PHIL 308S: Voting Theory and Fair Division

Lecture 19

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R. Sanver. *Approval as an intrinsic part of preference*. Handbook of Approval Voting, 2010.

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Assumes the existence of an absolute scale over which the utilities of individuals are measured and compared.

At the other extreme, rule out any kind of cardinal information and interpersonal comparability, partitions $U(A)^n$ into cells which are *ordinally equivalent*:

$$F : W(A)^n \rightarrow \wp(A) - \{\emptyset\}$$

where $W(A)$ is the set of *weak orderings* on A.

Approval Voting

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"approval is not a strategic action but has an intrinsic meaning: It refers to the alternatives which are qualified as good."

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What about asking for **both** pieces of information?

Brams and Sanver. *Voting Systems that Combine Approval and Preference*. in Mathematics and Democracy, by S. Brams.

Assumptions

Assume each voter has a (linear) preference over the candidates.

Each voter is asked to rank the candidates from most preferred to least preferred (ties are not allowed).

Voters are then asked to specify which candidates are acceptable.

Consistency Assumption Given two candidates a and b, if a is approved and b is disapproved then a is ranked higher than b.

For example, we denote this approval ranking for a set $\{a, b, c, d\}$ of candidates as follows

$$a\ d \mid c\ b$$

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Rule 2 allows for candidates to be elected that are not the most approved.

PAV vs. Condorcet

Rule 1

I.	1 voter:	$a\ b \mid c$
II.	1 voter:	$b \mid a\ c$
III.	1 voter:	$c \mid a\ b$

PAV vs. Condorcet

Rule 1

I.	1 voter:	a b | c
II.	1 voter:	b | a c
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b is the AV winner.

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b is also the PAV winner.

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PAV vs. Condorcet

Rule 2(a)

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a is the Condorcet winner.

PAV vs. Condorcet

Rule 2(b)

I.	1 voter:	$d\ a\ b\ c \mid e$
II.	1 voter:	$d\ b\ c\ a \mid e$
III.	1 voter:	$e \mid d\ c\ a\ b$
IV.	1 voter:	$a\ b\ c \mid d\ e$
V.	1 voter:	$c \mid b\ a\ d\ e$

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a (3 votes), *b* (3 votes), and *c* (4 votes) are all majority approved.

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a (3 votes), *b* (3 votes), and *c* (4 votes) are all majority approved.
c is the PAV winner (a, b, c for a Condorcet cycle).
d is the Condorcet winner.

Example

I.	3 voters:	$a\ b\ c \mid d$
II.	3 voters:	$d\ a\ c \mid b$
III.	2 voters:	$b\ d\ c \mid a$

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c is approved by all 8 voters.

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c is approved by all 8 voters.
There is a top cycle $a > b > d > a$ which are all preferred by majorities to c (the AV winner).
a is the PAV winner

Example

I.	3 voters:	$a\ b\ c \mid d$
II.	3 voters:	$d\ a\ c \mid b$
III.	2 voters:	$b\ d\ c \mid a$

a is the PAV winner.
c is the AV winner.
d is the STV winner.

Example

I.	2 voters:	$a\ c\ b \mid d$
II.	2 voters:	$a\ c\ d \mid b$
III.	3 voters:	$b\ c\ d \mid a$

Example

I.	2 voters:	a c b | d
II.	2 voters:	a c d | b
III.	3 voters:	b c d | a

c is approve by all 7 voters.
a is the least approved candidate.
a is the PAV winner.
$BC(a) = 12$
$BC(c) = 14$

Fallback Voting (FV)

1. Voters indicate all candidates of whom they approve, who may range from no candidate to all candidate. Voters rank only those candidates whom they approve.

2. The highest-ranked candidate of all voters is considered. If a majority agrre on the highest-ranked candidate, this candidate is the FV winner (level 1).

Fallback Voting (FV)

1. If there is no level 1 winner, the next-highest ranked candidate of all voters in considered. If a majority of voters agree on one candidate as either their highest or their next-highest ranked candidate, this candidate is the FV winner (level 2). If more than one receive majority approval, then the candidate with the largest majority is the FV winner.
2. If no level 2 winner, the voters descend – one level at a time — to lower ranks of *approved* candidates stopping when one or more candidates receives majority approval. If more than one receives majority approval then the candidate with the largest majority is the FV winner. If the descent reaches the bottom and no candidate has won, then the candidate with the most approval is the FV winner.

Example

I.	4 voters:	$a\ b\ c \mid d$
II.	3 voters:	$b\ c \mid a\ d$
III.	2 voters:	$d\ a\ c \mid b$

Example

I.	4 voters:	$a\ b\ c \mid d$
II.	3 voters:	$b\ c \mid a\ d$
III.	2 voters:	$d\ a\ c \mid b$

b is the FV winner.
c is the AV winner.
a is the PAV winner.

