PHIL 308S: Voting Theory and Fair Division

Lecture 22

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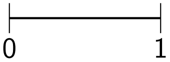
The Cake-Cutting Problem

The cake is the unit interval $[0, 1]$



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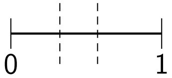

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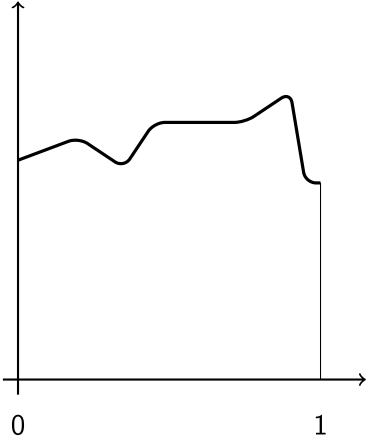

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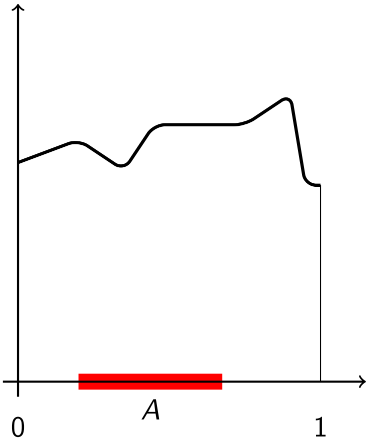

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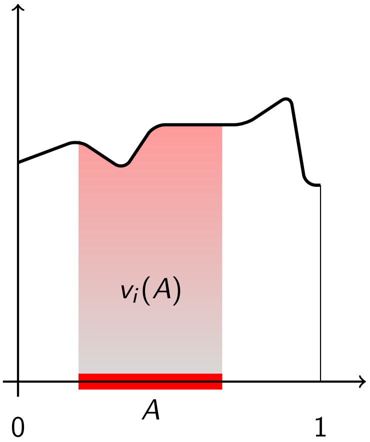

A division of a cake $[0, 1]$ for n players is a partition $(S_1, \ldots, S_n)$ (i.e., each $S_i \subseteq [0, 1]$, $\cup_i S_i = [0, 1]$ and $S_i \cap S_j \neq \emptyset$). We are typically interested in divisions where each S_i is contiguous (i.e., a subinterval of $[0, 1]$).

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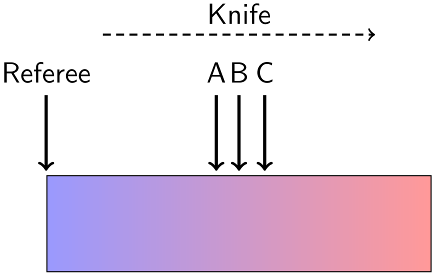
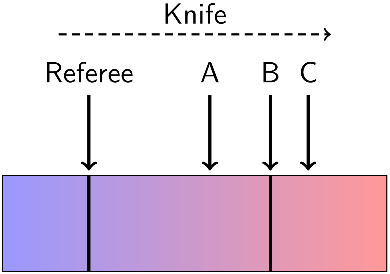
- **Equitable**: for each i, j, $v_i(S_i) = v_j(S_j)$
- **Efficient**: there is no other division $(T_1, \ldots, T_n)$ such that $v_i(T_i) \geq v_i(S_i)$ for all i and there is some j such that $v_j(T_j) > v_j(S_j)$.

Envy-free procedures for 3 players.

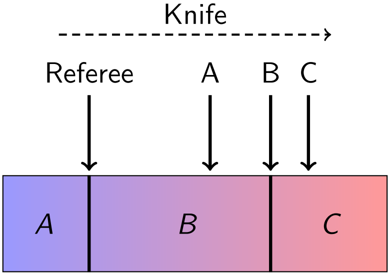
Stromquist Procedure

- A referee slowly moves a knife across the cake, from left to right (supposed to cut somewhere around the 1/3 mark)
- At the same time, each agent is moving her own knife (in the same direction as the referee) so that it would cut the righthand piece in half (with respect to her own valuation)
- The first agent to call "stop" receives the piece to the left of the referee's knife. The righthand part is cut by the middle one of the three agent knifes. If neither of the other two agents hold the middle knife, they each obtain the piece at which their knife is pointing. If one of them does hold the middle knife, then the other one gets the piece at which her knife is pointing.

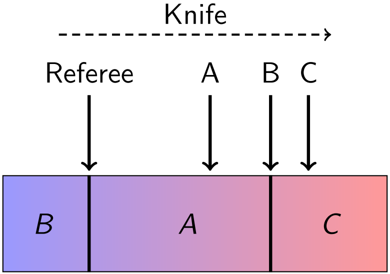

Knife
Referee
A B C



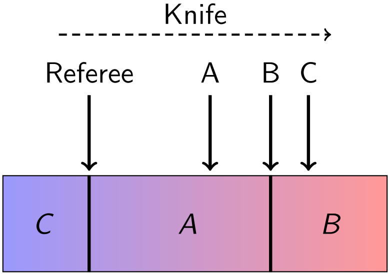
Strategy: Yell "stop" when the left piece as at least as big as each of the right pieces.



A yells "stop".



B yells “stop”.



B yells "stop".

