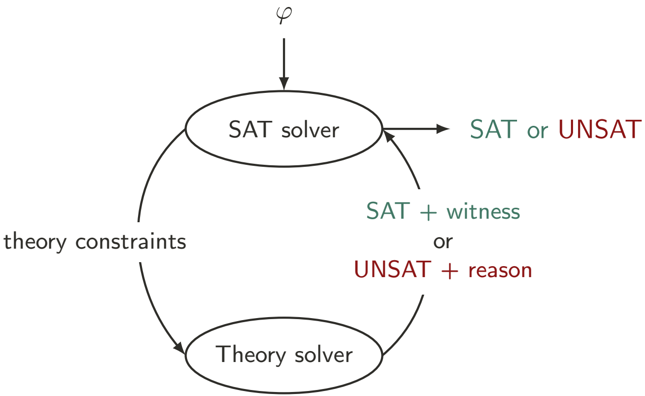
Extensions:

- model generation
- unsat cores
- quantifiers
- optimization queries
- interpolants
- formal proofs
- ...

SMT solving – CDCL(T)



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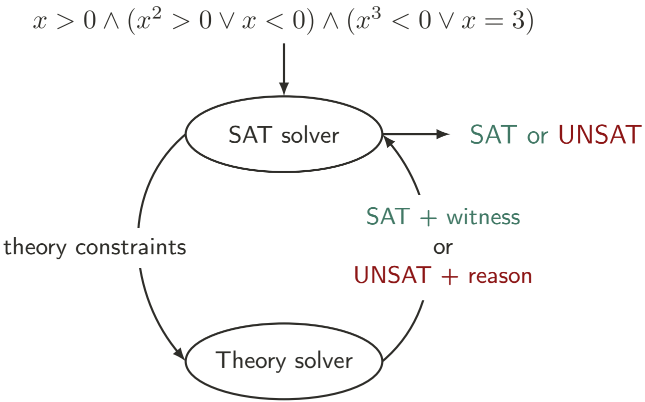

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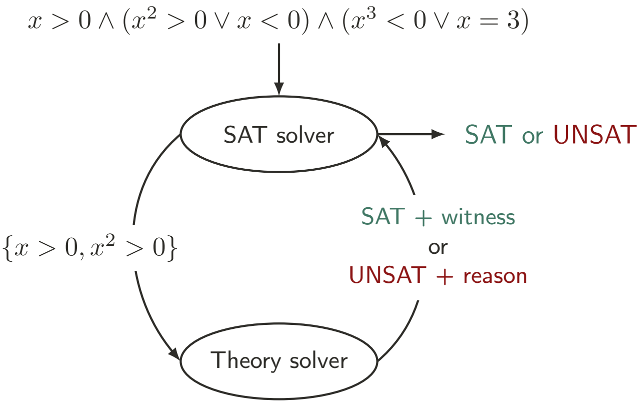

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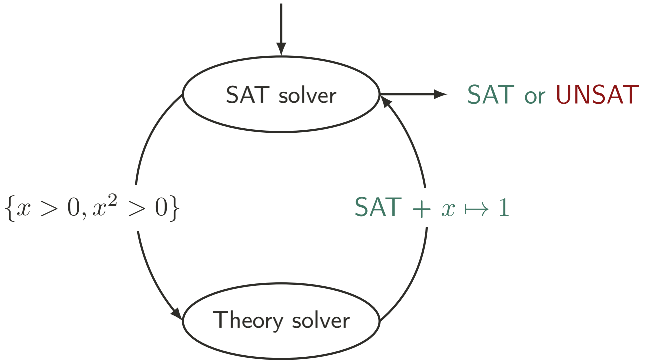

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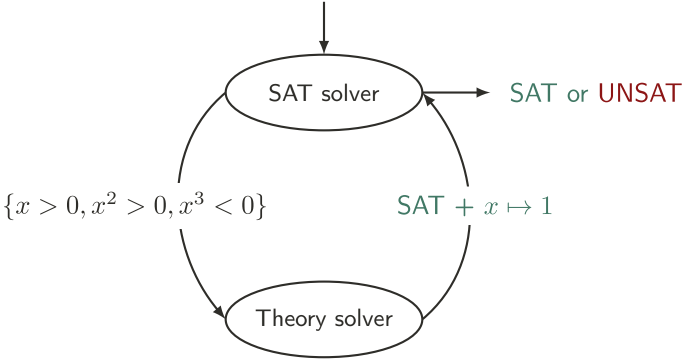

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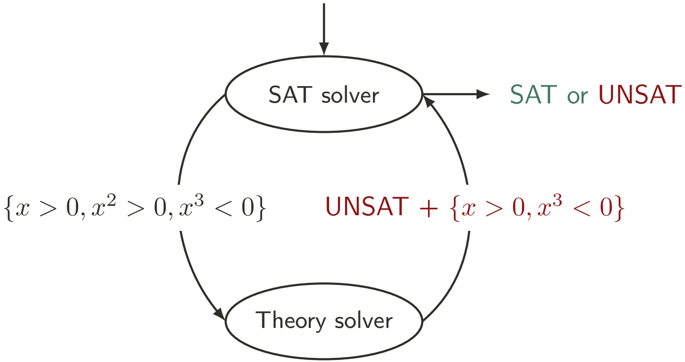

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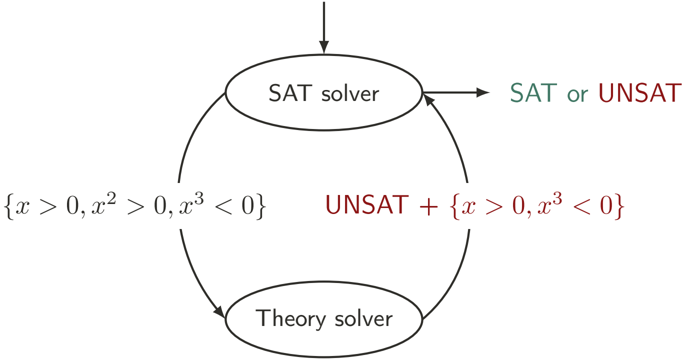

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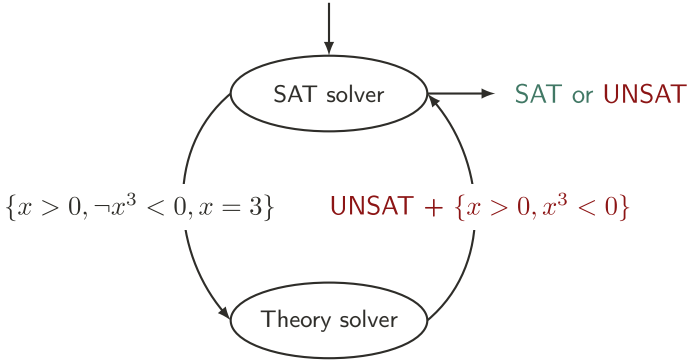

