PHIL 308S: Voting Theory and Fair Division

Lecture 14

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Distance-Base Rationalization

- For some profiles, there is a clear winner (eg., Condorcet winner, unanimous top choice, unanimous rankings, majority winner)
- If the profile P is not a consensus profile, then find the *closest* consensus profile, according to some notion of *distance*

Kemeny Metric

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Examples:

$d(a > b > c > d, d > a > b > c) = 3$

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Examples:

$d(a > b > c > d, d > a > b > c) = 3$

$d(a > b > c > d, c > d > a > b) = 4$

S. Nitzan. *Some Measures of Closeness to Unanimity and Their Implications*. Theory and Decision, 13, 129 - 138, 1981.

A **top-ranked** *B* **top-ranked** *C* **top-ranked** *D* **top-ranked**

3	5	7	6
A	A	B	C
B	C	D	B
C	B	C	D
D	D	A	A

***A* top-ranked** | ***B* top-ranked** | ***C* top-ranked** | ***D* top-ranked**

3	5	7	6
A	A	A	A
B	C	B	C
C	B	D	B
D	D	C	D

39

3	5	7	6
A	A	B	C
B	C	D	B
C	B	C	D
D	D	A	A

$3 \cdot 0 + 5 \cdot 0 + 7 \cdot 3 + 6 \cdot 3$

***A* top-ranked**   ***B* top-ranked**   ***C* top-ranked**   ***D* top-ranked**

3	5	7	6
A	A	A	A
B	C	B	C
C	B	D	B
D	D	C	D

3	5	7	6
B	B	B	B
A	A	D	C
C	C	C	D
D	D	A	A

39

19

3	5	7	6
A	A	B	C
B	C	D	B
C	B	C	D
D	D	A	A

$3 \cdot 1 + 5 \cdot 2 + 7 \cdot 0 + 6 \cdot 1$

A top-ranked **B top-ranked** **C top-ranked** **D top-ranked**

3	5	7	6
A	A	A	A
B	C	B	C
C	B	D	B
D	D	C	D

3	5	7	6
B	B	B	B
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3	5	7	6
C	C	C	C
A	A	B	B
B	B	D	D
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39 19 25

3	5	7	6
A	A	B	C
B	C	D	B
C	B	C	D
D	D	A	A

$3 \cdot 2 + 5 \cdot 1 + 7 \cdot 2 + 6 \cdot 0$

A **top-ranked**			
3	5	7	6
A	A	A	A
B	C	B	C
C	B	D	B
D	D	C	D

B **top-ranked**			
3	5	7	6
B	B	B	B
A	A	D	C
C	C	C	D
D	D	A	A

C **top-ranked**			
3	5	7	6
C	C	C	C
A	A	B	B
B	B	D	D
D	D	A	A

D **top-ranked**			
3	5	7	6
D	D	D	D
A	A	B	C
B	C	C	B
C	B	A	A

39 19 25 42

3	5	7	6
A	A	B	C
B	C	D	B
C	B	C	D
D	D	A	A

$$3 \cdot 3 + 5 \cdot 3 + 7 \cdot 1 + 6 \cdot 2$$

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Distance to closest profile with X at the top
$D(X) = 0 \cdot N_1(X) + 1 \cdot N_2(X) + \cdots + (m-1) \cdot N_m(X)$

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Borda score of X
$B(X) = (m-1) \cdot N_1(X) + (m-2) \cdot N_2(X) + \cdots + 0 \cdot N_m(X)$

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Fact. $D(X) \leq D(Y)$ iff $B(X) \geq B(Y)$

So, X minimizes $D(\cdot)$ iff X maximizes $B(\cdot)$

$$d_2(P_i, Q_i) = \begin{cases} 0 & \text{if } top(P_i) = top(Q_i) \\ 1 & \text{otherwise} \end{cases}$$

Fact An alternative is the plurality winner iff it is closest to the unanimous profile using the d_2 measure.

Distance-Based Judgement Aggregation

G. Pigozzi. *Belief merging and the discursive dilemma: an argument-based account of paradoxes in judgement aggregation*. Synthese 152, pgs. 285 - 298, 2006.

