PHIL 308S: Voting Theory and Fair Division

Lecture 8

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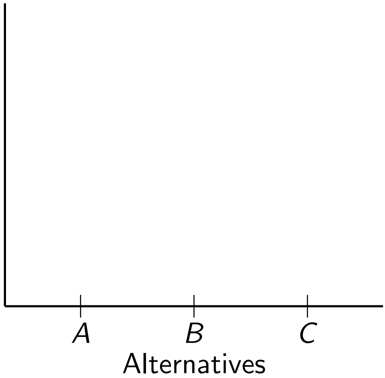
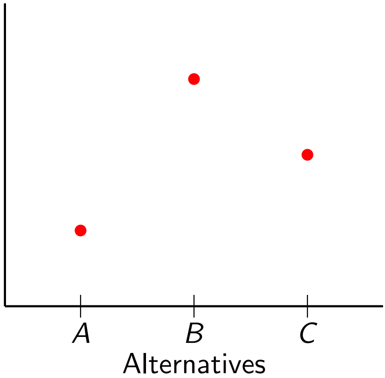
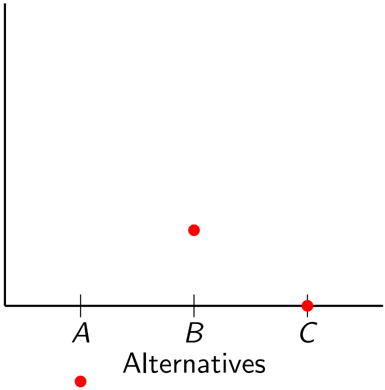
1	1	1
A	B	C
B	C	A
C	A	B



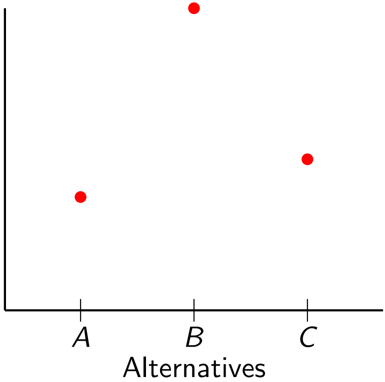
A
B
C
Alternatives



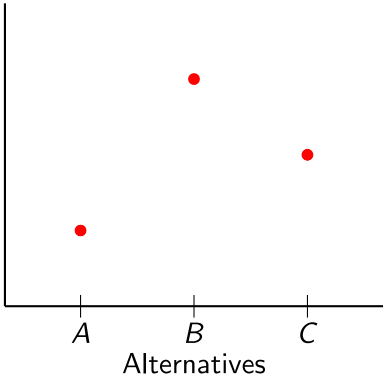
Plot $B\ P_2\ C\ P_2\ A$



Plot $B \ P_2 \ C \ P_2 \ A$



Plot $B \; P_2 \; C \; P_2 \; A$



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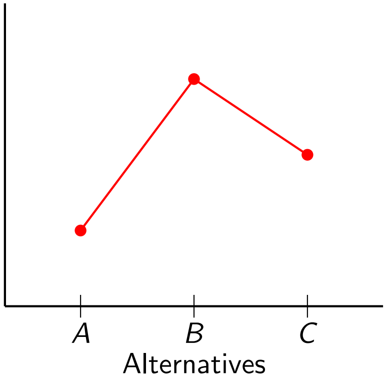
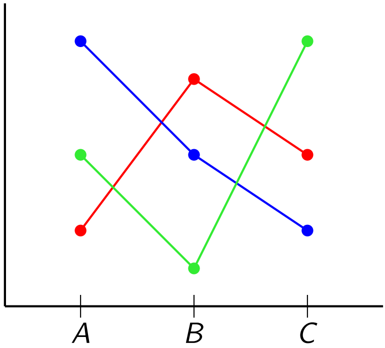