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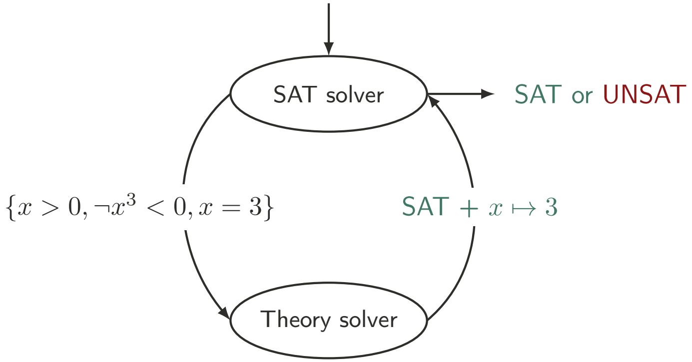

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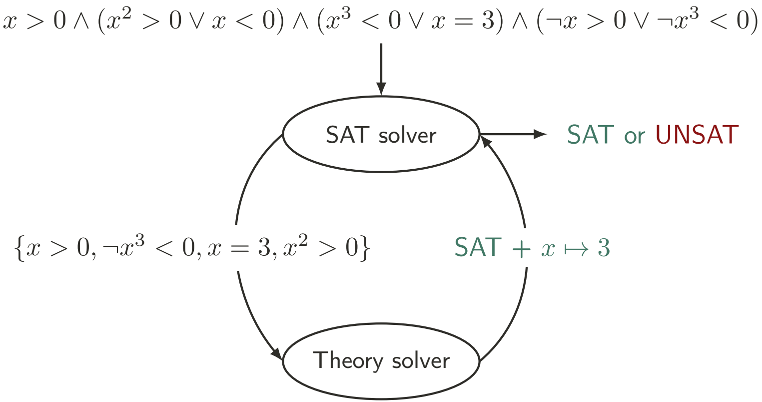

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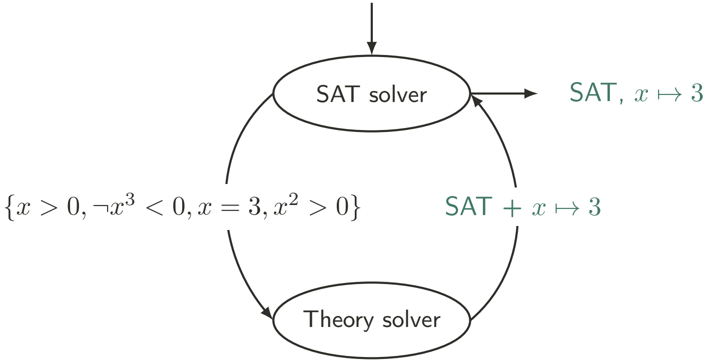

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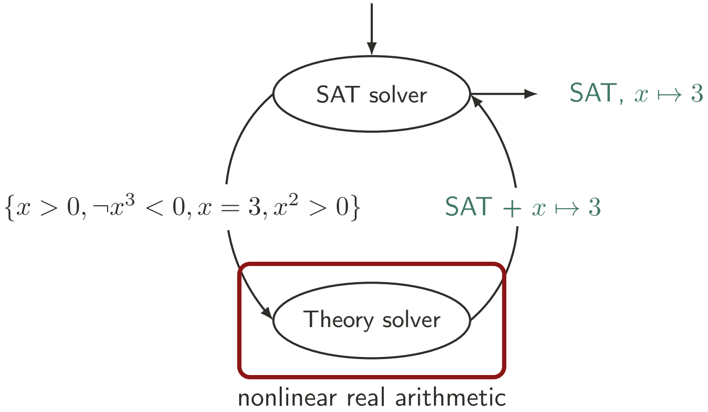

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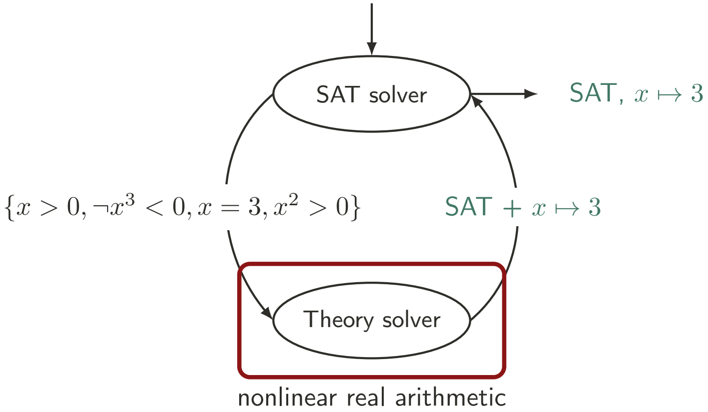

Also: NLSAT/MCSAT [Jovanović et al. 2012] [Moura et al. 2013]

SMT for Nonlinear Real Arithmetic

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Nonlinear Real Arithmetic:

- real variables $v := x_i \in \mathbb{R}$
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Linear arithmetic: essentially a solved problem.
Use Simplex (or sometimes Fourier-Motzkin)

Theory of the Reals in a nutshell

- complete (we have decision procedures that are sound and complete)
- admits quantifier elimination (quantifiers are conceptually easy)

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Some methods:

- [Tarski 1951] Tarski: first complete method, non-elementary complexity
- [Buchberger 1965] Gröbner bases: limited applicability, standard tool in CA
- [Collins 1974] CAD: complete, doubly exponential complexity
- [Weispfenning 1988] VS: up to bounded degree, singly exponential complexity
- [Gao et al. 2013] ICP: heuristic interval reasoning, incomplete
- [Fontaine et al. 2017] Subtropical satisfiability: incomplete reduction to LRA
- [Irfan 2018] Linearization: incomplete, axiom instantiation
- [Ábrahám et al. 2021] CDCAC: conflict-driven CAD
- and some more...

SC-Square

SC²

Satisfiability Checking and Symbolic Computation

Bridging Two Communities to Solve Real Problems

Consortium of the EU-CSA project

University of Bath	James Davenport; Russell Bradford
RWTH Aachen	Erika Ábrahám; Viktor Levandovskyy
Fondazione Bruno Kessler	Alberto Griggio; Alessandro Cimatti
Università degli Studi di Genova	Anna Bigatti
Maplesoft Europe Ltd	Jürgen Gerhard; Stephen Forrest
Université de Lorraine (LORIA)	Pascal Fontaine
Coventry University	Matthew England
University of Oxford	Daniel Kroening; Martin Brain
Universität Kassel	Werner Seiler; John Abbott
Max Planck Institut für Informatik	Thomas Sturm
Universität Linz	Tudur Jebelean; Bruno Buchberger; Wolfgang Windsteiger; Roxana-Maria Holom

Overview

Incremental linearization

[Irfan 2018] [Cimatti et al. 2018]

implicitly linearize: $x \cdot y \rightsquigarrow a_{x \cdot y}$

$$x > 2 \land y > -1 \land x \cdot y < 2$$

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$$\begin{aligned} & x > 2 \land y > -1 \land x \cdot y < 2 \\ \text{Model: } & x \mapsto 3, y \mapsto 0, x \cdot y \mapsto 1 \\ \text{Lemma: } & y = 0 \Rightarrow x \cdot y = 0 \end{aligned}$$

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Lemma: $y = 0 \Rightarrow x \cdot y = 0$

