PHIL 308S: Voting Theory and Fair Division

Lecture 4

Eric Pacuit

Department of Philosophy
University of Maryland, College Park
ai.stanford.edu/∼epacuit
epacuit@umd.edu

September 14, 2012

Characterizing voting methods.

What properties do we want?

What properties do we want?

- **Condorcet Candidate**: Always choose the candidate that beats every other candidate in head-to-head elections
- **Monotonicity** A candidate receiving more support shouldn't make her worse off
- **Independence**: The winner should not depend on "irrelevant" spoiler candidates
- **Unanimity**: If everyone agrees that candidate X is preferred to candidate Y, then candidate Y should not win.
- **Anonymity**: The names of the voters do not matter (if two voters change votes, then the outcome is unaffected)
- **Neutrality**: The names of the candidates, or options, do not matter (if two candidate are exchanged in every ranking, then the outcome changes accordingly)

Monotonicity A candidate receiving more support shouldn't make her worse off

Failure of monotonicity: plurality with runoff

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

# voters	6	5	4	2
	A	C	B	A
	B	A	C	B
	C	B	A	C

Failure of monotonicity: plurality with runoff

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

# voters	6	5	4	2
	A	C	B	A
	B	A	C	B
	C	B	A	C

Failure of monotonicity: plurality with runoff

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

# voters	6	5	4	2
	A	C	B	A
	B	A	C	B
	C	B	A	C

Failure of monotonicity: plurality with runoff

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

# voters	6	5	4	2
	A	C	B	A
	B	A	C	B
	C	B	A	C

Failure of monotonicity: plurality with runoff

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

Winner: *A*

# voters	6	5	4	2
	A	C	B	A
	B	A	C	B
	C	B	A	C

Failure of monotonicity: plurality with runoff

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

Winner: *A*

# voters	6	5	4	2
	A	C	B	A
	B	A	C	B
	C	B	A	C

Failure of monotonicity: plurality with runoff

# voters	6	5	4	2
	A	C	B	B
	B	A	C	A
	C	B	A	C

Winner: *A*

# voters	6	5	4	2
	A	C	B	A
	B	A	C	B
	C	B	A	C

Winner: *C*

Failure of monotonicity: plurality with runoff

# voters	6	5	4	2
	A	C	B	**B**
	B	A	C	**A**
	C	B	A	**C**

Winner: *A*

# voters	6	5	4	2
	A	C	B	**A**
	B	A	C	**B**
	C	B	A	**C**

Winner: *C*

No-show paradox

Totals	Rankings	H over W	W over H
417	B H W	417	0
82	B W H	0	82
143	H B W	143	0
357	H W B	357	0
285	W B H	0	285
324	W H B	0	324
1608		**917**	**691**

Fishburn and Brams. *Paradoxes of Preferential Voting*. Mathematics Magazine (1983).

No-show paradox

Totals	Rankings	H over W	W over H
417	B H W	417	0
82	B W H	0	82
143	H B W	143	0
357	H W B	357	0
285	W B H	0	285
324	W H B	0	324
1608		**917**	**691**

B: $417 + 82 = 499$
H: $143 + 357 = 500$
W: $285 + 324 = 609$

No-show paradox

Totals	Rankings	H over W	W over H
417	X H W	417	0
82	X W H	0	82
143	H X W	143	0
357	H W X	357	0
285	W X H	0	285
324	W H X	0	324
1608		**917**	**691**

H Wins

No-show paradox

Totals	Rankings	H over W	W over H
419	B H W	417	0
82	B W H	0	82
143	H B W	143	0
357	H W B	357	0
285	W B H	0	285
324	W H B	0	324
1610		**917**	**691**

Suppose two more people show up with the ranking B H W

No-show paradox

Totals	Rankings	H over W	W over H
419	B H W	417	0
82	B W H	0	82
143	H B W	143	0
357	H W B	357	0
285	W B H	0	285
324	W H B	0	324
1610		**917**	**691**

B: $419 + 82 = 501$
H: $143 + 357 = 500$
W: $285 + 324 = 609$

No-show paradox

Totals	Rankings	B over W	W over B
419	B X W	419	0
82	B W X	82	0
143	X B W	143	0
357	X W B	0	357
285	W B X	0	285
324	W X B	0	324
1610		**644**	**966**

B: $419 + 82 = 501$

H: $143 + 357 = 500$

W: $285 + 324 = 609$

No-show paradox

Totals	Rankings	B over W	W over B
419	B X W	419	0
82	B W X	82	0
143	X B W	143	0
357	X W B	0	357
285	W B X	0	285
324	W X B	0	324
1610		**644**	**966**

W Wins!

Characterizing Majority Rule

If there are only **two** options, then majority voting is the "best" procedure:

Characterizing Majority Rule

If there are only **two** options, then majority voting is the "best" procedure: Choosing the outcome with the most votes (allowing for ties) is the *only* group decision method satisfying the previous properties.

K. May. *A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision*. Econometrica, Vol. 20 (1952).

May's Theorem: Details

Suppose there are only two candidates A and B and n voters (let $N = \{1, \ldots, n\}$ be the set of voters).

Then the voters' preferences can be represented by elements of $\{-1, 0, 1\}$ (where 1 means A is preferred to B, -1 means B is preferred to A and 0 means indifference between A and B).

A **social decision method** is a function $F : \{-1, 0, 1\}^n \to \{-1, 0, 1\}$.

