# PHIL 308S: Voting Theory and Fair Division

## Lecture 13

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What does it *mean* to vote strategically?

- Voting as a game vs. voting as an act of communication

K. Dowding and M. van Hees. *In Praise of Manipulation*. British Journal of Political Science, 38 : pp 1-15, 2008.

# Distance

“...in many situations , differences of opinion arise from differences in values, not erroneous judgments. In this case it seems better to adopt the view that group choice is an exercise in finding a compromise between conflicting opinions.” (Young, p. 60)

H. P. Young. *Optimal Voting Rules*. The Journal of Economic Perspectives, 9:1, pgs. 51 - 64, 1995.

# Dodgson Method

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F. Brandt. *Some Remarks on Dodgson's Voting Rule*. Mathematical Logic Quarterly, 55:4, pp. 460-463, 2009.

# Distance-Base Rationalization

- For some profiles, there is a clear winner (eg., Condorcet winner, unanimous top choice, unanimous rankings, majority winner)
- If the profile $P$ is not a consensus profile, then find the *closest* consensus profile, according to some notion of *distance*

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Examples:

$d(a > b > c > d, d > a > b > c) = 3$

$d(a > b > c > d, c > d > a > b) = 4$

S. Nitzan. *Some Measures of Closeness to Unanimity and Their Implications*. Theory and Decision, 13, 129 - 138, 1981.

# Reaching Consensus

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$x$ is a **relative unanimous winner** provided the *distance* between $P$ and $U(x)$ is no larger than the distance between $P$ and $U(y)$ for all other alternatives $y$.

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$U^*(x) = \{P \in \mathcal{P}^n \mid d(P, U(x)) \leq d(P, U(y)) \text{ for all } x \in X\}$

**Fact**. An alternative $x$ has the highest Borda score iff it is a relative unanimous winner.

$$\delta_2(P_i, Q_i) = \begin{cases} 0 & \text{if } top(P_i) = top(Q_i) \\ 1 & \text{otherwise} \end{cases}$$

**Fact** An alternative is the plurality winner iff it is closest to the unanimous profile using the $\delta_2$ measure.

# Distance-Based Judgement Aggregation

G. Pigozzi. *Belief merging and the discursive dilemma: an argument-based account of paradoxes in judgement aggregation*. Synthese 152, pgs. 285 - 298, 2006.

M. Miller and D. Osherson. *Methods for distance-based judgement aggregation*. Social Choice and Welfare, 32, pgs. 575 - 601, 2009.

C. Duddy and A. Piggins. *A measure of distance between judgement sets*. Manuscript, 2011.

Given $(A_1, \ldots, A_n)$, select the set consistent and complete $A$ that minimizes the total distance from the individual judgement sets: find $A$ such that $\sum_{i \in N} d(A, A_i)$ is minimized.

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**Hamming Metric**: $d(A, A') =$ the number of propositions for which $A$ and $A'$ disagree

$d_H(\{p, q, p \wedge q\}, \{p, \neg q, \neg(p \wedge q)\}) = 2$

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Duddy and Piggins: shouldn't
$d(\{p, q, p \wedge q\}, \{p, \neg q, \neg(p \wedge q)\} = 1$?

# Duddy and Piggins Measure

Judgement set $C$ is between judgement sets $A$ and $B$ if $A, B$ and $C$ are distinct and, on each proposition $C$ agrees with $A$ or with $B$ (or both). (*$C$ is a compromise between $A$ and $B$*)