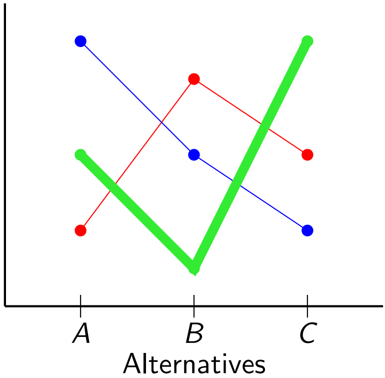
Plot $B \; P_2 \; C \; P_2 \; A$



Plot $A \; P_1 \; B \; P_1 \; C$

Plot $B \; P_2 \; C \; P_2 \; A$

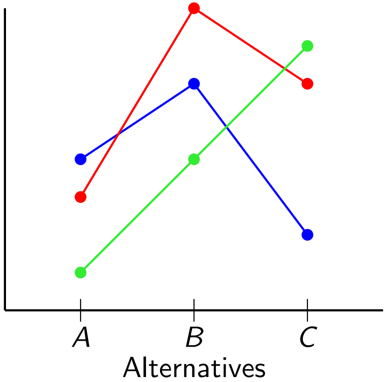
Plot $C \; P_3 \; A \; P_3 \; B$



Plot $A \; P_1 \; B \; P_1 \; C$

Plot $B \; P_2 \; C \; P_2 \; A$

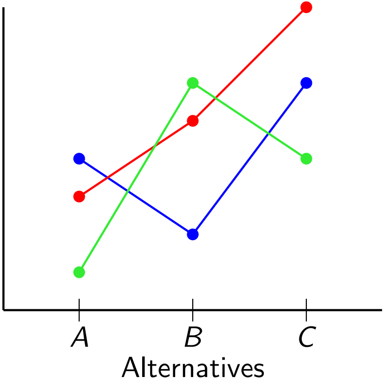
Plot C $\mathbf{P_3}$ A $\mathbf{P_3}$ B



Plot $B \ P_1 \ A \ P_1 \ C$

Plot $B \ P_2 \ C \ P_2 \ A$

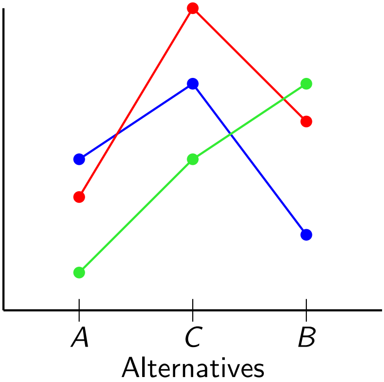
Plot $C \ P_3 \ B \ P_3 \ A$



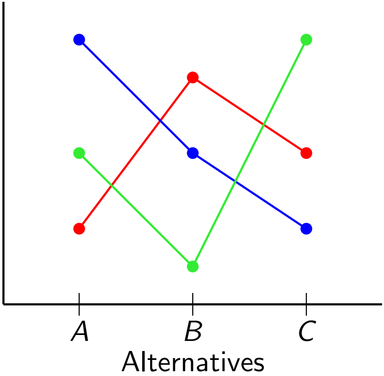
Plot $C \; P_1 \; A \; P_1 \; B$

Plot $C \; P_2 \; B \; P_2 \; A$

Plot $B \; P_3 \; C \; P_3 \; A$



Plot $C \; P_1 \; A \; P_1 \; B$
Plot $C \; P_2 \; B \; P_2 \; A$
Plot $B \; P_3 \; C \; P_3 \; A$



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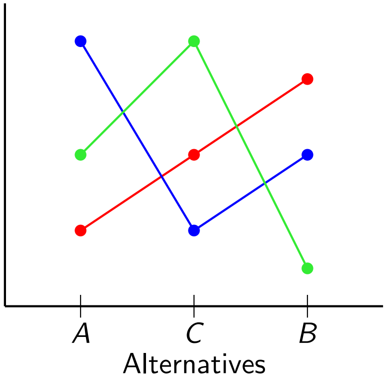

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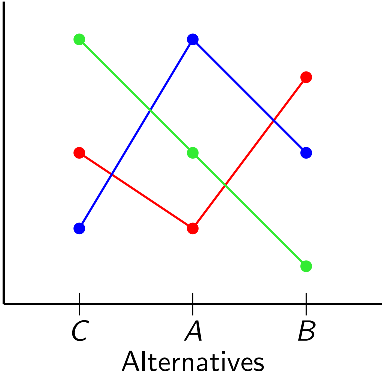

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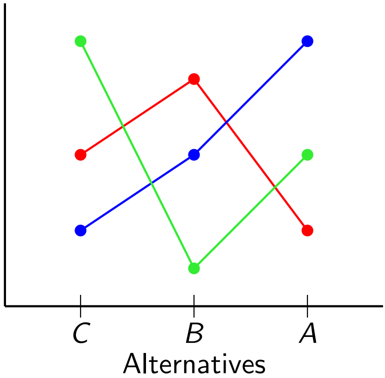

Plot $A\ P_1\ B\ P_1\ C$

Plot $B\ P_2\ C\ P_2\ A$

Plot $C\ P_3\ A\ P_3\ B$

D. Black. *On the rationale of group decision-making*. Journal of Political Economy, 56:1, pgs. 23 - 34, 1948.

Single-Peakedness: the preferences of group members are said to be single-peaked if the alternatives under consideration can be represented as points on a line and each of the utility functions representing preferences over these alternatives has a maximum at some point on the line and slopes away from this maximum on either side.