May's Theorem: Details

- **Unanimity**: unanimously supported alternatives must be the social outcome.
- **Anonymity**: all voters should be treated equally.
- **Neutrality**: all candidates should be treated equally.
- **Pos. response**: unidirectional shift in voters' opinions should not harm the alternative toward which this shift occurs

May's Theorem: Details

- **Unanimity**: unanimously supported alternatives must be the social outcome.

  If for all $i \in N$, $v_i = x$ then $F(v) = x$ (for $x \in \{-1, 0, 1\}$).

- **Anonymity**: all voters should be treated equally.

- **Neutrality**: all candidates should be treated equally.

- **Pos. response**: unidirectional shift in voters' opinions should not harm the alternative toward which this shift occurs

May's Theorem: Details

- **Unanimity**: unanimously supported alternatives must be the social outcome.

  If for all $i \in N$, $v_i = x$ then $F(v) = x$ (for $x \in \{-1, 0, 1\}$).

- **Anonymity**: all voters should be treated equally.

  $F(v_1, v_2, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$ where π is a permutation of the voters.

- **Neutrality**: all candidates should be treated equally.

- **Pos. response**: unidirectional shift in voters' opinions should not harm the alternative toward which this shift occurs

May's Theorem: Details

- **Unanimity**: unanimously supported alternatives must be the social outcome.

  If for all $i \in N$, $v_i = x$ then $F(v) = x$ (for $x \in \{-1, 0, 1\}$).

- **Anonymity**: all voters should be treated equally.

  $F(v_1, v_2, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$ where π is a permutation of the voters.

- **Neutrality**: all candidates should be treated equally.

  $F(-v) = -F(v)$ where $-v = (-v_1, \ldots, -v_n)$.

- **Pos. response**: unidirectional shift in voters' opinions should not harm the alternative toward which this shift occurs

May's Theorem: Details

- **Unanimity**: unanimously supported alternatives must be the social outcome.

  If for all $i \in N$, $v_i = x$ then $F(v) = x$ (for $x \in \{-1, 0, 1\}$).

- **Anonymity**: all voters should be treated equally.

  $F(v_1, v_2, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$ where π is a permutation of the voters.

- **Neutrality**: all candidates should be treated equally.

  $F(-v) = -F(v)$ where $-v = (-v_1, \ldots, -v_n)$.

- **Pos. response**: unidirectional shift in voters' opinions should not harm the alternative toward which this shift occurs

  If $F(v) = 0$ or $F(v) = 1$ and $v \prec v'$, then $F(v') = 1$ (where $v \prec v'$ means for all $i \in N$ $v_i \leq v'_i$ and there is some $i \in N$ with $v_i < v'_i$) then $F(v') = 1$.

May's Theorem: Details

May's Theorem (1952) A social decision method F satisfies unaniminity, neutrality, anonminity and positive responsiveness iff F is majority rule.

Proof Idea

If $(1, 1, -1)$ is assigned 0 or -1 then

Proof Idea

If $(1, 1, -1)$ is assigned 0 or -1 then

- Neutrality implies $(-1, -1, 1)$ is assigned 0 or 1

Proof Idea

If $(1, 1, -1)$ is assigned 0 or -1 then

- Neutrality implies $(-1, -1, 1)$ is assigned 0 or 1
- Anonymity implies $(1, -1, -1)$ is also assigned 0 or 1

Proof Idea

If $(1, 1, -1)$ is assigned 0 or -1 then

- Neutrality implies $(-1, -1, 1)$ is assigned 0 or 1
- Anonymity implies $(1, -1, -1)$ is also assigned 0 or 1
- Positive Responsiveness implies $(1, 0, -1)$ is assigned 1

Proof Idea

If $(1, 1, -1)$ is assigned 0 or -1 then

- Neutrality implies $(-1, -1, 1)$ is assigned 0 or 1
- Anonymity implies $(1, -1, -1)$ is also assigned 0 or 1
- Positive Responsiveness implies $(1, 0, -1)$ is assigned 1
- Positive Responsiveness implies $(1, 1, -1)$ is assigned 1, Contradiction.

Other characterizations

Weak path independence: If $|F(R_1) - F(R_2)| \neq 2$ then $F(R_1 \oplus R_2) = F(F(R_1) \oplus F(R_2))$

G. Asan and R. Sanver. *Another Characterization of the Majority Rule*. Economics Letters, 75 (3), 409-413, 2002.

E. Maskin. *Majority rule, social welfare functions and game forms*. in *Choice, Welfare and Development*, The Clarendon Press, pgs. 100 - 109, 1995.

G. Woeginger. *A new characterization of the majority rule*. Economic Letters, 81, pgs. 89 - 94, 2003.

Independence: The winner should not depend on "irrelevant" spoiler candidates

Failure of Independence

# voters	3	2	2
	A	B	C
	B	C	A
	C	A	B

Failure of Independence

# voters	3	2	2
	A	B	C
	B	C	A
	C	A	B

- The BC ranking is: A (8) $>$ B (7) $>$ C (6)

Failure of Independence

# voters	3	2	2
	A	B	C
	B	C	X
	C	X	A
	X	A	B

- The BC ranking is: A (8) $>$ B (7) $>$ C (6)
- Add a new (undesirable) candidate X

Failure of Independence

# voters	3	2	2
	A	B	C
	B	C	X
	C	X	A
	X	A	B

- The BC ranking is: A (8) $>$ B (7) $>$ C (6)
- Add a new (undesirable) candidate X
- The new BC ranking is: C (13) $>$ B (12) $>$ A (11) $>$ X (6)