PHIL 308S: Voting Theory and Fair Division

Lecture 19

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November 21, 2012

Fair Division

S. Brams, P. Edelman and P. Fishburn. *Paradoxes of Fair Division*. Journal of Philosophy, **98:6**, pgs. 300-314.

J. Robertson and W. Webb. *Cake-Cutting Algorithms: Be Fair if You Can*. A.K. Peters, 1998.

S. Brams and A. Taylor. *Fair Division: From cake-cutting to dispute resolution*. Cambridge University Press, 1998.

S. Brams and A. Taylor. *The Win-Win Solution*. W. W. Norton & Company, 2000.

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- **Equitable:** each player values its allocation the same *according to its own valuation function*.
- **Efficiency:** there is no other division better for everybody, or better for some players and not worse for the others

Unique Efficient, EF Division May Loose in Voting

$$
\begin{array}{lccccccccccc}
\text{Ann:} & 1 & \succ & 2 & \succ & 3 & \succ & 4 & \succ & 5 & \succ & 6 \\
\text{Bob:} & 5 & \succ & 6 & \succ & 2 & \succ & 1 & \succ & 4 & \succ & 3 \\
\text{Cath:} & 3 & \succ & 6 & \succ & 5 & \succ & 4 & \succ & 1 & \succ & 2
\end{array}
$$

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- Both Ann and Bob prefer $(12, 56, 34)$ to $(14, 25, 36)$
- The division $(12, 45, 36)$ would tie $(14, 25, 36)$ in a vote. (Cath is indifferent, Ann prefers the former and Bob the latter)

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- The only envy-free and efficient division is $(14, 25, 36)$
- Both Ann and Bob prefer $(12, 56, 34)$ to $(14, 25, 36)$
- The division $(12, 45, 36)$ would tie $(14, 25, 36)$ in a vote. (Cath is indifferent, Ann prefers the former and Bob the latter)
- The unique envy-free division would loose in a vote to any of the other efficient divisions

Adjusted Winner

Adjusted winner (AW) is an algorithm for dividing n divisible goods among two people (invented by Steven Brams and Alan Taylor).

For more information see

- *Fair Division: From cake-cutting to dispute resolution* by Brams and Taylor, 1998
- *The Win-Win Solution* by Brams and Taylor, 2000
- `www.nyu.edu/projects/adjustedwinner`

Item	Ann	Bob
A		
B		
C		

Suppose Ann and Bob are dividing three goods (A, B, C)

Item	Ann	Bob
A	5	4
B	65	46
C	30	50
Total	100	100

Suppose Ann and Bob are dividing three goods (A, B, C)

Point Assignment: Both Ann and Bob distribute 100 points among the three items

Item	Ann	Bob
A	5	4
B	65	46
C	30	50
Total	100	100

Item	Ann	Bob
A	5	0
B	65	0
C	0	50
Total	70	50

Suppose Ann and Bob are dividing three goods (A, B, C)

Point Assignment: Both Ann and Bob distribute 100 points among the three items

Winner Take All: The person who assigned the most points is given that good

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Equitability Adjustment: Transfer all or part of the goods from the person with the most points until both receive the same number of points

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Find the item whose ratio is closes to 1: $65/46 \geq 5/4 \geq 1 \geq 30/50$

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Still not equal, so give (some of) B to Bob: $65p = 100 - 46p$ yielding $p = \frac{100}{111} = 0.901$

Item	Ann	Bob
A	5	4
B	65	46
C	30	50
Total	100	100

Item	Ann	Bob
A	0	4
B	58.56	4.56
C	0	50
Total	58.56	58.56

Suppose Ann and Bob are dividing three goods (A, B, C)

Point Assignment: Both Ann and Bob distribute 100 points among the three items

Winner Take All: The person who assigned the most points is given that good

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Easy Observations

- For two-party disputes, proportionality and envy-freeness are equivalent.
- *AW* only produces equitable allocations (equitability is essentially built in to the procedure).
- *AW* produces allocations σ that in which at most one good is split.

Adjusted Winner is Fair

Theorem (Brams and Taylor) *AW produces allocations that are efficient, equitable and envy-free (with respect to the announced valuations)*

Strategizing

Can the agents improve their allocation by misrepresenting their preferences?

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However, while honesty may not always be the best policy it is the only **safe** *one, i.e., the only one which will guarantee 50%.*

Strategizing

Item	Ann	Bob
Matisse	75	25
Picasso	25	75

Ann will get the Matisse and Bob will get the Picasso and each gets 75 of his or her points.

Strategizing: Example

Suppose Ann knows Bob's preferences, but Bob does not know Ann's.

Item	Ann	Bob
M	75	25
P	25	75

Item	Ann	Bob
M	26	25
P	74	75

So Ann will get M plus a portion of P.

According to Ann's announced allocation, she receives 50 points

According to Ann's actual allocation, she receives $75 + 0.33 * 25 = 83.33$ points.

Strategizing: Example

Suppose *both* players know each other's preferences but neither knows that the other knows their own preference.

Item	Ann	Bob
M	75	25
P	25	75

Item	Ann	Bob
M	26	74
P	74	26

Each will get 74 of his or her announced points, but each one is really getting only 25 of his or her *true* points.

Strategizing: Example

Suppose *both* players know each other's preferences. Moreover, Ann knows that Bob knows her preference and Bob doesn't know that Ann knows.

Item	Ann	Bob
M	26	74
P	74	26

Item	Ann	Bob
M	73	74
P	27	26

What happens as the level of knowledge increases?