M. Miller and D. Osherson. *Methods for distance-based judgement aggregation*. Social Choice and Welfare, 32, pgs. 575 - 601, 2009.

C. Duddy and A. Piggins. *A measure of distance between judgement sets*. Manuscript, 2011.

Given $(A_1, \ldots, A_n)$, select the set consistent and complete A that minimizes the total distance from the individual judgement sets: find A such that $\sum_{i \in N} d(A, A_i)$ is minimized.

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Hamming Metric: $d(A, A') =$ the number of propositions for which A and A' disagree

$d_H(\{p, q, p \wedge q\}, \{p, \neg q, \neg(p \wedge q)\}) = 2$

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Duddy and Piggins: shouldn't
$d(\{p, q, p \wedge q\}, \{p, \neg q, \neg(p \wedge q)\} = 1$?

Duddy and Piggins Measure

Judgement set C is between judgement sets A and B if A, B and C are distinct and, on each proposition C agrees with A or with B (or both). (*C is a compromise between A and B*)

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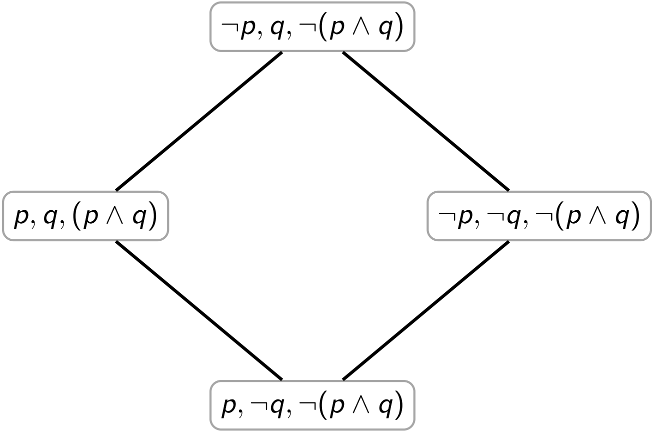
Draw a graph where the nodes are possible judgement sets and there is an edge between A and B provided there is no judgement set between them.

The distance between A and B is the length of the shortest path from A to B.



$\neg p, q, \neg(p \wedge q)$
$p, q, (p \wedge q)$
$\neg p, \neg q, \neg(p \wedge q)$
$p, \neg q, \neg(p \wedge q)$

Axioms

Axiom 1 $d(A, B) = 0$ iff $A = B$
Axiom 2 $d(A, B) = d(B, A)$
Axiom 3 $d(A, B) \leq d(A, C) + d(C, B)$

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Axiom 5 If there is no judgement set between A and B with $A \neq B$ then $d(A, B) = 1$

Theorem (Duddy & Piggins) The previously defined metric is the unique metric satisfying Axioms 1 - 5.

	p	q	$p \wedge q$

	p	q	$p \wedge q$
1	T	T	T

	p	q	$p \wedge q$
1	T	T	T
2	T	F	F

	p	q	$p \wedge q$
1	T	T	T
2	T	F	F
3	F	T	F

	p	q	$p \wedge q$
1	T	T	T
2	T	F	F
3	F	T	F
Majority	T	T	F

	p	q	$p \wedge q$
1	T	T	T
2	T	F	F
3	F	T	F
Majority	T	T	F
DP-metric	T	T	T

	p	q	$p \wedge q$
1	T	T	T
2	T	F	F
3	F	T	F
Majority	T	T	F
DP-metric	T	T	T
Hamming	F	T	F

	p	q	$p \wedge q$
1	T	T	T
2	T	F	F
3	F	T	F
Majority	T	T	F
DP-metric	T	T	T
Hamming	F	T	F
Premise	T	T	T

M. Miller and D. Osherson. *Methods for distance-based judgement aggregation*. Social Choice and Welfare, 32, pgs. 575 - 601, 2009.

Differing on $\{a, b \wedge c\}$ may be considered more consequential than differing on $\{a, a \wedge b\}$.

