

PHIL 308S: Voting Theory and Fair Division

Lecture 23

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The Cake-Cutting Problem

A division of a cake $[0, 1]$ for n players is a partition (S_1, \dots, S_n) (i.e., each $S_i \subseteq [0, 1]$, $\cup_i S_i = [0, 1]$ and $S_i \cap S_j \neq \emptyset$). We are typically interested in divisions where each S_i is contiguous (i.e., a subinterval of $[0, 1]$).

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- ▶ **Equitable**: for each i, j , $v_i(S_i) = v_j(S_j)$
- ▶ **Efficient**: there is no other division (T_1, \dots, T_n) such that $v_i(T_i) \geq v_i(S_i)$ for all i and there is some j such that $v_j(T_j) > v_j(S_j)$.

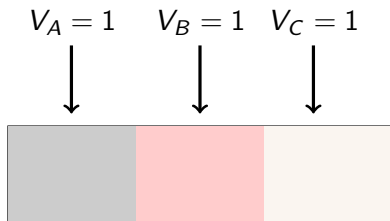
Selfridge-Conway Procedure

1. Agent 1 cuts the cake in three pieces (she considers equal).
2. Agent 2 either “passes” (if she thinks at least two pieces are tied for largest) or trims one piece (to get two tied for largest pieces). If she passed, then let agents 3, 2, 1 pick (in that order).
3. If agent 2 did trim, then let 3, 2, 1 pick (in that order), but require 2 to take the trimmed piece (unless 3 did). Keep the trimmings unallocated for now.
4. Now divide the trimmings. Whoever of 2 and 3 received the untrimmed piece does the cutting. Let agents choose in this order: non-cutter, agent 1, cutter.

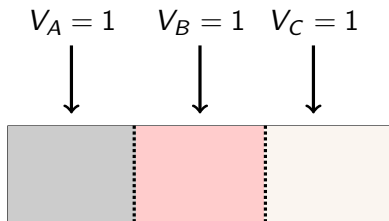
Selfridge-Conway is not Pareto Optimal

| | chocolate | strawberry | vanilla |
|---|-----------|------------|---------|
| A | 1 | 0 | 0 |
| B | 0 | 1 | 0 |
| C | 0 | 0 | 1 |

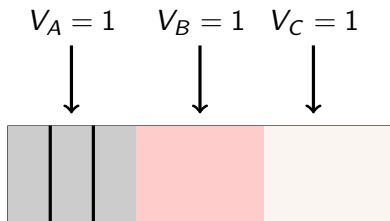
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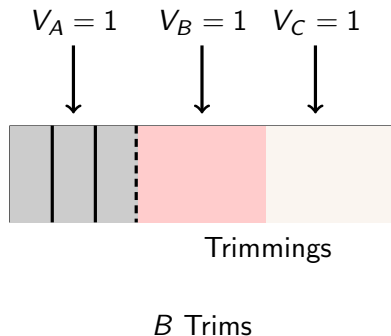


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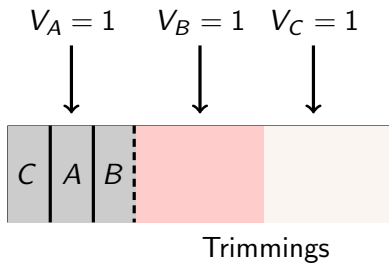


A Cuts

Selfridge-Conway is not Pareto Optimal

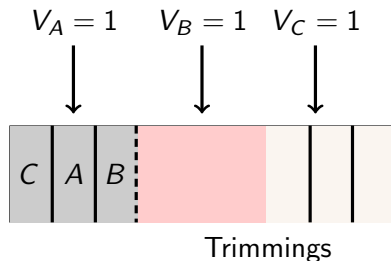


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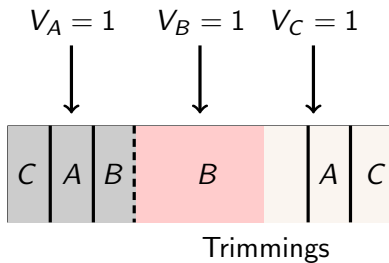
Choose in the order C, B, A

Selfridge-Conway is not Pareto Optimal



C divides the trimmings

Selfridge-Conway is not Pareto Optimal



Choose in the order B, A, C

Selfridge-Conway is not Equitable

| | chocolate | strawberry | vanilla |
|---|-----------|------------|---------|
| A | $1/3$ | $1/3$ | $1/3$ |
| B | 0 | $1/2$ | $1/2$ |
| C | 0 | $1/2$ | $1/2$ |

Selfridge-Conway is not Equitable



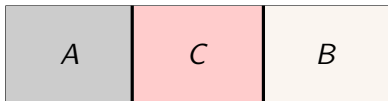
A divides the cake

Selfridge-Conway is not Equitable



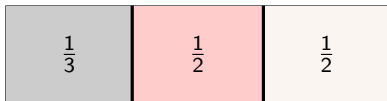
B does not trim

Selfridge-Conway is not Equitable



Choose in the order *C*, *B*, *A*

Selfridge-Conway is not Equitable



The division is not equitable

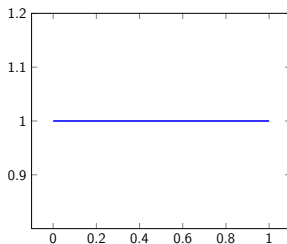
We say that players' measures are **absolutely continuous** with respect to one another if, whenever one player assigns value 0 to a particular piece of a cake or to a sector of a pie, all players do so.