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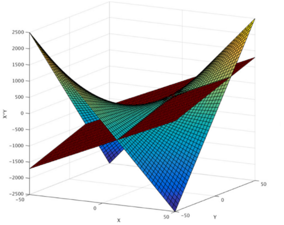
Lemma: ~~$(x = 3 \land y = 1) \Rightarrow x \cdot y = 3$~~



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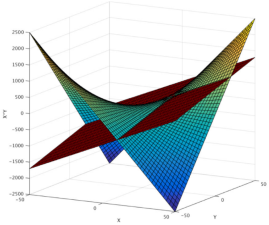
\text{Lemma: } & \cancel{(x = 3 \wedge y = 1) \Rightarrow x \cdot y = 3} \\
& (x \leq 3 \wedge y \leq 1) \vee (x \geq 3 \wedge y \geq 1) \\
\Leftrightarrow & (x \cdot y \geq 1 \cdot x + 3 \cdot y - 3 \cdot 1)
\end{aligned}
$$



[Cimatti et al. 2018]

Incremental linearization – schemas

split zero	$\top \Rightarrow (t = 0 \vee t \neq 0)$
sign	$x > 0 \wedge y > 0 \Rightarrow xy > 0$
	$x = 0 \Rightarrow xyz = 0$
magnitude	$\lvert x\rvert > \lvert y\rvert \Rightarrow \lvert xz\rvert > \lvert yz\rvert$
	$\lvert z\rvert > \lvert y\rvert \wedge \lvert u\rvert > \lvert w\rvert \wedge \lvert x\rvert \geq 1 \Rightarrow \lvert zuxx\rvert > \lvert yw\rvert$
bounds	$x > 0 \wedge y > z + w \Rightarrow xy > x(z + w)$
resolution bounds	$y \geq 0 \wedge s \leq xz \wedge xy \leq t \Rightarrow ys \leq zt$
tangent plane	$(x \leq 3 \wedge y \leq 1) \vee (x \geq 3 \wedge y \geq 1) \Rightarrow xy \geq x + 3y - 3$

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[Irfan 2018] [Cimatti et al. 2018]

Intuition: iteratively teach the linear solver about the nonlinear parts, add lemmas that cut away unsatisfiable regions.

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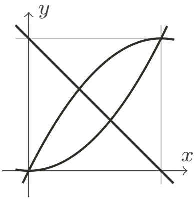
Extensions:

- Repair model (if easily possible)
- Transcendental functions ($\texttt{sin}$, $\texttt{cos}$, $\dots$)
- extended operators in general

Question

Better linearization lemmas? Linearization lemmas for other functions?

Interval Constraint Propagation



$y > x^2 \;\wedge\; y < -x^2 + 2x \;\wedge\; y \leq 1 - x$ $\qquad x \times y$

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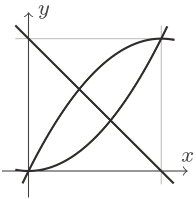

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$y > x^2 \Rightarrow y \in (0, \infty)$ $\qquad (-\infty, \infty) \times (0, \infty)$

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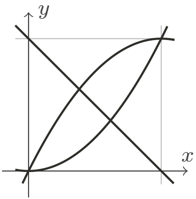

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$x > 0.5x^2 + y \Rightarrow x \in (0, \infty)$	$(0, \infty) \times (0, \infty)$

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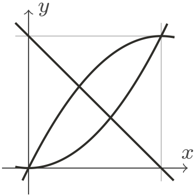

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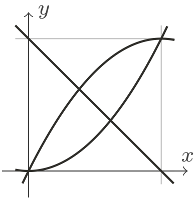

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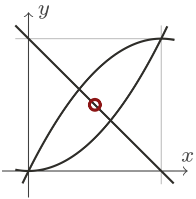

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guess midpoint $(0.5, 0.5) \in (0, 1) \times (0, 1)$	

Interval Constraint Propagation

[Benhamou et al. 2006] [Gao et al. 2013] [Scheibler et al. 2013] [Schupp 2013] [Tung et al. 2017]

Core idea:

- Maintain interval assignment (that represents the current box)
- Perform over-approximating contractions until
  - the current box is empty (UNSAT),
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- Incomplete solving procedure
- Used as preprocessor for other techniques [Loup et al. 2013]
- Delicate tuning of heuristics (splitting, thresholds, model guessing)

Question

Sensible initial bounds? Better propagation schemas?

Subtropical satisfiability

[Fontaine et al. 2017] [Fontaine et al. 2018]

Core idea: reduce $p = 0$ to a linear problem in the exponents of p

- Assume $p(1, \ldots, 1) < 0$ (otherwise consider $-p$)
- Find $x \in \mathbb{R}_+^n$ such that $p(x) > 0$
- Solve $p(y) = 0$ with y on the line $(1, \ldots, 1)$ — x

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Core problem: How to find $x \in \mathbb{R}_+^n$ such that $p(x) > 0$?

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Core problem: How to find $x \in \mathbb{R}_+^n$ such that $p(x) > 0$?

For $n = 1$: $\lim_{x \to \infty} p(x) = \infty$ if $\mathrm{lcoeff}(p) > 0$. Increase x as necessary.

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For $n \geq 2$: search direction in exponent space such that the largest exponent in this direction is positive. Increase x in this direction as necessary.

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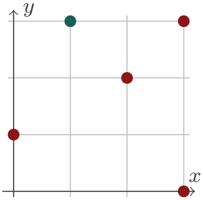
$$p = -y + 2xy^3 - 3x^2y^2 - x^3 - 4x^3y^3$$



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Core idea: reduce $p = 0$ to a linear problem in the exponents of p

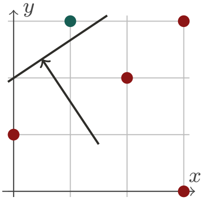
- Assume $p(1, \ldots, 1) < 0$ (otherwise consider $-p$)
- Find $x \in \mathbb{R}_+^n$ such that $p(x) > 0$
- Solve $p(y) = 0$ with y on the line $(1, \ldots, 1)$ — x
- Incomplete (no such y exists, though $p = 0$ has a solution)

Core problem: How to find $x \in \mathbb{R}_+^n$ such that $p(x) > 0$?

For $n = 1$: $\lim_{x\to\infty} p(x) = \infty$ if $\mathrm{lcoeff}(p) > 0$. Increase x as necessary.
For $n \geq 2$: search direction in exponent space such that the largest exponent in this direction is positive. Increase x in this direction as necessary.

$$p = -y + 2xy^3 - 3x^2y^2 - x^3 - 4x^3y^3$$

Find hyperplane that separates a positive node



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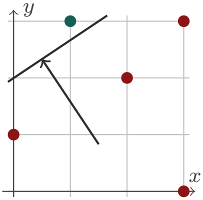
Find hyperplane that separates a positive node
Encoding in `QF_LRA`
Growing degree only impacts coefficient size



Gröbner basis

[Buchberger 1965] [Junges 2012]

- Canonical generators for a polynomial ideal
- For us: Normal form for sets of polynomials
- Maintains set of common complex roots
- The workhorse of computer algebra for polynomial equalities
- Mature implementations (every CAS)
- Doubly exponential in worst case, but usually much faster.

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But: What about inequalities? How to go from $\mathbb{C}$ to $\mathbb{R}$?
see [Junges 2012] for some approaches.