Selfridge-Conway Procedure

1. Agent 1 cuts the cake in three pieces (she considers equal).
2. Agent 2 either "passes" (if she thinks at least two pieces are tied for largest) or trims one piece (to get two tied for largest pieces). If she passed, then let agents 3, 2, 1 pick (in that order).
3. If agent 2 did trim, then let 3, 2, 1 pick (in that order), but require 2 to take the trimmed piece (unless 3 did). Keep the trimmings unallocated for now.
4. Now divide the trimmings. Whoever of 2 and 3 received the untrimmed piece does the cutting. Let agents choose in this order: non-cutter, agent 1, cutter.

Selfridge-Conway is not Pareto Optimal

	chocolate	strawberry	vanilla
A	1	0	0
B	0	1	0
C	0	0	1

Selfridge-Conway is not Equitable

	chocolate	strawberry	vanilla
A	1/3	1/3	1/3
B	0	1/2	1/2
C	0	1/2	1/2

We say that players' measures are **absolutely continuous** with respect to one another if, whenever one player assigns value 0 to a particular piece of a cake or to a sector of a pie, all players do so.

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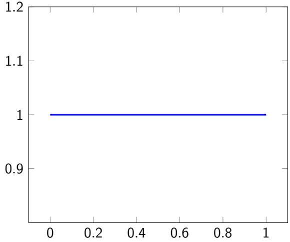
Theorem.

1. Any envy-free allocation of a cake among two or more players whose measures are absolutely continuous with respect to one another is also efficient.
2. For two or more players and any cake and corresponding measures, if the measures are absolutely continuous with respect to one another, then there exists an allocation that is envy-free and undominated.

	$[0, 1/6)$	$[1/6, 1/3)$	$[1/3, 1)$
Player A	$\frac{1}{3}$	0	$\frac{2}{3}$
Player B	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$
Player C	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{2}{3}$



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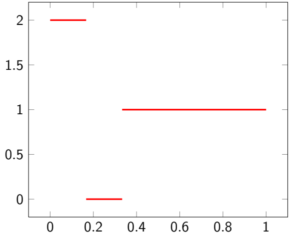

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Suppose that A receives the leftmost piece $[0, x)$ for some x.

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- If $x > 1/3$, then there is not enough left for B and C to receive $1/3$.
- If $x < 1/3$, then B and C must divide the right piece in half to ensure envy-freeness. If, say, C receives the rightmost piece, then $v_A(S_A) \leq 1/3$, but $v_A(S_C) = (1-x)/2 > 1/3$, violating envy-freeness.

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- If $x = 1/3$, then to ensure envy freeness the rightmost piece must be cut at the $2/3$ point. This division is envy free (note that $v_A([0, 1/3)) = \frac{1}{3} = v_A([2/3, 1))$.

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  However, this division is Pareto dominated by the division $S_A=[0,1/6), S_B=[1/6,7/12), S_C=[7/12,1)$.

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Suppose that B (or C) receives the leftmost piece $[0, x)$ for some x.

- Then, in order to ensure envy freeness, the division must be $[0, 1/3), [1/3, 2/3), [2/3, 1)$.

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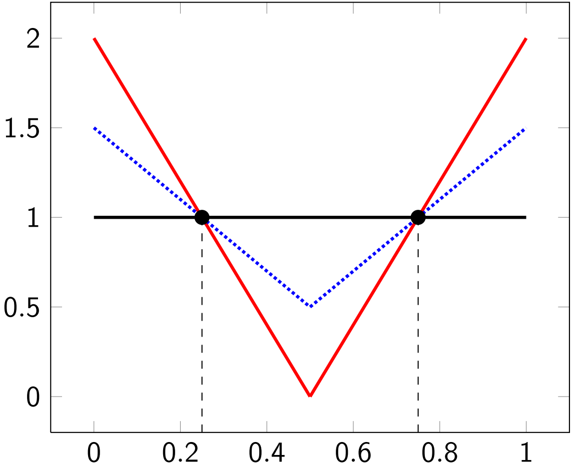
Envy Freeness vs. Equitability



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- 4-person, 3-cut envy free procedure? (Unknown)
  - (Barbanel and Brams, 2004): no more than 5 cuts are needed to ensure 4-person envy-freeness.
- Beyond 4 players, *no procedure is known* that yields an envy-free division of a cake unless an *unbounded* number of cuts is allowed (Brams and Taylor, 1995)

How about some pie?

A cake is a line segment and becomes a pie when its endpoints are connected to form a circle.

The cuts divide the pie into sectors each one of which is given to a player

Gale (1993): Is there an allocation of the pie that is envy-free and undominated?

Barabanel and Brams: for 2 players yes, for 3 players envy-free but not necessarily undominated, for 4 players no.

J. Barbanel and S. Brams. *Cutting a Pie Is Not a Piece of Cake*. 2005.

References

F. Su. *Review of* Cake-Cutting Algorithms: Be Fair If You Can. American Mathematical Monthly (2000).

S. Brams, M. Jones and C. Klamler. *Better Ways to Cut a Cake*. Notices of the AMS (2006).

S. Brams and A. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution*. 1996.

S. Brams, P. Edelman and P. Fishburn. *Paradoxes of Fair Division*. *Journal of Philosophy*, **98:6** (2001).