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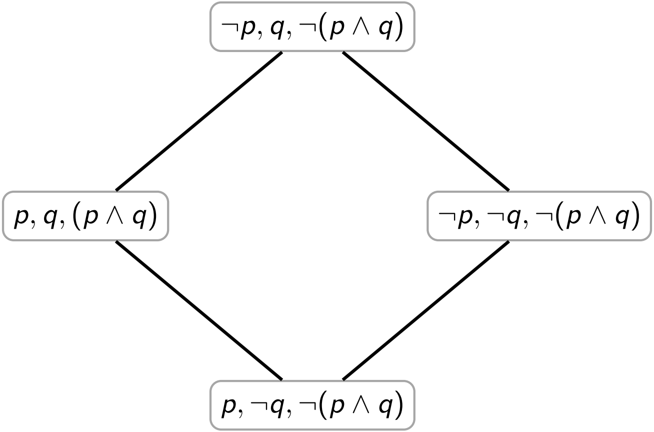
Draw a graph where the nodes are possible judgement sets and there is an edge between $A$ and $B$ provided there is no judgement set between them.

The distance between $A$ and $B$ is the length of the shortest path from $A$ to $B$.


$\neg p, q, \neg(p \wedge q)$
$p, q, (p \wedge q)$
$\neg p, \neg q, \neg(p \wedge q)$
$p, \neg q, \neg(p \wedge q)$

# Axioms

**Axiom 1** $d(A, B) = 0$ iff $A = B$
**Axiom 2** $d(A, B) = d(B, A)$
**Axiom 3** $d(A, B) \leq d(A, C) + d(C, B)$

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**Axiom 5** If there is no judgement set between $A$ and $B$ with $A \neq B$ then $d(A, B) = 1$

**Theorem** (Duddy & Piggins) The previously defined metric is the unique metric satisfying Axioms 1 - 5.

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| 1 | $T$ | $T$ | $T$ |

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| 1 | $T$ | $T$ | $T$ |
| 2 | $T$ | $F$ | $F$ |

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| 1 | $T$ | $T$ | $T$ |
| 2 | $T$ | $F$ | $F$ |
| 3 | $F$ | $T$ | $F$ |

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| 1 | $T$ | $T$ | $T$ |
| 2 | $T$ | $F$ | $F$ |
| 3 | $F$ | $T$ | $F$ |
| Majority | $T$ | $T$ | $F$ |

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| 1 | $T$ | $T$ | $T$ |
| 2 | $T$ | $F$ | $F$ |
| 3 | $F$ | $T$ | $F$ |
| Majority | $T$ | $T$ | $F$ |
| DP-metric | $T$ | $T$ | $T$ |

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| 1 | $T$ | $T$ | $T$ |
| 2 | $T$ | $F$ | $F$ |
| 3 | $F$ | $T$ | $F$ |
| Majority | $T$ | $T$ | $F$ |
| DP-metric | $T$ | $T$ | $T$ |
| Hamming | $F$ | $T$ | $F$ |

| | $p$ | $q$ | $p \wedge q$ |
|---|---|---|---|
| 1 | $T$ | $T$ | $T$ |
| 2 | $T$ | $F$ | $F$ |
| 3 | $F$ | $T$ | $F$ |
| Majority | $T$ | $T$ | $F$ |
| DP-metric | $T$ | $T$ | $T$ |
| Hamming | $F$ | $T$ | $F$ |
| Premise | $T$ | $T$ | $T$ |

M. Miller and D. Osherson. *Methods for distance-based judgement aggregation*. Social Choice and Welfare, 32, pgs. 575 - 601, 2009.

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Let $\mathcal{F}$ be the set of *all* judgement sets and $\mathcal{F}^\circ$ the set of all consistent judgement sets.

$d : \mathcal{F} \times \mathcal{F} \rightarrow \mathbb{R}$

**Axiom 1** $d(A, B) = 0$ iff $A = B$
**Axiom 2** $d(A, B) = d(B, A)$
**Axiom 3** $d(A, B) \leq d(A, C) + d(C, B)$

$d(J, J') = \sum_{i \leq n} d(J_i, J'_i)$

For a profile $P$, $M(P) \in \mathcal{F}$ the judgement set resulting from majority rule. $P$ is majority consistent provided $M(P) \in \mathcal{F}^{\circ}$