Question

How to construct models? How to obtain infeasible subsets?

Virtual Substitution

[Weispfenning 1988] [Weispfenning 1997] [Košta et al. 2015] [Košta 2016] [Nalbach 2017]

Core idea:

- Use solution formula to solve polynomial equation for x
- Substitute value for x into remaining equations
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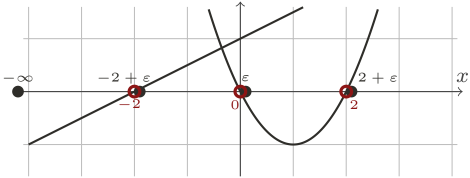
What about inequalities?

- Construct test candidates for all sign-invariant regions in x
- Always try the roots and the smallest values of the intermediate intervals



- Introduces special terms $t+\varepsilon$ and $-\infty$

Virtual Substitution

Algorithmic core: a collection of substitution rules
Example: Substitute $e + \varepsilon$ for x into $a \cdot x^2 + b \cdot x + c > 0$:

$$
\begin{array}{lllllll}
 & (& (ax^2+bx+c>0)[e//x] & & & &) \\
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Not always applicable:

- Solution formulas only exist up to degree four
- The above rule may introduce a degree growth
- Efficient if applicable
- [Košta et al. 2015] uses FO formulas, allows arbitrary but fixed degrees (needs precomputed substitution rules obtained by quantifier elimination)

Cylindrical Algebraic Decomposition

The core idea: sign-invariance (or rather truth-table equivalence)

$$sgn(p(a)) = sgn(p(b)) \; \forall p \in \varphi \quad \Rightarrow \quad \varphi(a) = \varphi(b)$$

For our purpose, a and b are equivalent!

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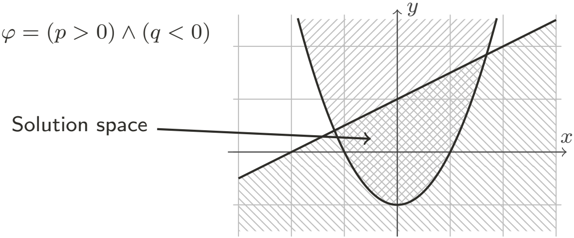

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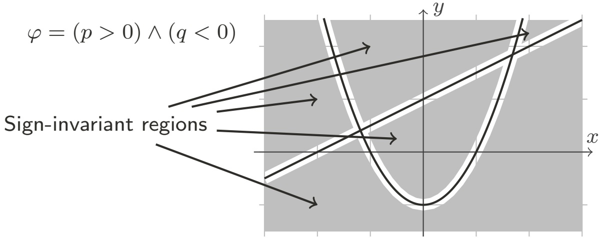

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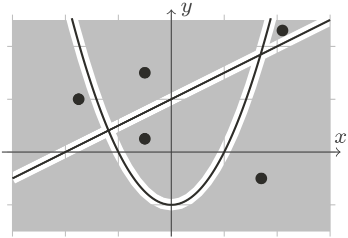
Sample points



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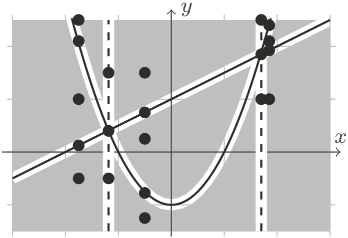
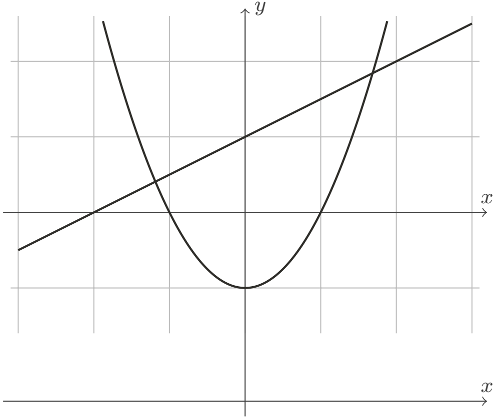
Actual sample points

Arranged in cylinders



Cylindrical Algebraic Decomposition in $\mathbb{R}^2$

Proceed dimension-wise: project to lower-dimensional problem, lift results.



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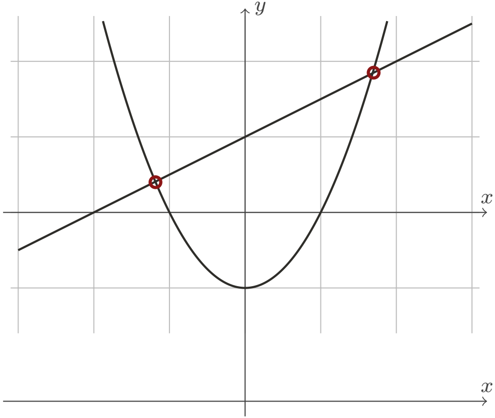
Intuition

Critical points



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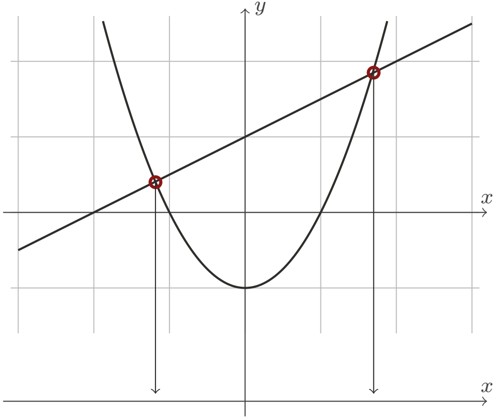

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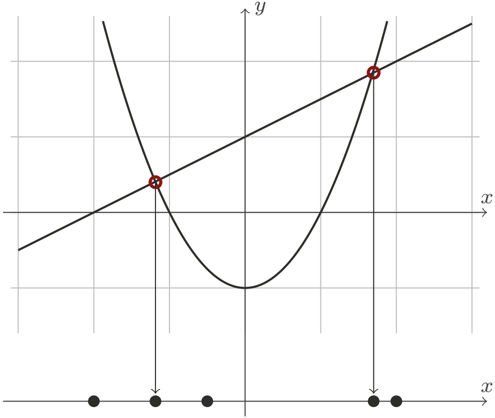
Solve 1-dim



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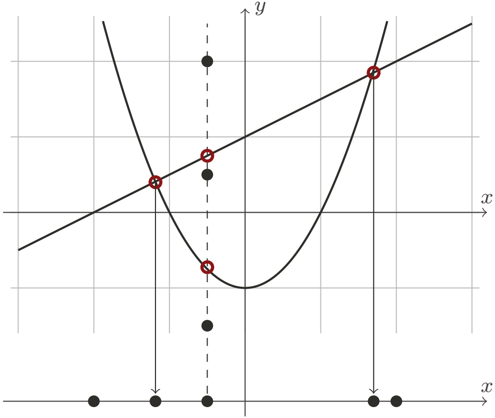
Lift to 2-dim



