PHIL 308S: Voting Theory and Fair Division

Lecture 22

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Main References

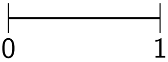
S. Brams and A. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution*. 1996.

J. Robertson and W. Webb. *Cake-Cutting Algorithms: Be Fair If You Can*. 1998.

J. Barbanel. *The Geometry of Efficient Fair Division*. 2005.

The Cake-Cutting Problem

The cake is the unit interval $[0, 1]$



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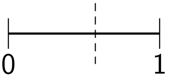

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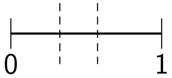

The Cake-Cutting Problem

Each player i has a continuous value measure $v_i(x)$ on $[0, 1]$ such that

- $v_A(x) \geq 0$ and $v_B(x) \geq 0$ for $x \in [0, 1]$
- v_A and v_B are finitely additive, non-atomic, absolutely continuous measures
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value of finite number of disjoint pieces equals the value of their union (hence, no subpieces have greater value than the larger piece containing them).

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a single cut (which defines the border of a piece) has no area and so has no value.

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no portion of cake is of positive measure for one player and zero measure for another player.

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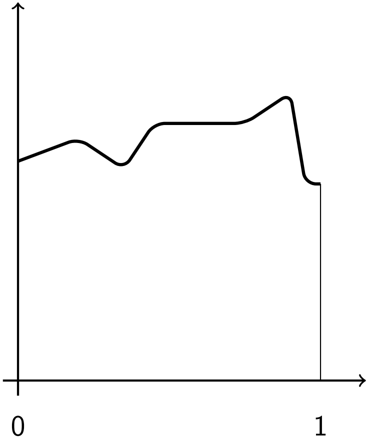

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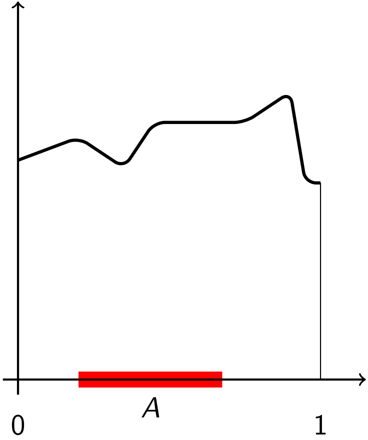

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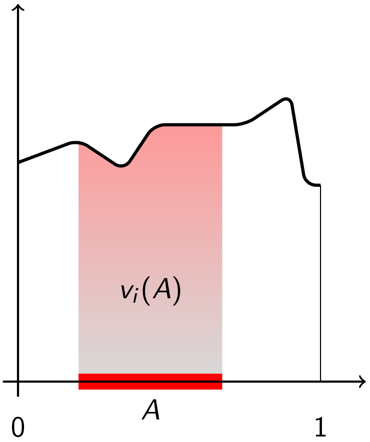

Cutting a Cake: Divide and Choose

Procedure: one player cuts the cake into two portions and the other player chooses one.

Suppose A is the cutter.
If A has no information about the other player's preferences, then A should cut the cake at some point x so that the value of the portion to the left of x is equal to the value of the portion to the right.

This strategy creates an **envy-free** and **efficient** allocation, but it is not necessarily **equitable**.

Cutting a Cake: Divide and Choose

Suppose A values the vanilla half twice as much as the chocolate half. Hence,

$$v_A(x) = \begin{cases} 4/3 & x \in [0, 1/2] \\ 2/3 & x \in (1/2, 1] \end{cases}$$

$$v_B(x) = \begin{cases} 1/2 & x \in [0, 1/2] \\ 1/2 & x \in (1/2, 1] \end{cases}$$

A should cut the cake at $x = 3/8$:

$$(4/3)(x - 0) = 4/3(1/2 - x) + 2/3(1 - 1/2)$$

Note that the portions are not equitable (B receive $5/8$ according to his valuation)

Surplus Procedure

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3. If a and b coincide, the cake is cut at $a = b$. One player is randomly assigned the piece to the left and the other to the right. The procedure ends.