Fix a metric $d$ and a profile $J \in \mathcal{F}^{\circ}$

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Fix a metric $d$ and a profile $J \in \mathcal{F}^\circ$

- $Full_d(J)$ is the collection of $M(J') \in \mathcal{F}^\circ$ such that $J'$ minimizes $d(J, J')$ over all majority consistent profiles $J'$ in $\mathcal{F}^\circ$

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- $Endpoint_d(J)$ is the collection of $K \in \mathcal{F}^\circ$ that minimize $d(J, J')$ over all majority consistent profiles $J'$

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- $Endpoint_d(J)$ is the collection of $K \in \mathcal{F}^\circ$ that minimize $d(J, J')$ over all majority consistent profiles $J'$
- $Prototype_d(J)$ is the collection of $K \in \mathcal{F}^\circ$ that minimize $\sum_{i \leq n} d(J_i, K)$ over all $K \in \mathcal{F}^\circ$

For $J, K$ let $Ham(J, K)$ denote the Hamming distance (the number of items on which $J$ and $K$ disagree)

$$d(J, K) = \begin{cases} 0.9 & \text{if } J \text{ and } K \text{ disagree only on } a \wedge b \\ \sqrt{Ham(p, q)} & \text{otherwise} \end{cases}$$

| | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | T | T | T | T | T | T | T | T | T |
| 2 | T | T | T | T | T | T | T | T | T |
| 3 | T | F | F | T | F | F | **T** | **F** | **T** |
| 4 | T | F | F | T | F | F | T | F | F |
| 5 | F | T | F | **F** | **F** | **F** | F | T | F |
| M | T | T | F | T | F | F | T | T | T |

| | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | T | T | T | T | T | T | T | T | T |
| 2 | T | T | T | T | T | T | T | T | T |
| 3 | T | F | F | T | F | F | **T** | **F** | **T** |
| 4 | T | F | F | T | F | F | T | F | F |
| 5 | F | T | F | **F** | **F** | **F** | F | T | F |
| M | T | T | F | T | F | F | T | T | T |

- $Full_d(J) = TFF \ (d(FTF, FFF) = 1)$

| | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | T | T | T | T | T | T | T | T | T |
| 2 | T | T | T | T | T | T | T | T | T |
| 3 | T | F | F | T | F | F | **T** | **F** | **T** |
| 4 | T | F | F | T | F | F | T | F | F |
| 5 | F | T | F | **F** | **F** | **F** | F | T | F |
| M | T | T | F | T | F | F | T | T | T |

- $Full_d(J) = TFF$ $(d(FTF, FFF) = 1)$
- $Output_d(J) = TTT$ $(d(TFF, TFT) = 0.9)$

| | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | T | T | T | T | T | T | T | T | T |
| 2 | T | T | T | T | T | T | T | T | T |
| 3 | T | F | F | T | F | F | **T** | **F** | **T** |
| 4 | T | F | F | T | F | F | T | F | F |
| 5 | F | T | F | **F** | **F** | **F** | F | T | F |
| M | T | T | F | T | F | F | T | T | T |

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| | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ | $a$ | $b$ | $a \wedge b$ |
|---|---|---|---|---|---|---|---|---|---|
| 1 | T | T | T | T | T | T | T | T | T |
| 2 | T | T | T | T | T | T | T | T | T |
| 3 | T | F | F | T | F | F | **T** | **F** | **T** |
| 4 | T | F | F | T | F | F | T | F | F |
| 5 | F | T | F | **F** | **F** | **F** | F | T | F |
| M | T | T | F | T | F | F | T | T | T |

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- $Prototype_d(J) = \{TTT, TFF\}\ (\sum_i d(J_i, TTT) = 3\sqrt{2}$, $\sum_i d(J_i, TFF) = 3\sqrt{2}$, $\sum_i d(J_i, FTF) = 4\sqrt{2}$, $\sum_i d(J_i, FFF) = 2\sqrt{3} + 3)$