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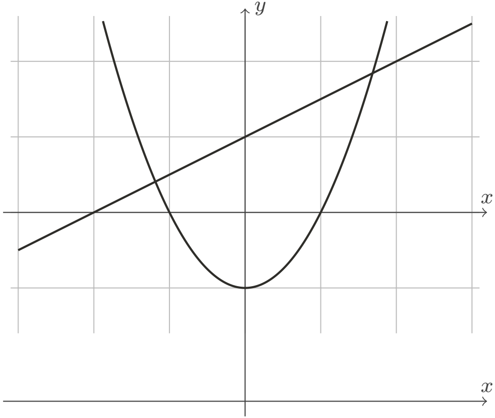

Implementation

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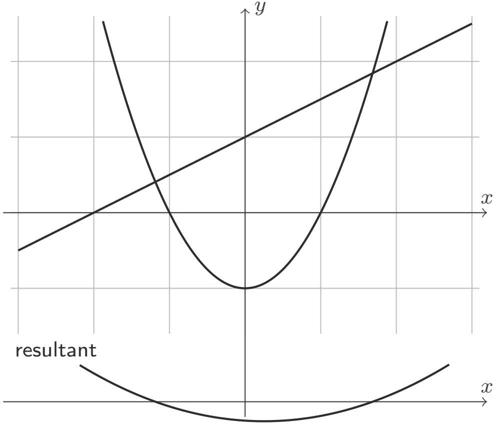

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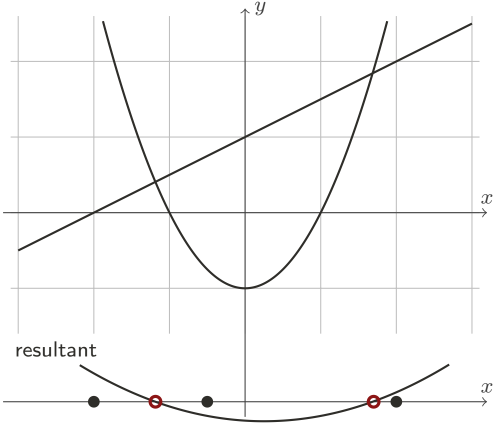

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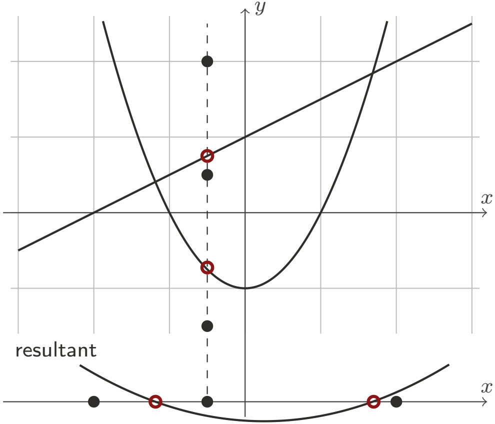

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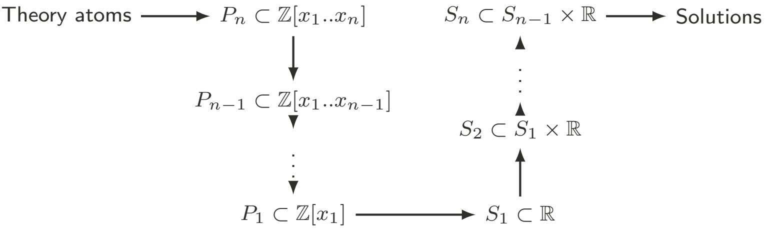

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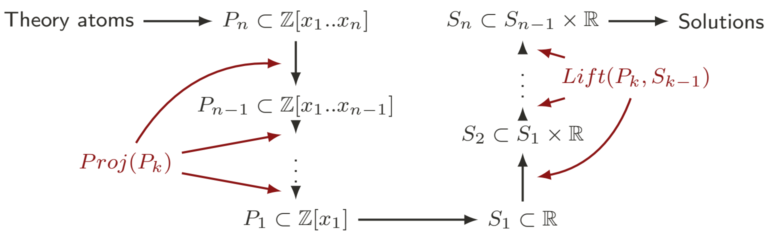

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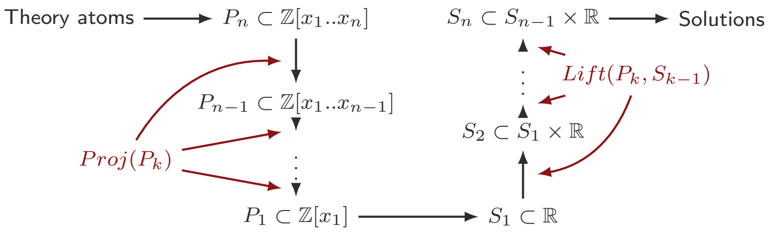

Projection:

- Intersections (resultants)
- Flipping points (discriminants)
- Singularities (coefficients)

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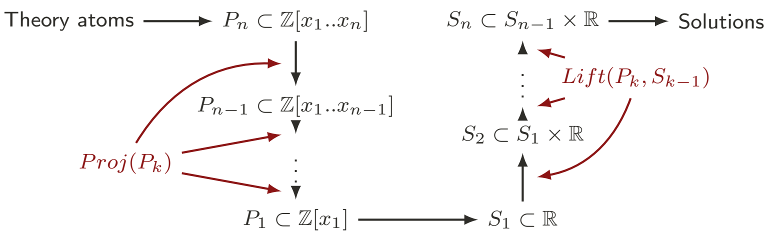

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Lifting:

- Substitution $s \in S_k$, $p \in P_{k+1}$
  $p(s) \to p' \in \mathbb{Z}[x_{k+1}]$ ***
- Isolate real roots of p'

Final notes on CAD

- Asymptotic complexity: $(n \cdot m)^{2^r}$ (r variables, m polynomials of degree n)
- Oftentimes way faster, but worst-case occurs in practice!
- Best complete method that is known and implemented. [Hong 1991]
- Active research:
  - Projection [McCallum 1984] [McCallum 1988] [Hong 1990] [Lazard 1994] [Brown 2001] [McCallum 2001] [McCallum et al. 2016] [McCallum et al. 2019];[Strzeboński 2000] [Seidl et al. 2003] [Jovanović et al. 2012] [Brown 2013] [Strzeboński 2014] [Brown et al. 2015]
  - Lifting [Collins 1974] [Lazard 1994] [McCallum et al. 2016] [McCallum et al. 2019]
  - Equational constraints [Collins 1998] [McCallum 1999] [McCallum 2001] [England et al. 2015] [Haehn et al. 2018] [Nair et al. 2019]
  - Variable ordering [England et al. 2014] [Huang et al. 2014] [Nalbach et al. 2019] [Florescu et al. 2019]
  - Adaptions [Jovanović et al. 2012] [Brown 2013] [Brown 2015] [Ábrahám et al. 2021]
- Implementation needs groundwork: polynomial computation (resultants, multivariate gcd, optionally multivariate factorization), real algebraic numbers (representation, multivariate root isolation)

Conflict-Driven Cylindrical Algebraic Coverings

[Ábrahám et al. 2021]

Core idea: use CAD techniques in a conflict-driven way.

My intuition: MCSAT turned into a theory solver.