PAV and FV

- Neither PAV nor FV may elect the Condorcet winner.
- Both PAV and FV are monotonic (approval-monotonic and rank-monotonic)
- Truth-telling strategies of voters under PAV and FV may not be in equilibrium

$q_i(x; R) = G$ iff $xP_i\mathbf{0}$

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$f : W(A \cup \{\mathbf{0})^n \rightarrow \wp(A) - \emptyset$ satisfies **majoritarian approval** iff we have $f(R) \subseteq \gamma(R)$ for every $R \in W(A \cup \{\mathbf{0})^n$ where $\gamma(R) \neq \emptyset$

$f : W(A \cup \{\mathbf{0}\})^n \to \wp(A) - \emptyset$ satisfies **approval independence** iff we have $f(R) = f(R')$ for every $R, R' \in W(A \cup \{\mathbf{0}\})^n$ where xR_iy iff xR'_iy for all $x, y \in A$ and $i \in N$.

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Fact. Majoritarian approval and approval independence are logically incompatible.

Two Questions

1. How to refine the set of socially good alternatives, when this set contains more than one alternative?
2. Which alternative to choose when none of them is socially good?

Majoritarian Compromise

1. The highest-ranked candidate of all voters is considered. If a majority of voters agree on one highest-ranked candidate, this candidate is the MC winner.
2. If there is no level 1 winner, the next-highest ranked candidate of all voters is considered. If a majority of voters agree on one candidate as either their highest or their next-highest ranked candidate, this candidate is the MC winner. If more than one candidate receives a majority support, then the candidate with highest support is the MC winner. The procedure stops.
3. If there is no level 2 winner, the voters descend one level at a time to lower and lower ranks, stopping when, for the first time, one or more candidates receive a majority support. If more than one candidate receives a majority support, then the candidate with the highest majority support is the MC winner.

Majoritarian Approval Compromise

3 voters $a \mid b\ c\ d$
2 voters $b\ a\ c \mid d$
2 voters $c \mid a\ b\ d$
2 voters $d\ b\ c \mid a$

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a is the MAC winner.

Approval Voting with Runoff

Given $R \in W(A \cup \{\mathbf{0}\})^n$, let $\rho(R) = \{x, y\}$ be the pair of alternatives which receive the highest approval. AVR picks the pairwise majority winner among the runoff winners.

R' is a lifting of x with respect to R provided for all $i \in N$,
xR_iy implies xR'_iy for all $y \in A$
xP_iy implies xP'_iy for all $y \in A$
$xP_i\mathbf{0}$ implies $xP'_i\mathbf{0}$
and yP_iz iff yP'_iz for all $y, z \in A - \{x\} \cup \{\mathbf{0}\}$.

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f is monotonic iff $x \in f(R)$ implies $x \in f(R')$ for all R' that are liftings of x with respect to R.

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f is monotonic iff $x \in f(R)$ implies $x \in f(R')$ for all R' that are liftings of x with respect to R.

Theorem. Approval Voting, Majoritarian Approval Compromise, Preference- Approval Voting and Approval Voting with a runoff are all monotonic.

$B = A \cup \{x^*\}$.

We say that $R \in W(A \cup \{\mathbf{0}\})^n$ and $R' \in W(B \cup \{\mathbf{0}\})^n$ **agree** provided for all $x, y \in A$, xR_iy iff xR'_iy and $xR_i\mathbf{0}$ iff $xR'_i\mathbf{0}$.

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We call x^* a spoiler if its presence can change the social choice without x^* being chosen.

$x^* \notin f(R') \neq f(R)$ at some $R \in W(A \cup \{\mathbf{0}\})$ that agrees with $R' \in W(B \cup \{\mathbf{0}\})$

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f is independent if it does not admit a **spoiler**.

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f is independent if it does not admit a **spoiler**.

Theorem. Majoritarian Approval Compromise, Preference-Approval Voting and Approval Voting with a runoff fail independence.

Fair Division

S. Brams, P. Edelman and P. Fishburn. *Paradoxes of Fair Division*. Journal of Philosophy, **98:6**, pgs. 300-314.

J. Robertson and W. Webb. *Cake-Cutting Algorithms: Be Fair if You Can*. A.K. Peters, 1998.

S. Brams and A. Taylor. *Fair Division: From cake-cutting to dispute resolution*. Cambridge University Press, 1998.

S. Brams and A. Taylor. *The Win-Win Solution*. W. W. Norton & Company, 2000.

Fairness Conditions

- **Proportional:** (for two players) each player receives at least 50% of their valuation.

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- **Equitable:** each player values its allocation the same *according to its own valuation function*.
- **Efficiency:** there is no other division better for everybody, or better for some players and not worse for the others

Paradoxes of Fair Division

- The conflict between efficiency and envy-freeness;
- The failure of a unique efficient and envy-free division to satisfy other fair-division criteria;
- The desirability, on occasion, of dividing items unequally.

Envy-Freeness and Efficiency

Ann:	1	$\succ$	2	$\succ$	3	$\succ$	4	$\succ$	5	$\succ$	6
Bob:	4	$\succ$	3	$\succ$	2	$\succ$	1	$\succ$	5	$\succ$	6
Cath:	5	$\succ$	1	$\succ$	2	$\succ$	6	$\succ$	3	$\succ$	4

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Ann: $\{1, 3\}$
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There is no other division that guarantees *envy freeness*

No Envy-Free Division

Ann:	1	$\succ$	2	$\succ$	3
Bob:	1	$\succ$	3	$\succ$	2
Cath:	2	$\succ$	1	$\succ$	2

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There are no envy-free divisions.