PHIL 308S: Voting Theory and Fair Division

Lecture 16

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E. Aragones, I. Gilboa and A. Weiss. *Making Statements and Approval Voting*. Theory and Decision, Volume 71, Number 4 (2011), pp. 461 – 472.

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Abstract. We assume that people have a need to make statements, and construct a model in which this need is the sole determinant of voting behavior. In this model, an individual selects a ballot that makes as close a statement as possible to her ideal point, where abstaining from voting is a possible (null) statement. We show that in such a model, a political system that adopts approval voting may be expected to enjoy a significantly higher rate of participation in elections than a comparable system with plurality rule.

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The voter considers every possible ballot she may cast as a vectot $x = (x_1, \ldots, x_m) \in \mathbb{R}_+^m$, where x_j is the degree to which the ballot supports party j. The voting system determines a subset $F \subset \mathbb{R}_+^m$ of feasible values for x.

1. Abstention corresponds to the null statement $0 \in \mathbb{R}^n$;
2. A voter for a single party corresponds to the position $v^j \in \mathbb{R}^n$ of that party; and
3. A voter approving of a non-empty set of parties corresponds ot the arithmetic average of their positions.

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A ballot $x \in \mathbb{R}^m_+$ makes the statement

$$Vx = \sum_{j \in T} x_j v^j \in \mathbb{R}^n$$

Plurality rule, in which a voter selects a single party. In this case the degree of support is $x^j = 1$ for the selected party and $x^j = 0$ for the others. $F^M = \{0\} \cup \{e^j\}_{j \leq m}$ where e^j is the vector with 1 in the jth position.

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Approval voting, in which a voter may choose any subset of parties as her vote. We model an individual who selects a non-empty subset $S \subset T$ as choosing the vector

$$x = \frac{1}{|S|} \sum_{j \in S} e^j$$

that is, as supporting each party to degree $\frac{1}{|S|}$.

This reflects the fact that the strength of the statement made in favor of a party by endorsing it depends on the other parties one endorses.

A voter with an ideal point $w \in \mathbb{R}^n$ may be modeled as solving the following problem:

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If the solution to this problem is $x = 0 \in \mathbb{R}^m$, the voter will abstain.

Abstention does not result from the cost of voting, but rather from the fact that the voter feels dissatisfied with any statement that the political systems allows her to make.

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Consider a voter with ideal point $w = (0, 0.8)$.

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Conclusion: the individual will participate under Approval Voting, but not under Plurality.

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The voter will prefer to abstain rather than to vote to any single party (for instance,
$||(0, 0, 0) - (1, 1, 1)|| = \sqrt{3} < ||(-1, 1, 1) - (1, 1, 1)|| = 2$)

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$||(0,0,0)-(1,1,1)|| = \sqrt{3} < ||(-1,1,1)-(1,1,1)|| = 2$)

By contrast, approving of all three parties yields the vector $(1/3, 1/3, 1/3)$ which is closer to $(1,1,1)$ than is $(0,0,0)$.

Assumptions: (1) Each party has to take a position on each issue (without it a party may position itself at the origin, in which case no voter would strictly prefer to abstain under either voting system)

(2) each individual has an ideal point $w \in \mathbb{R}^n$ that reflects a position on every issue ($w_i \in \{-1, 1\}$ for every i)

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Under these assumptions, there are two results that attempt to capture the fact that a richer set of possible statements will make more individuals vote.

For a given number of issues n, let $K_M(n)$ ($K_A(n)$) be the minimal number of parties that guarantee that each individual casts a ballot in a plurality (approval) system.

Theorem. $K_M(n)$ is bounded below by an exponential function of n, whereas $K_A(n)$ is bounded above by 4.

R. Sanver. *Approval as an intrinsic part of preference*. Handbook of Approval Voting, 2010.

Let $U(A)$ be the set of real-valued "utility functions" defined over A, an *aggregation function* is

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At the other extreme, rule out any kind of cardinal information and interpersonal comparability, partitions $U(A)^n$ into cells which are *ordinally equivalent*:

$$F : W(A)^n \to \wp(A) - \{\emptyset\}$$

where $W(A)$ is the set of *weak orderings* on A.

