# PHIL 308S: Voting Theory and Fair Division

## Lecture 5

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September 18, 2012

Characterizing voting methods.

# What properties do we want?

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- **Condorcet Candidate**: Always choose the candidate that beats every other candidate in head-to-head elections
- **Monotonicity** A candidate receiving more support shouldn't make her worse off
- **Independence**: The winner should not depend on "irrelevant" spoiler candidates
- **Unanimity**: If everyone agrees that candidate $X$ is preferred to candidate $Y$, then candidate $Y$ should not win.
- **Anonymity**: The names of the voters do not matter (if two voters change votes, then the outcome is unaffected)
- **Neutrality**: The names of the candidates, or options, do not matter (if two candidate are exchanged in every ranking, then the outcome changes accordingly)

- Condorcet's paradox
- Condorcet's other paradox
- Failures of monotonicity, no-show paradox
- May's theorem: a social decision method satisfies neutrality, anonymity, unanimity, positive responsiveness iff it is majority rule
- Independence of irrelevant alternatives

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- Indifference: society's ranking of $X$ and $Y$ doesn't change if all voters who rank $X$ and $Y$ equally are removed from society (assuming some voters are left) ence vs. value judgements (judge $b$ to be better than $a$ but prefer $a$ over $b$)

If a population is divided up into districts, and a candidate wins all the districts, then that candidate should win the overall election.

# Multiple Districts Paradox

| Totals | Rankings | H over W | W over H |
|---|---|---|---|
| 417 | B H W | 417 | 0 |
| 82 | B W H | 0 | 82 |
| 143 | H B W | 143 | 0 |
| 357 | H W B | 357 | 0 |
| 285 | W B H | 0 | 285 |
| 324 | W H B | 0 | 324 |
| **1608** | | **917** | **691** |

B: $417 + 82 = 499$
H: $143 + 357 = 500$
W: $285 + 324 = 609$

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| 357 | H W X | 357 | 0 |
| 285 | W X H | 0 | 285 |
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**H Wins**

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| 357 | H W B | 0 | 357 |
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G. Malinas and J. Bigelow. *Simpson's Paradox*. The Stanford Encyclopedia of Philosophy, Fall 2012 Edition, Edward N. Zalta (ed.).

# Scoring Rules: Young's Theorem

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**Reinforcement**: Suppose that $X$ and $Y$ are disjoint sets of voters. Let $W_X$ be the set of winners for $X$ and $W_Y$ the set of winners for $Y$. If there is at least one candidate that wins both elections, then the winner(s) for the entire population is $W_X \cap W_Y$.

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**Continuity**: Suppose that a group of voters $X$ elects a candidate $A$ and a disjoint group of voters $Y$ elects a different candidate $B$. Then there must be some number $m$ such that the population consisting of the subgroup $Y$ together with $m$ copies of $X$ will elect $A$.

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**Theorem (Young 1975)**.A social decision method satisfies anonymity, neutrality, reinforcement and continuity if and only if the method is a scoring rule.

# Borda Count

**Cancellation**: For a profile $R$, Suppose that $N_{a\ b} = \{i \mid aP_ib\}$. If $N_{a\ b} = N_{b\ a}$ then $aI_{F(R)}b$.

H. P. Young. *An axiomatization of Borda's rule*. Journal of Economic Theory, 9, pgs. 43 - 52, 1974.

S. Nitzan and A. Rubinstein. *A further characterization of Borda ranking method*. Public Choice, 36, pgs. 153 - 158, 1981.

# Approval Voting

**Fact** There is no fixed rule that always elects a unique Condorcet winner.

| # voters | 2 | 2 | 1 |
|---|---|---|---|
| | a | b | c |
| | d | d | a |
| | b | a | b |
| | c | c | d |

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The unique Condorcet winner is *a*.

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Vote-for-1 elects $\{a, b\}$, vote-for-2 elects $\{d\}$, vote-for-3 elects $\{a, b\}$.

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$(\{a\}, \{b\}, \{c, a\})$ elects $a$ under AV.

# Approval Voting

**Fact** Condorcet winners are always AV outcomes, but a Condorcet looser may or may not be an AV outcome.

# Fishburn's Theroem

**Theorem (Fishburn 1978)**. A social decision method is approval voting if and only if the method satisfies anonymity, neutrality, reinforcement and the following technical property:

- If there are exactly two voters who approve of disjoint sets of candidates, then the methods selects as winners all the candidates chosen by the two voters (i.e., the union of the ballots chosen by the voters).

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Assume each voter has a reflexive, transitive and complete preference over $X$ (ties are allowed).

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- Let $R_i$ is voter $i$'s (weak) preference ordering over $X$. $P_i$ is the strict component of $R_i$.
- Let $\mathcal{P}$ be the set of all reflexive, transitive and complete orderings over $X$. $\mathcal{P}^n$ is sthe set of all $n$-tuples of orderings. Denote elements of $\mathcal{P}^n$ by $\vec{R}$.

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- An **social welfare function** maps an ordering for each agent to a "social ordering" ($F : \mathcal{P}^n \to \mathcal{P}$ is a function from the voters' preferences to a preference, so $F(\vec{R})$ is an ordering over $X$.)

# Unanimity

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# Universal Domain

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$F$ is a total function (the domain of $F$ is $\mathcal{P}^n$).

# Independence of Irrelevant Alternatives

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If for each $i \in \mathcal{A}$, $xR_iy$ iff $xR'_iy$, then $xF(\vec{R})y$ iff $xF(\vec{R}')y$.

# Dictatorship

There is an individual $d \in \mathcal{A}$ such that the society strictly prefers $x$ over $y$ whenever $d$ strictly prefers $x$ over $y$.

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There is a $d \in \mathcal{A}$ such that for each profile $\vec{R}$, if $xP_iy$, then $xP_{F(\vec{R})}y$

# Arrow's Theorem

**Theorem** (Arrow, 1951) Any social welfare function that satisfies universal domain, independence of irrelevant alternatives and unanimity is a dictatorship.

# Arrow's Theorem

K. Arrow. *Social Choice & Individual Values*. 1951.

Also, see

J. Geanakoplos. *Three Brief Proofs of Arrow's Impossibility Theorem*. Economic Theory, **26**, 2005.

A. Taylor. *Social Choice and The Mathematics of Manipulation*. Cambridge University Press, 2005.

W. Gaertner. *A Primer in Social Choice Theory*. Oxford University Press, 2006.

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