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Theorem. If there is an odd number of voters that display single-peaked preferences, then a Condorcet winner exists.

A. Sen. *A Possibility Theorem on Majority Decisions*. Econometrica 34, 1966, pgs. 491 - 499.

Assume n voters (n is odd).

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Triplewise value-restriction: For every triple of distinct candidates x_1, x_2, x_3 there exists an $x_i \in \{x_1, x_2, x_3\}$ and $r \in \{1, 2, 3\}$ such that no voter ranks x_i has her rth preference among x_1, x_2, x_3.

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Theorem (Sen, 1966). For every profile satisfying triplewise value-restriction, pairwise majority voting generates a transitive group preference ordering.

For simplicity, assume all voters have *linear preferences*

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Define a matrix $M = (m_{ij})$ where

$$m_{ij} = \begin{cases} 1 & \text{if } x_i > x_j \\ 0 & \text{otherwise} \end{cases}$$

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Example: $x_1 > x_2 > x_3$ is defined as follows:

$$M_{123} = \begin{matrix} & x_1 & x_2 & x_3 \\ x_1 & & & \\ x_2 & & & \\ x_3 & & & \end{matrix}$$

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Example: $x_1 > x_2 > x_3$ is defined as follows:

$$M_{123} = \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ x_1 & & 1 & \\ x_2 & & & \\ x_3 & & & \end{array}$$

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Example: $x_1 > x_2 > x_3$ is defined as follows:

$$M_{123} = \begin{array}{c|ccc} & x_1 & x_2 & x_3 \\ x_1 & 0 & 1 & 1 \\ x_2 & 0 & 0 & 1 \\ x_3 & 0 & 0 & 0 \end{array}$$

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Define a matrix $M = (m_{ij})$ where

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$$M_{123} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad M_{132} = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad M_{213} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

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$$M_{231} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad M_{312} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} \quad M_{321} = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

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Let a_{hij} denote the number of voters holding the preference ordering $x_h > x_i > x_j$

Pairwise majority:

$$S = a_{123}M_{123} + a_{132}M_{132} + a_{213}M_{213} + a_{231}M_{231} + a_{312}M_{312} + a_{321}M_{321}$$

3	2	5	1
x_1	x_2	x_3	x_2
x_2	x_3	x_1	x_1
x_3	x_1	x_2	x_3

$$S = a_{123}M_{123} + a_{231}M_{231} + a_{312}M_{312} + a_{213}M_{213}$$

3	2	5	1
x_1	x_2	x_3	x_2
x_2	x_3	x_1	x_1
x_3	x_1	x_2	x_3

$$S = a_{123}M_{123} + a_{231}M_{231} + a_{312}M_{312} + a_{213}M_{213}$$

$$= 3M_{123} + 2M_{231} + 5M_{312} + 1M_{213}$$

3	2	5	1
x_1	x_2	x_3	x_2
x_2	x_3	x_1	x_1
x_3	x_1	x_2	x_3

$$S = a_{123}M_{123} + a_{231}M_{231} + a_{312}M_{312} + a_{213}M_{213}$$

$$= 3\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + 5\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

3	2	5	1
x_1	x_2	x_3	x_2
x_2	x_3	x_1	x_1
x_3	x_1	x_2	x_3

$$S = a_{123}M_{123} + a_{231}M_{231} + a_{312}M_{312} + a_{213}M_{213}$$

$$= 3\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + 5\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ 2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 5 & 0 \\ 0 & 0 & 0 \\ 5 & 5 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

3	2	5	1
x_1	x_2	x_3	x_2
x_2	x_3	x_1	x_1
x_3	x_1	x_2	x_3

$$S = a_{123}M_{123} + a_{231}M_{231} + a_{312}M_{312} + a_{213}M_{213}$$

$$= 3\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + 2\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} + 5\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix} + 1\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 3 & 3 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 2 & 0 & 2 \\ 2 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 5 & 0 \\ 0 & 0 & 0 \\ 5 & 5 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 8 & 4 \\ 3 & 0 & 6 \\ 7 & 5 & 0 \end{pmatrix}$$

$$S = \begin{pmatrix} 0 & a_{123} + a_{132} + a_{312} & a_{123} + a_{132} + a_{213} \\ a_{213} + a_{231} + a_{321} & 0 & a_{123} + a_{213} + a_{231} \\ a_{231} + a_{312} + a_{321} & a_{132} + a_{312} + a_{321} & 0 \end{pmatrix}$$