Approval Voting

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"approval is not a strategic actions but has an intrinsic meaning: It refers to the alternatives which are qualified as good."

M. Balinski and R. Laraki. *Majority Judgement: Measuring, Ranking and Electing*. The MIT Press, 2010.

Voting Mechanism

- the specific form of the voters' and judges' *inputs*, the *messages* used to exert their wills, and
- the procedure by which the inputs or messages are amalgamated or transformed into a final decision, social choice, or *output*.

Grading

In many situations, people use measures or grades that are well defined **absolute common languages of evaluation** to define decision mechanisms:

- in figure skating (new system), diving and gymnastics competitions;
- in piano, ute and orchestra competitions;
- in classifying wines at wine competitions;
- in ranking university students;
- in classifying hotels and restaurants, e.g., the Michelin *

Grading

"Judges and voters have complex aims, ends, purposes, and wishes: their *preferences* or *utilities*. A judge's or a voter's preferences may depend on many factors, including his beliefs about what is right and wrong, about the common language, about the method that transforms input messages into decisions, about the other judges' or voters' acts and behaviors, in addition to his evaluations of the competitors or candidates. But the judges' or voters' input messages—the grades they give—are assuredly not their preferences: a judge may dislike a wine, a dive, or a part of a skater's performance yet give it a high grade because of its merits; or a judge may like it yet give it a low grade because of its demerits." (pg. 3)

Majority Grade

The *majority-grade* is the (assuming an odd number of judges) *median* grade (the grade that is in the middle of the list when the grades or ordered). If there is an even number of judges, then the majority grade is the lowest grade in the middle interval.

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Suppose A's grades are $\{7, 7, 8, 8, 11, 11, 11, 13\}$
Suppose B's grades are $\{9, 9, 9, 9, 9, 10, 10, 10\}$
B should be ranked higher than A

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When two competitors have different majority-grades, the competitor with the higher grade is naturally ranked higher.

A's grades: $\{7, 9, \mathbf{9}, 11, 11\}$
B's grades: $\{8, 9, \mathbf{9}, 10, 11\}$

The second majority grade is found:
A's grades: $\{7, \mathbf{9}, 9, 11, 11\}$
B's grades: $\{8, \mathbf{9}, 9, 10, 11\}$

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The third majority grade is found:
A's grades: $\{7, 9, 9, \mathbf{11}, 11\}$
B's grades: $\{8, 9, 9, \mathbf{10}, 11\}$

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The third majority grade is found:
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B's grades: $\{8, 9, 9, \mathbf{10}, 11\}$

So, A is ranked above B.

The Model

A common language Λ: a set of strictly ordered grades: $\alpha, \beta, \gamma, \ldots$;

A finite set of m competitors (alternatives, candidates, performances, competing goods) $C = \{A, B, ..., I, ..., Z\}$; and

A finite set of n judges $J = \{1, ..., j, ..., n\}$.

A problem is specified by a profile $\Phi = \Phi(C, J)$: $m \times n$ matrix of grades assigned to the competitors (rows) by the judges (columns).

Example

Suppose that there are five voters, $1, \ldots, 5$ and three candidates I,II, and III. The grades that the voters may assign are A, B, C, D, or F (from best to worst).

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	1	2	3	4	5
I	A	A	C	D	D
II	B	B	F	B	F
III	D	C	B	A	D

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I	A	A	**C**	D	D
II	B	B	F	**B**	F
III	D	**C**	B	A	D

Candidates I, II, and III receive *majority grades* (median grades) C, B and C, respectively. Thus Candidate II would be declared the winner.