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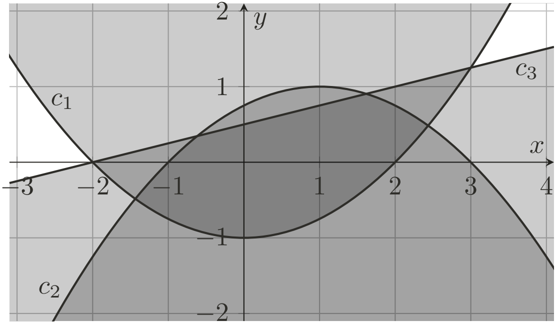
My intuition: MCSAT turned into a theory solver.

- Fix a variable ordering
- For the kth variable
  - Use constraints to exclude unsatisfiable intervals
  - Guess a value for the kth variable
  - Recurse to $k + 1$st variable and obtain
    - a full variable assignment ($\rightarrow$ return SAT)
    - or a covering for the $k + 1$st variable
  - Use CAD machinery to infer an interval for the kth variable
- Until the collected intervals form a covering for the kth variable

An example

$$c_1 : 4 \cdot y < x^2 - 4 \qquad c_2 : 4 \cdot y > 4 - (x-1)^2 \qquad c_3 : 4 \cdot y > x + 2$$



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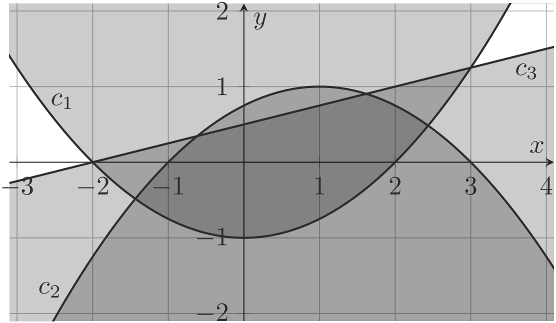

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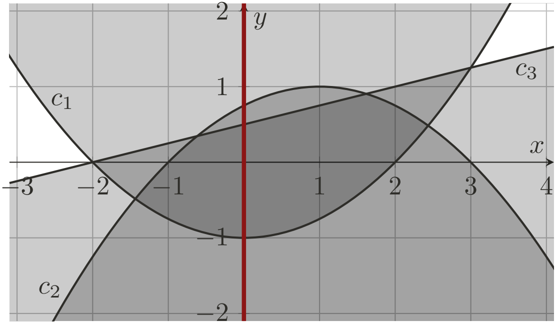

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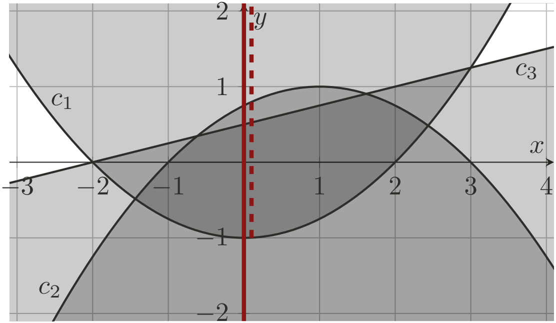

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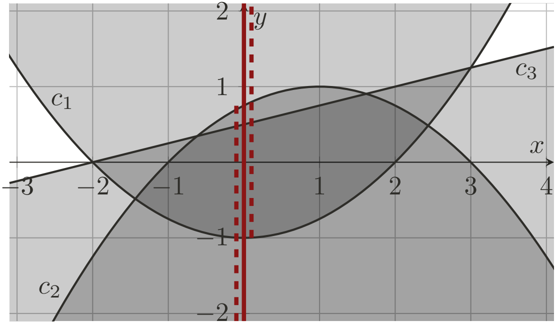

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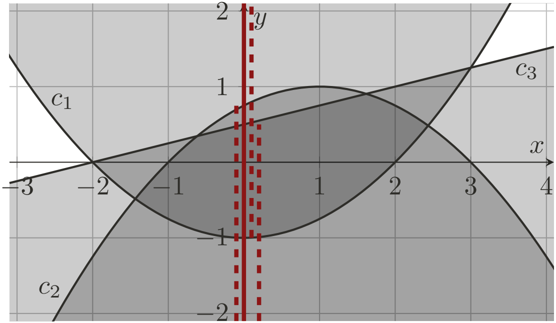

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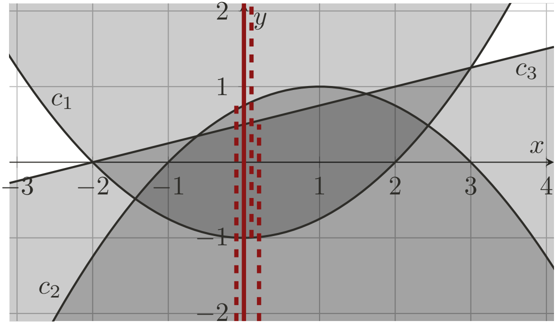

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$(-\infty, 0.5), (-1, \infty)$

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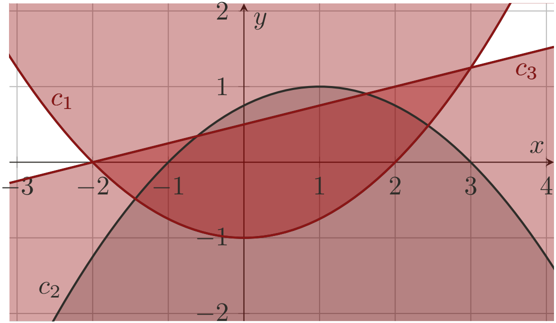

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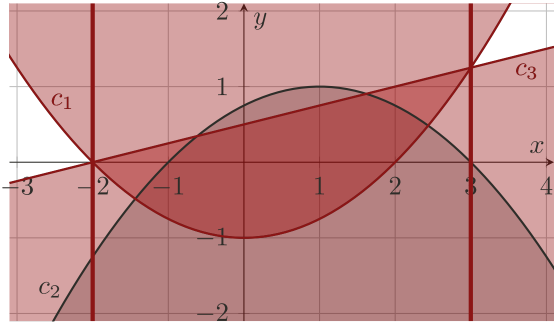

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Construct covering
$(-\infty, 0.5), (-1, \infty)$
Construct interval for x
$x \notin (-2, 3)$

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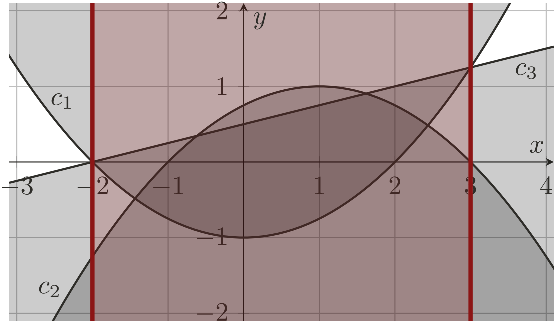


