PHIL 308S: Voting Theory and Fair Division

Lecture 3

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Characterizing voting methods.

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- **Anonymity**: The names of the voters do not matter (if two voters change votes, then the outcome is unaffected)
- **Neutrality**: The names of the candidates, or options, do not matter (if two candidate are exchanged in every ranking, then the outcome changes accordingly)

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"When a set of axioms regarding social choice can all be simultaneously satisfied, there may be several possible procedures that work, among which we have to choose. In order to choose between different possibilities through the use of discriminating axioms, we have to introduce *further* axioms, until only and only one possible procedure remains. This is something of an exercise in brinkmanship. We have to go on and on cutting alternative possibilities, moving—implicitly—*towards* an impossibility, but then stop just before all possibilities are eliminated, to wit, when one and only one options remains." (pg. 354)

A. Sen. *The Possibility of Social Choice*. The American Economic Review, 89:3, pgs. 349 - 378, 1999 (reprint of his Nobel lecture).

Condorcet Paradox

Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
C	B	A

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- Does the group prefer A over B?

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Voter 1	Voter 2	Voter 3
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C	B	A

- Does the group prefer A over B? Yes
- Does the group prefer B over C? Yes

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Voter 1	Voter 2	Voter 3
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B	A	C
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- Does the group prefer A over B? Yes
- Does the group prefer B over C? Yes
- Does the group prefer A over C? No

Condorcet Paradox

Voter 1	Voter 2	Voter 3
A	C	B
B	A	C
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- Does the group prefer A over B? Yes
- Does the group prefer B over C? Yes
- Does the group prefer A over C? No
  (this conflicts with transitivity)

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(1)	(2)	(3)	(4)	(5)	(6)
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B	C	A	C	A	B
C	B	C	A	B	A

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A	A	B	B	C	C
B	C	A	C	A	B
C	B	C	A	B	A

(7)	(8)	(9)	(10)	(11)	(12)	(13)
$A\ B$	$A\ C$	$B\ C$	A	B	C	
						$A\ B\ C$
C	B	A	$A\ B$	$A\ C$	$B\ C$	

The probability of a Condorcet paradox, I

Consider a group of size n voting on m alternatives

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$A\ P_G\ B\ P_G\ C\ P_G\ A$ and $C\ P_G\ B\ P_G\ A\ P_G\ C$

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G_1	G_2	G_3
A	B	C
B	C	A
C	A	B

G_1	G_2	G_3
C	B	A
B	A	C
A	C	B

The probability of a Condorcet paradox, II

The probability of intransitive majority preferences of a group of size n voting on m alternatives.

$$Pr(m, n) = [\# \text{ of "problem" preference configurations}]/(m!)^n$$

The probability of a Condorcet paradox, III

Display 5-1

Values of $p(n, m)$: Proportion of Possible Profiles Without a Condorcet Winner

	n = Number of Voters						
m = Number of Alternatives	3	5	7	9	11	...	Limit
3	.056	.069	.075	.078	.080		.088
4	.111	.139	.150	.156	.160		.176
5	.160	.200	.215				.251
6	.202						.315
Limit	1.000	1.000	1.000	1.000	1.000		1.000

W. Riker. *Liberalism Against Populism*. Freeman, 1982.

Assumptions

- each ordering is equally likely
- the voter's choices are independent

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More information

W. Gehrlein. *Condorcet's Paradox*. Springer, 2006.

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
2	A	A	B	B	C	C
1	B	C	A	C	A	B
0	C	B	C	A	B	A

$$BS(A) = 2 \times 31 + 1 \times 39 + 0 \times 11 = 101$$
$$BS(B) = 2 \times 39 + 1 \times 31 + 0 \times 11 = 109$$
$$BS(C) = 2 \times 11 + 1 \times 11 + 0 \times 59 = 33$$

$B >_{BC} A >_{BC} C$

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

$B >_{BC} A >_{BC} C$ $A >_{M} B >_{M} C$

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	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

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	B	C	A	C	A	B
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$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
s_2	A	A	B	B	C	C
s_1	B	C	A	C	A	B
s_0	C	B	C	A	B	A

Condorcet's Other Paradox: No *scoring rule* will work...

$B >_{BC} A >_{BC} C$ $\quad\quad$ $A >_M B >_M C$

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s_0	C	B	C	A	B	A

Condorcet's Other Paradox: No *scoring rule* will work...
$Score(A) = s_2 \times 31 + s_1 \times 39 + s_0 \times 11$
$Score(B) = s_2 \times 39 + s_1 \times 31 + s_0 \times 11$

$B >_{BC} A >_{BC} C$ $A >_M B >_M C$

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s_0	C	B	C	A	B	A

Condorcet's Other Paradox: No *scoring rule* will work...

$Score(A) = s_2 \times 31 + s_1 \times 39 + s_0 \times 11$
$Score(B) = s_2 \times 39 + s_1 \times 31 + s_0 \times 11$
$Score(A) > Score(B) \Rightarrow 31s_2 + 39s_1 > 39s_2 + 31s_1 \Rightarrow s_1 > s_2$

$B >_{BC} A >_{BC} C$ $\qquad$ $A >_M B >_M C$

Condorcet's Other Paradox

# voters	30	1	29	10	10	1
s_2	A	A	B	B	C	C
s_1	B	C	A	C	A	B
s_0	C	B	C	A	B	A

Theorem (Fishburn 1974). For all $m \geq 3$, there is some voting situation with a Condorcet winner such that every weighted scoring rule will have at least $m - 2$ candidates with a greater score than the Condorcet winner.

Electing the Condorcet Winner, I

Condorcet Rule: Each voter submits a linear ordering over all the candidates. If there is a Condorcet winner, then that candidate wins the election. Otherwise, all candidates tie for the win.

Copeland's Rule: Each voter submits a linear ordering over all the candidates. A win-loss record for candidate B is calculated as follows:

$$WL(B) = |\{C \mid B >_M C\}| - |\{C \mid C >_M B\}|$$

The Copeland winner is the candidate that maximizes WL.

Electing the Condorcet Winner, II

Dodgson's Method: Each voter submits a linear ordering over all the candidates. For each candidate, determine the fewest number of pairwise swaps needed to make that candidate the Condorcet winner. The candidate(s) with the fewest swaps is(are) declared the winner(s).

Black's Procedure: Each voter submits a linear ordering over all the candidates. If there is a Condorcet winner, then that candidate is the winner. Otherwise, let the winners be the Borda Count winners.

Should a Condorcet winner *always* win?

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	B	C	A	C	A	B
	C	B	C	A	B	A

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# voters	30	1	29	10	10	1
	A	A	B	B	C	C
	B	C	A	C	A	B
	C	B	C	A	B	A

10	10	10
A	B	C
B	C	A
C	A	B

1	1	1
A	C	B
C	B	A
B	A	C

20	28
A	B
B	A
C	C