Arrow's Theorem

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Notation

- X is a (finite or infinite) set of **alternatives** (or **candidates**.
- $N = \{1, \ldots, n\}$ is a set of **voters**
- **Preferences**: $\mathcal{P} = \{R \mid R \subseteq W \times W$ where R is reflexive, transitive and connected$\}$.
- Given $R \in \mathcal{P}$, let P be the **strict preference generated by** R: xPy iff xRy and not yRx (we write P_R if necessary)
- A **profile** is a tuple $(R_1, \ldots, R_n) \in \mathcal{P}^n$
- **Social Welfare Function**: $F : \mathcal{D} \to \mathcal{P}$ where $\mathcal{D} \subseteq \mathcal{P}^n$ is the domain.

Axioms

- **Universal Domain** (UD): The domain of F is $\mathcal{P}^n$ (i.e., $\mathcal{D} = \mathcal{P}^n$)
- **Independence of Irrelevant Alternatives** (IIA): F satisfies IIA provide for all profiles $\vec{R}, \vec{R}' \in \mathcal{D}$, if xR_iy iff xR'_iy for all $i \in N$, then $xF(\vec{R})y$ iff $xF(\vec{R}')y$
- **(weak) Pareto** (P): For all profiles $\vec{R} \in \mathcal{D}$, if xP_iy for all $i \in N$ then $xP_{F(\vec{R})}y$
- Agent i is a **dictator** for F provided for every preference profile and every pair $x, y \in X$, xP_iy implies $xP_{F(\vec{R})}y$.

Arrows Impossibility Theorem: Suppose that $|X| \geq 3$ and F satisfies UD, IIA and P. Then there is some $i \in N$ that is a dictator for F.