No constraint for x

Guess $x \mapsto 0$

$c_1 \to y \notin (-1, \infty)$

$c_2 \to y \notin (-\infty, 0.75)$

$c_3 \to y \notin (-\infty, 0.5)$

Construct covering

$(-\infty, 0.5), (-1, \infty)$

Construct interval for x

$x \notin (-2, 3)$

New guess for x

The main algorithm

function get_unsat_cover($(s_1, \ldots, s_{i-1})$)

```
𝕀 := get_unsat_intervals(s)
while ⋃_{I∈𝕀} I ≠ ℝ do
    s_i := sample_outside(𝕀)
    if i = n then  return (SAT, (s_1, ..., s_{i-1}, s_i))
    (f, O) := get_unsat_cover((s_1, ..., s_{i-1}, s_i))
    if f = SAT then  return (SAT, O)
    else if f = UNSAT then
        R := construct_characterization((s_1, ..., s_{i-1}, s_i), O)
        J := interval_from_characterization((s_1, ..., s_{i-1}), s_i, R)
        𝕀 := 𝕀 ∪ {J}
    end
end
return (UNSAT, 𝕀)
```

The main algorithm

function get_unsat_cover($(s_1, \ldots, s_{i-1})$)

$\mathbb{I}$:= get_unsat_intervals(s)

Real root isolation over a partial sample point

while $\bigcup_{I \in \mathbb{I}} I \neq \mathbb{R}$ **do**

- s_i := sample_outside($\mathbb{I}$)
- **if** $i = n$ **then** **return** $(\mathrm{SAT}, (s_1, \ldots, s_{i-1}, s_i))$
- (f, O) := get_unsat_cover($(s_1, \ldots, s_{i-1}, s_i)$)
- **if** $f = \mathrm{SAT}$ **then** **return** (SAT, O)
- **else if** $f = \mathrm{UNSAT}$ **then**
  - R := construct_characterization($(s_1, \ldots, s_{i-1}, s_i), O$)
  - J := interval_from_characterization($(s_1, \ldots, s_{i-1}), s_i, R$)
  - $\mathbb{I} := \mathbb{I} \cup \{J\}$
- **end**

end

return $(\mathrm{UNSAT}, \mathbb{I})$

The main algorithm

function `get_unsat_cover`$((s_1, \ldots, s_{i-1}))$

$\mathbb{I} :=$ `get_unsat_intervals(s)`
while $\bigcup_{I \in \mathbb{I}} I \neq \mathbb{R}$ **do**
  $s_i :=$ `sample_outside`$(\mathbb{I})$
  if $i = n$ **then return** $(\mathrm{SAT}, (s_1, \ldots, s_{i-1}, s_i))$
  $(f, O) :=$ `get_unsat_cover`$((s_1, \ldots, s_{i-1}, s_i))$
  if $f = \mathrm{SAT}$ **then return** (SAT, O)
  else if $f = \mathrm{UNSAT}$ **then**
    $R :=$ `construct_characterization`$((s_1, \ldots, s_{i-1}, s_i), O)$
    $J :=$ `interval_from_characterization`$((s_1, \ldots, s_{i-1}), s_i, R)$
    $\mathbb{I} := \mathbb{I} \cup \{J\}$
  end
end
return $(\mathrm{UNSAT}, \mathbb{I})$

Real root isolation over a partial sample point

Select sample from $\mathbb{R} \setminus \mathbb{I}$

The main algorithm

function get_unsat_cover($(s_1, \ldots, s_{i-1})$)

$\mathbb{I} :=$ get_unsat_intervals(s)
while $\bigcup_{I \in \mathbb{I}} I \neq \mathbb{R}$ **do**
  $s_i :=$ sample_outside($\mathbb{I}$)
  if $i = n$ **then** **return** $(\mathrm{SAT}, (s_1, \ldots, s_{i-1}, s_i))$
  $(f, O) :=$ get_unsat_cover($(s_1, \ldots, s_{i-1}, s_i)$)
  if $f = \mathrm{SAT}$ **then** **return** (SAT, O)
  else if $f = \mathrm{UNSAT}$ **then**
    $R :=$ construct_characterization($(s_1, \ldots, s_{i-1}, s_i), O$)
    $J :=$ interval_from_characterization($(s_1, \ldots, s_{i-1}), s_i, R$)
    $\mathbb{I} := \mathbb{I} \cup \{J\}$
  end
end
return $(\mathrm{UNSAT}, \mathbb{I})$

Real root isolation over a partial sample point

Select sample from $\mathbb{R} \setminus \mathbb{I}$

Recurse to next variable

The main algorithm

`function get_unsat_cover`$((s_1, \ldots, s_{i-1}))$

$\mathbb{I}$:= `get_unsat_intervals(s)`
while $\bigcup_{I \in \mathbb{I}} I \neq \mathbb{R}$ **do**
  s_i := `sample_outside`$(\mathbb{I})$
  if $i = n$ **then return** $(\mathrm{SAT}, (s_1, \ldots, s_{i-1}, s_i))$
  (f, O) := `get_unsat_cover`$((s_1, \ldots, s_{i-1}, s_i))$
  if $f = \mathrm{SAT}$ **then return** (SAT, O)
  else if $f = \mathrm{UNSAT}$ **then**
    R := `construct_characterization`$((s_1$
    J := `interval_from_characterization`$((s_1,$
    $\mathbb{I} := \mathbb{I} \cup \{J\}$
  end
end
return $(\mathrm{UNSAT}, \mathbb{I})$

Real root isolation over a partial sample point

Select sample from $\mathbb{R} \setminus \mathbb{I}$

Recurse to next variable

CAD-style projection: Roots of polynomials restrict where covering is still applicable

The main algorithm

`function get_unsat_cover`$((s_1, \ldots, s_{i-1}))$

$\mathbb{I} :=$ `get_unsat_intervals(s)`
while $\bigcup_{I \in \mathbb{I}} I \neq \mathbb{R}$ **do**
  $s_i :=$ `sample_outside`$(\mathbb{I})$
  if $i = n$ **then return** $(\mathrm{SAT}, (s_1, \ldots, s_{i-1}, s_i))$
  $(f, O) :=$ `get_unsat_cover`$((s_1, \ldots, s_{i-1}, s_i))$
  if $f = \mathrm{SAT}$ **then return** (SAT, O)
  else if $f = \mathrm{UNSAT}$ **then**
    $R :=$ `construct_characterization`$((s_1$
    $J :=$ `interval_from_characterization`$((s_1,$
    $\mathbb{I} := \mathbb{I} \cup \{J\}$
  end
end
return $(\mathrm{UNSAT}, \mathbb{I})$

Real root isolation over a partial sample point

Select sample from $\mathbb{R} \setminus \mathbb{I}$

Recurse to next variable

CAD-style projection: Roots of polynomials restrict where covering is still applicable

Extract interval from polynomials

The main algorithm

```
function get_unsat_cover((s_1, ..., s_{i-1}))

𝕀 := get_unsat_intervals(s)
while ⋃_{I∈𝕀} I ≠ ℝ do
  s_i := sample_outside(𝕀)
  if i = n then return (SAT, (s_1, ..., s_{i-1}, s_i))
  (f, O) := get_unsat_cover((s_1, ..., s_{i-1}, s_i))
  if f = SAT then return (SAT, O)
  else if f = UNSAT then
    R := construct_characterization((s_1, ..., s_{i-1}, s_i), O)
    J := interval_from_characterization((s_1, ..., s_{i-1}), s_i, R)
    𝕀 := 𝕀 ∪ {J}
  end
end
return (UNSAT, 𝕀)
```