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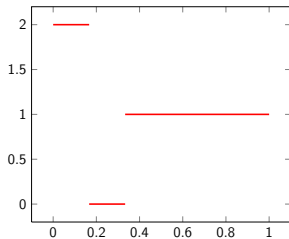
Theorem.

1. Any envy-free allocation of a cake among two or more players whose measures are absolutely continuous with respect to one another is also efficient.
2. For two or more players and any cake and corresponding measures, if the measures are absolutely continuous with respect to one another, then there exists an allocation that is envy-free and undominated.

| | $[0, 1/6)$ | $[1/6, 1/3)$ | $[1/3, 1)$ |
|----------|---------------|---------------|---------------|
| Player A | $\frac{1}{3}$ | 0 | $\frac{2}{3}$ |
| Player B | $\frac{1}{6}$ | $\frac{1}{6}$ | $\frac{2}{3}$ |
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Suppose that A receives the leftmost piece $[0, x)$ for some x .

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- ▶ If $x > 1/3$, then there is not enough left for B and C to receive $1/3$.
- ▶ If $x < 1/3$, then B and C must divide the right piece in half to ensure envy-freeness. If, say, C receives the rightmost piece, then $v_A(S_A) \leq 1/3$, but $v_A(S_C) = (1 - x)/2 > 1/3$, violating envy-freeness.

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Suppose that A receives the leftmost piece $[0, x)$ for some x .

- ▶ If $x = 1/3$, then to ensure envy freeness the rightmost piece must be cut at the $2/3$ point. This division is envy free (note that $v_A([0, 1/3)) = \frac{1}{3} = v_A([2/3, 1))$).

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However, this division is Pareto dominated by the division $S_A = [0, 1/6), S_B = [1/6, 7/12), S_C = [7/12, 1)$.

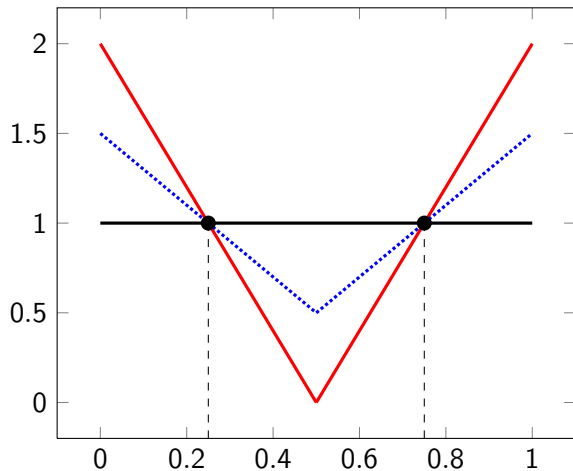
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Suppose that B (or C) receives the leftmost piece $[0, x)$ for some x .

- ▶ Then, in order to ensure envy freeness, the division must be $[0, 1/3), [1/3, 2/3), [2/3, 1)$.

However, this division is Pareto dominated by the division $S_A = [0, 1/6), S_B = [1/6, 7/12), S_C = [7/12, 1)$.

Envy Freeness vs. Equitability



Open Questions

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- ▶ 4-person, 3-cut envy free procedure? (Unknown)
 - (Barbanel and Brams, 2004): no more than 5 cuts are needed to ensure 4-person envy-freeness.
- ▶ Beyond 4 players, *no procedure is known* that yields an envy-free division of a cake unless an *unbounded* number of cuts is allowed (Brams and Taylor, 1995)

How about some pie?

A cake is a line segment and becomes a pie when its endpoints are connected to form a circle.

The cuts divide the pie into sectors each one of which is given to a player

Gale (1993): Is there an allocation of the pie that is envy-free and undominated?

Barabanel and Brams: for 2 players yes, for 3 players envy-free but not necessarily undominated, for 4 players no.

J. Barbanel and S. Brams. *Cutting a Pie Is Not a Piece of Cake*. 2005.

References

F. Su. *Review of Cake-Cutting Algorithms: Be Fair If You Can.* American Mathematical Monthly (2000).

S. Brams, M. Jones and C. Klamler. *Better Ways to Cut a Cake.* Notices of the AMS (2006).

S. Brams and A. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution.* 1996.

S. Brams, P. Edelman and P. Fishburn. *Paradoxes of Fair Division.* *Journal of Philosophy*, **98:6** (2001).

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5. Exam: multiple choice questions, short answer questions (for voting method X , which of Arrow's axioms does X (not) satisfy), essay questions (see handout).