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II	B	B	F	**B**	F
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Candidate II is ranked above candidate I: Is this reasonable? *If asked their preference, then candidate I is preferred to II by 4 judges.*

The Thesis

1. The traditional model is a bad model, in theory and in practice
2. The majority judgement is a better alternative to all other known methods, in theory and in practice

Electing vs. Ranking: Condorcet's Ranking

Condorcet-ranking (also known as Kemeny's rule) associates a score to each possible rank ordering:

- A voter contributes k Condorcet-points to a rank-ordering if his input agrees in k pair-by-pair comparisons.
- The Condorcet-score of a rank-ordering is the sum of its Condorcet-points over all voters.
- The Condorcet-ranking is the ranking that maximizes the Condorcet-score.

Electing vs. Ranking

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- Given a method of ranking, the first-placed candidate is the winner.
- Given a method of designating a winner (or loser), he is the first-ranked (or last-ranked); the second-ranked is the winner among the remaining candidates, and so on.

Are ranking and designating winners two sides of one coin?

333	333	333	1
A	B	C	A
B	C	A	C
C	A	B	B

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According to Condorcet: $A \succ B \succ C$ and $C \succ A \succ B$ are tied for first.

- No reasonable **ranking function** must choose $A \succ C \succ B$
- Any reasonable **choice function** must choose $A \succ C \succ B$

There is a fundamental incompatibility between electing and ranking.

Conclusion: The traditional model's inputs are inadequate messages and must be reformulated.

D. Felsenthal and M. Machover. *The Majority Judgement voting procedure: a critical evaluation*. Homo oeconomicus, 25, pp. 319 - 334.

S. Brams. *Grading Candidates*. Book Review, American Scientist, 2012.

M. Balinski. *A letter regarding Steven J. Bramss review of Majority Judgment* . American Scientist.

Strategic Grading

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Suppose a student submits a paper to a panel of examiners.

If an examiner were informed of the grades awarded by the other examiners, she would have no incentive to change her grade.

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If the examiner's grade turns out to be lower than the median, there is no incentive to change: lowering the grade would not change the result, raising the grade may result in a *higher* median grade.

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If an examiner were informed of the grades awarded by the other examiners, she would have no incentive to change her grade.

If the examiner's grade coincides with the median, there is no incentive to change

If the examiner's grade turns out to be lower than the median, there is no incentive to change: lowering the grade would not change the result, raising the grade may result in a *higher* median grade.

Similarly, if the examiner's grade turns out to be higher than the median grade.

	1	2	3
x	A	C	C
y	D	A	D
z	B	D	A

	1	2	3
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y	D	A	D
z	B	D	A

- If the judges act sincerely, then z is the winner

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z	B	D	A

- If the judges act sincerely, then *z* is the winner
- Both 1 and 2 would prefer that *x wins*
- 1 can down grade *z* to *D*, then *x* will win

	1	2	3
x	A	C	C
y	D	A	D
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- If the judges act sincerely, then z is the winner
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Region I

					Winner: y
x	21 A	31 B	48 C	1 D	
y	40 A	11 B	48 C	2 D	

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x	21 A	31 B	48 C	1 D	**Winner**: y
y	40 A	11 B	48 C	2 D	

Region II

x	1 A	46 B	14 C	40 D	**Winner**: y
y	1 A	45 B	33 C	22 D	

Region I

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Region II

x	1 A	46 B	14 C	40 D	**Winner**: y
y	1 A	45 B	33 C	22 D	

Region III

x	40 B	20 C	41 D	**Winner**: y
y	48 B	3 C	50 D	

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y	1 A	45 B	33 C	22 D	

Region III

x	40 B	20 C	41 D	**Winner**: y
y	48 B	3 C	50 D	

Merged

x	22 A	117 B	82 C	82 D	**Winner**: x
y	41 A	104 B	84 C	74 D	

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x	3	4	4
y	2	5	5

Winner: y

Voters who are indifferent between all candidates may decide to abstain from voting, or to express their indifference actively.

x	3	4	4
y	2	5	5

Winner: y

x	1	1	3	4	4
y	1	1	2	5	5

Winner: x