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3. If a and b coincide, the cake is cut at $a = b$. One player is randomly assigned the piece to the left and the other to the right. The procedure ends.
4. Suppose a is to the left of b (Then A receives $[0, a]$ and B receives $[b, 1]$). Cut the cake a point c in $[a, b]$ at which the players receive the *same proportion* p of the cake in this interval.

Surplus Procedure

A procedure is **strategy-proof** if maximin players always have an incentive to let $f_A = v_A$ and $f_B = v_B$.

Let c be the cut-point that guarantees proportional equitability and e the cut-point that guarantees equitability of the surplus.

Theorem The Surplus Procedure is strategy-proof, whereas any procedure that makes e the cut-point is strategy-vulnerable.

3 Players, 2 Cuts

Fact It is not always possible to divide a cake among three players into **envy-free and equitable** portions using 2 cuts.

More than 2 Players

A division is **super-envy free** if every player feels all other players received strictly less that $1/n$ of the total value of the cake.

Theorem (Barbenel) A super envy-free division *exists* if and only if the player measures are linearly independent. (in fact, there are infinitely many such divisions)

J. Barbanel. *Super envy-free cake division and independence of measures*. J. Math. Anal. Appl. (1996).

Banach-Knaster Last Diminisher Procedure

Suppose there are n different agents: $p_1, \ldots, p_n$.

Procedure:

- The first person (p_1) cuts out a piece which he claims is his fair share.
- Then, the piece goes around being inspected, in turn, by persons $p_2, p_3, \ldots, p_n$.
  - Anyone who thinks the piece is not too large just passes it. Anyone who thinks it is too big, may reduce it, putting some back into the main part.

Banach-Knaster Last Diminisher Procedure

- After the piece has been inspected by p_n, the last person who reduced the piece, takes it. If there is no such person, i.e., no one challenged p_1 , then the piece is taken by p_1.

- The algorithm continues with $n - 1$ participants.

This procedure is equitable but not envy-free

Dubins-Spanier Moving-Knife Procedure

Procedure: A referee holds a knife at the left edge of the cake and slowly moves it across the cake so that it remains parallel to its starting position.

At any time, any one of the three players (A, B or C) can call "cut".

When this occurs, the player who called cut receives the piece to the left of the knife and exits the game.

Dubins-Spanier Moving-Knife Procedure

The game now continues moving until a second player calls cut.

The second player receives the second piece and the third player gets the remainder.

If either two or three players call cut at the same time, the cut piece is given to one of the callers at random.

This procedure is equitable but not envy-free

Steinhaus Procedure

1. Agent 1 cuts the cake into three pieces (which she values equally).

2. Agent 2 "Passes" (if she thinks at least two of the pieces are $\geq 1/3$) or labels two of them as "bad" — If agent 2 passed, then agents 3, 2, 1 each choose a piece (in that order) and we are done.

3. If agent 2 did not pass, then agent 3 can also choose between passing and labeling – If agent 3 passed, then agents 2, 3, 1 each choose a piece (in that order) and we are done.

4. If neither agent 2 or agent 3 passed, then agent 1 has to take (one of) the piece(s) labelled as "bad" by both 2 and 3. The rest is reassembled and 2 and 3 play cut-and-choose.

Steinhaus Procedure

- Guarantees a proportional division of the cake
- Not envy-free
- Requires at most 3 cuts

Envy-free procedures for 3 players.

Stromquist Procedure

- A referee slowly moves a knife across the cake, from left to right (supposed to cut somewhere around the 1/3 mark)
- At the same time, each agent is moving her own knife so that it would cut the righthand piece in half (with respect to her own valuation)
- The first agent to call "stop" receives the piece to the left of the referee's knife. The righthand part is cut by the middle one of the three agent knifes. If neither of the other two agents hold the middle knife, they each obtain the piece at which their knife is pointing. If one of them does hold the middle knife, then the other one gets the piece at which her knife is pointing.