Real root isolation over a partial sample point

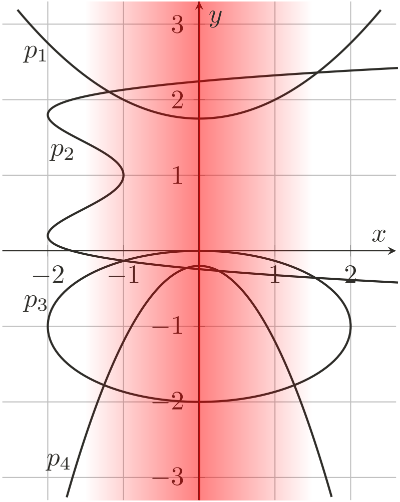
Select sample from $\mathbb{R} \setminus \mathbb{I}$

Recurse to next variable

CAD-style projection: Roots of polynomials restrict where covering is still applicable

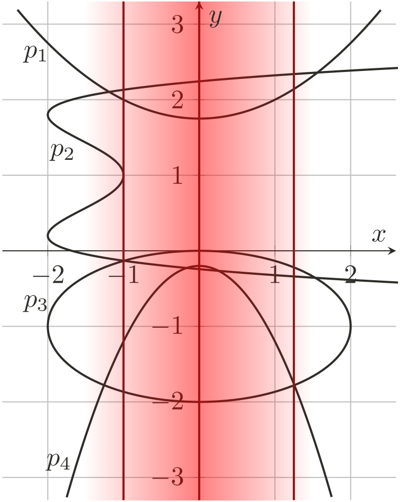
Extract interval from polynomials

construct_characterization



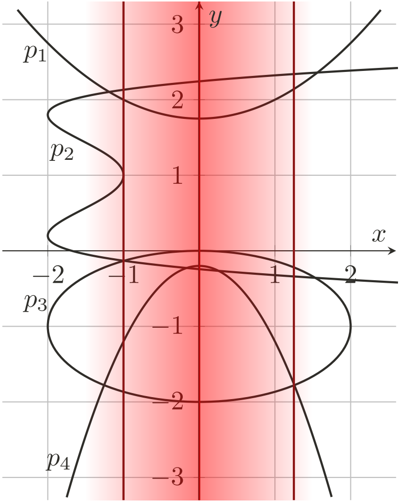
Identify region around sample

construct_characterization



Identify region around sample

construct_characterization

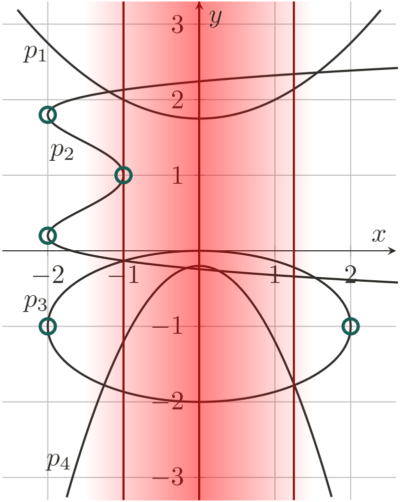


Identify region around sample

CAD projection:

- Discriminants (and coefficients)
- Resultants

construct_characterization

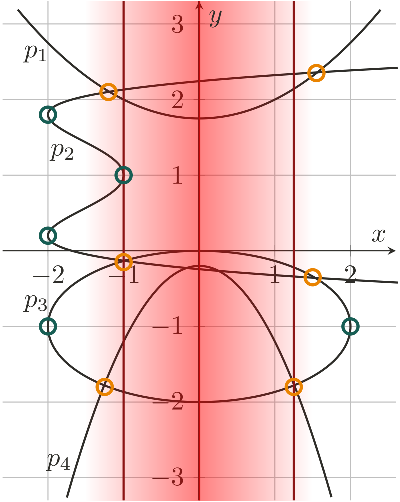


Identify region around sample

CAD projection:

- Discriminants (and coefficients)
- Resultants

construct_characterization

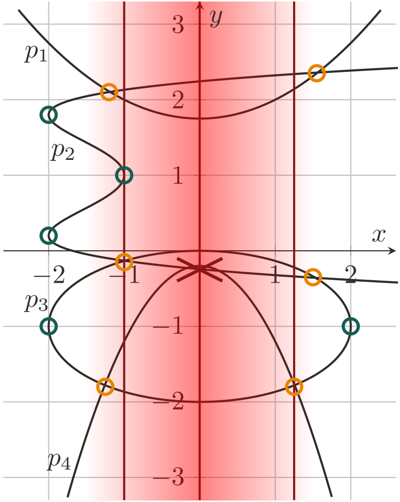


Identify region around sample

CAD projection:

- Discriminants (and coefficients)
- Resultants

construct_characterization



Identify region around sample

CAD projection:

- Discriminants (and coefficients)
- Resultants

Improvement over CAD:

- Resultants between neighbouring intervals only!

Other methods for (QF_)NRA

- Numerical methods [Kremer 2013]:
  focus on good approximation, but no formal guarantees
- Tarski's method [Tarski 1951]:
  theoretical breakthrough only, non-elementary complexity
- Grigor'ev and Vorobjov [Grigor'ev et al. 1988], Renegar [Renegar 1988]:
  singly exponentional, but impractical (see [Hong 1991])
- Basu, Pollack and Roy [Basu et al. 1996]:
  "realizable sign conditions", has not been implemented (yet)
- Other CAD-based methods:
  Regular Chains [Chen et al. 2009], NuCAD [Brown 2015]

Beyond QF_NRA

- Quantifiers:
  - Theory of the Reals admits quantifier elimination
  - CAD constructs φ' for $Q_x \varphi(x, y) \Leftrightarrow \varphi'(y)$
- Theory combination with Array, BV, FP, String, . . . [Nelson et al. 1979]
- Transcendentals: extend linearization [Cimatti et al. 2018] [Irfan 2018]
- Optimization: CAD can optimize for an objective [Kremer 2020]
- Integers: Branch&Bound complements BitBlasting [Kremer et al. 2016]

Beyond CDCL(T)-style SMT

Other approaches for (QF_)NRA:

- MCSAT / NLSAT:
  - Theory model construction integrated in the core solver
  - SMT-RAT, yices, z3 [Jovanović et al. 2012] [Jovanović et al. 2013] [Moura et al. 2013] [Nalbach et al. 2019] [Kremer 2020]
- CAD is a stand-alone tool:
  - Maple / RegularChains [Chen et al. 2009]
  - Mathematica [Strzeboński 2014]
  - QEPCAD B [Brown 2003]
  - Redlog / Reduce [Dolzmann et al. 1997]

These can be integrated as theory solvers [Fontaine et al. 2018] [Kremer 2018]

Some results...

Experiments on QF_NRA (11489 in total)

QF_NRA	sat	unsat	solved
cvc5	**5137**	**5596**	**10733**
Yices2	4966	5450	10416
z3	5136	5207	10343
cvc5.cov	5001	5077	10078
SMT-RAT	4828	5038	9866
veriT	4522	5034	9556
MathSAT	3645	5357	9002
cvc5.inclin	3421	5376	8797

Thank you for your attention!
Any questions?

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