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Let $\mathcal{F}$ be the set of *all* judgement sets and $\mathcal{F}^\circ$ the set of all consistent judgement sets.

$d : \mathcal{F} \times \mathcal{F} \to \mathbb{R}$

Axiom 1 $d(A, B) = 0$ iff $A = B$
Axiom 2 $d(A, B) = d(B, A)$
Axiom 3 $d(A, B) \leq d(A, C) + d(C, B)$

$d(J, J') = \sum_{i \leq n} d(J_i, J'_i)$

For a profile P, $M(P) \in \mathcal{F}$ the judgement set resulting from majority rule. P is majority consistent provided $M(P) \in \mathcal{F}^{\circ}$

Fix a metric d and a profile $J \in \mathcal{F}^{\circ}$

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Fix a metric d and a profile $J \in \mathcal{F}^\circ$

- $Full_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that J' minimizes $d(J, J')$ over all majority consistent profiles J' in $\mathcal{F}^\circ$

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- $\mathit{Full}_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that J' minimizes $d(J, J')$ over all majority consistent profiles J' in $\mathcal{F}^\circ$
- $\mathit{Output}_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that J' minimizes $d(J, J')$ over all majority consistent profiles J' in $\mathcal{F}$ (*allowing inconsistencies*)

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- $Output_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that J' minimizes $d(J, J')$ over all majority consistent profiles J' in $\mathcal{F}$ (*allowing inconsistencies*)
- $Endpoint_d(J)$ is the collection of $K \in \mathcal{F}^\circ$ that minimize $d(J, J')$ over all majority consistent profiles J'

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- $Output_d(J)$ is the collection of $M(J') \in \mathcal{F}^{\circ}$ such that J' minimizes $d(J, J')$ over all majority consistent profiles J' in $\mathcal{F}$ (*allowing inconsistencies*)
- $Endpoint_d(J)$ is the collection of $K \in \mathcal{F}^{\circ}$ that minimize $d(J, J')$ over all majority consistent profiles J'
- $Prototype_d(J)$ is the collection of $K \in \mathcal{F}^{\circ}$ that minimize $\sum_{i \leq n} d(J_i, K)$ over all $K \in \mathcal{F}^{\circ}$

For J, K let $Ham(J, K)$ denote the Hamming distance (the number of items on which J and K disagree)

$$d(J, K) = \begin{cases} 0.9 & \text{if } J \text{ and } K \text{ disagree only on } a \wedge b \\ \sqrt{Ham(p, q)} & \text{otherwise} \end{cases}$$

	a	b	$a \wedge b$	a	b	$a \wedge b$	a	b	$a \wedge b$
1	T	T	T	T	T	T	T	T	T
2	T	T	T	T	T	T	T	T	T
3	T	F	F	T	F	F	**T**	**F**	**T**
4	T	F	F	T	F	F	T	F	F
5	F	T	F	**F**	**F**	**F**	F	T	F
M	T	T	F	T	F	F	T	T	T

	a	b	$a \wedge b$	a	b	$a \wedge b$	a	b	$a \wedge b$
1	T	T	T	T	T	T	T	T	T
2	T	T	T	T	T	T	T	T	T
3	T	F	F	T	F	F	**T**	**F**	**T**
4	T	F	F	T	F	F	T	F	F
5	F	T	F	**F**	**F**	**F**	F	T	F
M	T	T	F	T	F	F	T	T	T

- $Full_d(J) = TFF \ (d(FTF, FFF) = 1)$

	a	b	$a \wedge b$	a	b	$a \wedge b$	a	b	$a \wedge b$
1	T	T	T	T	T	T	T	T	T
2	T	T	T	T	T	T	T	T	T
3	T	F	F	T	F	F	**T**	**F**	**T**
4	T	F	F	T	F	F	T	F	F
5	F	T	F	**F**	**F**	**F**	F	T	F
M	T	T	F	T	F	F	T	T	T