$x_i > x_j$ iff $s_{ij} > s_{ji}$

$$S = \begin{pmatrix} 0 & a_{123} + a_{132} + a_{312} & a_{123} + a_{132} + a_{213} \\ a_{213} + a_{231} + a_{321} & 0 & a_{123} + a_{213} + a_{231} \\ a_{231} + a_{312} + a_{321} & a_{132} + a_{312} + a_{321} & 0 \end{pmatrix}$$

$x_i > x_j$ iff $s_{ij} > s_{ji}$

Lemma. If there is a majority cycle over x_1, x_2, x_3 then ($a_{123} > a_{321}$ and $a_{312} > a_{213}$ and $a_{231} > a_{132}$) or ($a_{321} > a_{123}$ and $a_{213} > a_{312}$ and $a_{132} > a_{231}$)

$$S = \begin{pmatrix} 0 & a_{123} + a_{132} + a_{312} & a_{123} + a_{132} + a_{213} \\ a_{213} + a_{231} + a_{321} & 0 & a_{123} + a_{213} + a_{231} \\ a_{231} + a_{312} + a_{321} & a_{132} + a_{312} + a_{321} & 0 \end{pmatrix}$$

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Lemma. If there is a majority cycle over x_1, x_2, x_3 then ($a_{123} > a_{321}$ and $a_{312} > a_{213}$ and $a_{231} > a_{132}$) or ($a_{321} > a_{123}$ and $a_{213} > a_{312}$ and $a_{132} > a_{231}$)

Lemma. If ($a_{123} = 0$ or $a_{312} = 0$ or $a_{231} = 0$) and ($a_{321} = 0$ or $a_{213} = 0$ or $a_{132} = 0$), then there is no majority cycle over x_1, x_2, x_3.

Theorem Suppose that for every triple of distinct candidates x_1, x_2, x_3, we have ($a_{123} = 0$ or $a_{312} = 0$ or $a_{231} = 0$) and ($a_{321} = 0$ or $a_{213} = 0$ or $a_{132} = 0$), then pairwise majority generates a transitive ordering.

Theorem Suppose that for every triple of distinct candidates x_1, x_2, x_3, we have ($a_{123} = 0$ or $a_{312} = 0$ or $a_{231} = 0$) and ($a_{321} = 0$ or $a_{213} = 0$ or $a_{132} = 0$), then pairwise majority generates a transitive ordering.

Theorem A profile satisfies triplewise value-restriction iff for every triple of distinct candidates x_1, x_2, x_3, we have ($a_{123} = 0$ or $a_{312} = 0$ or $a_{231} = 0$) and ($a_{321} = 0$ or $a_{213} = 0$ or $a_{132} = 0$).

x_1	x_2	x_3
is not ranked 1st		
$a_{123} = 0$ & $a_{132} = 0$	$a_{213} = 0$ & $a_{231} = 0$	$a_{312} = 0$ & $a_{321} = 0$

x_1	x_2	x_3
is not ranked 2nd		
$a_{213} = 0$ & $a_{312} = 0$	$a_{123} = 0$ & $a_{321} = 0$	$a_{132} = 0$ & $a_{231} = 0$

x_1	x_2	x_3
is not ranked 3rd		
$a_{231} = 0$ & $a_{321} = 0$	$a_{132} = 0$ & $a_{312} = 0$	$a_{123} = 0$ & $a_{213} = 0$

C. List, R. Luskin, J. Fishkin and I. McLean. *Deliberation, Single-Peakedness, and the Possibility of Meaningful Democracy: Evidence from Deliberative Polls*. Manuscript, 2012.

Taking Stock

- Voting Methods

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- Voting Paradoxes: Condorcet Paradox, Condorcet's Other Paradox, No-Show Paradox, Failures of Monotonicity (Plurality of Runoff), Failures of Independence (Borda), Multiple Districts Paradox

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- Arrow's Impossibility Result
- "Escape routes" to Arrow's Impossibility Result: Domain restrictions

What's Next?

- "Epistemic Democracy" (Condorcet Jury Theorem) and distance-based rationalizations of voting methods
- Strategizing
- Changing the basic framework: Approval Voting, Voting by "grading"
- Voting on issues, judgement aggregation, and Sen's Liberal Paradox

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Fair Division