- $Full_d(J) = TFF$ $(d(FTF, FFF) = 1)$
- $Output_d(J) = TTT$ $(d(TFF, TFT) = 0.9)$

	a	b	$a \wedge b$	a	b	$a \wedge b$	a	b	$a \wedge b$
1	T	T	T	T	T	T	T	T	T
2	T	T	T	T	T	T	T	T	T
3	T	F	F	T	F	F	**T**	**F**	**T**
4	T	F	F	T	F	F	T	F	F
5	F	T	F	**F**	**F**	**F**	F	T	F
M	T	T	F	T	F	F	T	T	T

- $Full_d(J) = TFF\ (d(FTF, FFF) = 1)$
- $Output_d(J) = TTT\ (d(TFF, TFT) = 0.9)$
- $Endpoint_d(J) = TTT\ (d(TTF, TTT) = 0.9)$

	a	b	$a \wedge b$	a	b	$a \wedge b$	a	b	$a \wedge b$
1	T	T	T	T	T	T	T	T	T
2	T	T	T	T	T	T	T	T	T
3	T	F	F	T	F	F	**T**	**F**	**T**
4	T	F	F	T	F	F	T	F	F
5	F	T	F	**F**	**F**	**F**	F	T	F
M	T	T	F	T	F	F	T	T	T

- $Full_d(J) = TFF$ $(d(FTF, FFF) = 1)$
- $Output_d(J) = TTT$ $(d(TFF, TFT) = 0.9)$
- $Endpoint_d(J) = TTT$ $(d(TTF, TTT) = 0.9)$
- $Prototype_d(J) = \{TTT, TFF\}$ $(\sum_i d(J_i, TTT) = 3\sqrt{2}$, $\sum_i d(J_i, TFF) = 3\sqrt{2}$, $\sum_i d(J_i, FTF) = 4\sqrt{2}$, $\sum_i d(J_i, FFF) = 2\sqrt{3} + 3)$

Approval Voting

S. Brams and P. Fishburn. *Approval Voting*. Birkhauser, 1983.

J.-F. Laslier and M. R. Sanver (eds). *Handbook of Approval Voting*. Studies in Social Choice and Welfare, 2010.

Approval Voting: Each voter selects a subset of candidates. The candidate with the most “approvals” wins the election.

Why Approval Voting?

Because the voter is given the opportunity to provide more information about her opinion than with a single-name ballot, adoption of Approval Voting might increase voter turnout in general elections. Given the generally accepted view that the quality of a democracy is linked to the number of voters participating and their level of satisfaction with the electoral process, this suggests that Approval Voting can contribute to strengthening democracy.

Why Approval Voting?

By eliminating the wasted-vote effect, Approval Voting might broaden the span of candidates running for office, thereby contributing to the richness of the political debate. This point is related to the standard observation that the one-round Plurality system makes third parties nonviable, a critical point in U.S. politics.

Why Approval Voting?

By eliminating the squeezing effect, Approval Voting would encourage the election of consensual candidates. The squeezing effect is typically observed in multiparty elections with a runoff. The runoff tends to prevent extremist candidates from winning, but a centrist candidate who would win any pairwise runoff (the Condorcet winner) is also often squeezed between the left-wing and the right-wing candidates and so eliminated in the first round. This point is critical in countries using two-round Plurality.

Why Approval Voting?

AV will reduce negative campaigning. AV induces candidates to try to mirror the views of a majority of voters, not just cater to minorities whose votes could give them a slight edge in a crowded plurality contest. AV is therefore likely to cut down on negative campaigning, because candidates will have an incentive to broaden their appeals by reaching out for approval to voters who might have a different first choice. Lambasting such a choice, rather than being more expansive, risks alienating this candidates supporters, thereby losing their approval.

Why Approval Voting?

AV is eminently practicable. Unlike more complicated ranking systems, which suffer from a variety of theoretical as well as practical defects, AV is simple for voters to understand and use. Although more votes must be tallied under AV than under PV, AV can readily be implemented on existing voting machines. Because AV does not violate any state constitutions in the United States (or, for that matter, the constitutions of most countries in the world), it requires only an ordinary statute to enact