Liberalism against Populism

A Confrontation Between the Theory of Democracy and the Theory of Social Choice

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5

The Meaning of Social Choices

In Chapter 4 I showed that no method of voting could be said to amalgamate individual judgments truly and fairly because every method violates some reasonable canon of fairness and accuracy. All voting methods are therefore in some sense morally imperfect. Furthermore, these imperfect methods can produce different outcomes from the same profile of individual judgments. Hence it follows that sometimes—and usually we never know for sure just when—the social choice is as much an artifact of morally imperfect methods as it is of what people truly want. It is hard to have unbounded confidence in the *justice* of such results.

It is equally hard, as I will show in this chapter, to have unbounded confidence in the *meaning* of such results. Individual persons presumably can, if they think about it deeply enough, order their personal judgments transitively. Hence their valuations mean something, for they clearly indicate a hierarchy of preference that can guide action and choice in a sensible way. But the results of voting do not necessarily have this quality. It is instead the case that *no* method of voting can simultaneously satisfy several elementary conditions of fairness and also produce results that always satisfy elementary conditions of logical arrangement. Hence, not only may the results of voting fail to be fair, they may also fail to make sense. It is the latter possibility that will be analyzed in this chapter.

5.A. Arrow's Theorem

Kenneth Arrow published *Social Choice and Individual Values* in 1951. Although his theorem initially provoked some controversy among economists, its profound political significance was not immediately recognized by political scientists.[1] In the late 1960s, however, a wide variety of philosophers, economists, and political scientists began to appreciate how profoundly unsettling the theorem was and how deeply it called into question some conventionally accepted notions—not only about voting, the subject of this work, but also about the ontological validity of the concept of social welfare, a subject that, fortunately, we can leave to metaphysicians.

The essence of Arrow's theorem is that no method of amalgamating individual judgments can simultaneously satisfy some reasonable conditions of fairness on the method and a condition of logicality on the result. In a sense this theorem is a generalization of the paradox of voting (see section 1.H), for the theorem is the proposition that something like the paradox is possible in *any* fair system of amalgamating values. Thus the theorem is called the *General Possibility Theorem.*

To make the full meaning of Arrow's theorem clear, I will outline the situation and the conditions of fairness and of logicality that cannot simultaneously be satisfied.[2] The situation for amalgamation is:

1. There are n persons, $n \geq 2$, and n is finite. Difficulties comparable to the paradox of voting can arise in individuals who use several standards of judgment for choice. Our concern is, however, *social* choice, so we can ignore the Robinson Crusoe case.

2. There are three or more alternatives—that is, for the set $X = (x_1, \ldots, x_m)$, $m \geq 3$. Since transitivity or other conditions for logical choice are meaningless for fewer than three alternatives and since, indeed, simple majority decision produces a logical result on two alternatives, the conflict between fairness and logicality can only arise when $m \geq 3$.

3. Individuals are able to order the alternatives transitively: If $x\ R_i\ y$ and $y\ R_i\ z$, then $x\ R_i\ z$. If it is not assumed that individuals are able to be logical, then surely it is pointless to expect a group to produce logical results.

The conditions of fairness are:

1. *Universal admissibility of individual orderings (Condition U).* This is the requirement that the set, $\boldsymbol{D}$, includes all possible profiles, D, of individual orders, D_i. If each D_i is some permutation of possible orderings of X by preference and indifference, then this requirement is that individuals can choose any of the possible permutations. For ex-

ample, if $X = (x, y, z)$, the i
13 orderings:

1. $x\ y\ z$	7. $x\ (y\ z)$
2. $y\ z\ x$	8. $y\ (z\ x)$
3. $z\ x\ y$	9. $z\ (x\ y)$
4. $x\ z\ y$	
5. $z\ y\ x$	
6. $y\ x\ z$	

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4. *Unanimity or Pareto optimalit*
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ample, if $X = (x, y, z)$, the individual may choose any of the following 13 orderings:

1. $x\,y\,z$	7. $x\,(y\,z)$	10. $(x\,y)\,z$	13. $(x\,y\,z)$
2. $y\,z\,x$	8. $y\,(z\,x)$	11. $(y\,z)\,x$	
3. $z\,x\,y$	9. $z\,(x\,y)$	12. $(z\,x)\,y$	
4. $x\,z\,y$			
5. $z\,y\,x$			
6. $y\,x\,z$			(5-1)

The justification for this requirement is straightforward. If social outcomes are to be based exclusively on individual judgments—as seems implicit in any interpretation of democratic methods—then to restrict individual persons' judgments in any way means that the social outcome is based as much on the restriction as it is on individual judgments. Any rule or command that prohibits a person from choosing some preference order is morally unacceptable (or at least unfair) from the point of view of democracy.

2. *Monotonicity.* According to this condition, if a person raises the valuation of a winning alternative, it cannot become a loser; or, if a person lowers the valuation of a losing alternative, it cannot become a winner. The justification for monotonicity was discussed in section 3.B. Given the democratic intention that outcomes be based in some way on participation, it would be the utmost in perversity if the method of choice were to count individual judgments *negatively,* although, as I have shown, some real-world methods actually do so.
3. *Citizens' sovereignty or nonimposition.* Define a social choice as imposed if some alternative, x, is a winner for any set, D, of individual preferences. If x is always chosen, then what individuals want does not have anything to do with social choice. It might, for example, happen that x was everyone's least-liked alternative, yet an imposed choice of x would still select x. In such a situation, voters' judgments have nothing to do with the outcome and democratic participation is meaningless.
4. *Unanimity or Pareto optimality (Condition P).* This is the requirement that, if everyone prefers x to y, then the social choice function, F, does not choose y. (See Chapter 3, note 8, and Chapter 4, note 28.) This is the form in which monotonicity and citizens' sovereignty enter all proofs of Arrow's theorem. There are only two ways that a result contrary to unanimity could occur. One is that the system of amalgamation is not monotonic. Suppose in D' everybody but i prefers x to y

and $y\ P_i'\ x$. Then in D, i changes to $x\ P_i\ y$ so everybody has x preferred to y; but, if F is not monotonic, it may be that x does not belong to $F(\{x, y\}, D)$. The other way a violation of unanimity could occur is for F to impose y even though everybody prefers x to y. Thus the juncture of monotonicity and citizens' sovereignty implies Pareto optimality.

Many writers have interpreted the unanimity condition as purely technical—as, for example, in the discussion of the Schwartz method of completing the Condorcet rule (see section 4.C). But Pareto optimality takes on more force when it is recognized as the carrier of monotonicity and nonimposition, both of which have deep and obvious qualities of fairness.

5. *Independence from irrelevant alternatives (Condition I).* According to this requirement (defined in section 4.H), a method of amalgamation, F, picks the same alternative as the social choice every time F is applied to the same profile, D. Although some writers have regarded this condition simply as a requirement of technical efficiency, it actually has as much moral content as the other fairness conditions (see section 4.H). From the democratic point of view, one wants to base the outcome on the voters' judgments, but doing so is clearly impossible if the method of amalgamation gives different results from identical profiles. This might occur, for example, if choices among alternatives were made by some chance device. Then it is the device, not voters' judgments in D, that determines outcomes. Even if one constructs the device so that the chance of selecting an alternative is proportional in some way to the number of people desiring it (if, for example, two-thirds of the voters prefer x to y, then the device selects x with $p = 2/3$), still the expectation is that, of several chance selections, the device will choose x on p selections and y on $1 - p$ selections from the same profile, in clear violation of Condition I. In ancient Greece, election by lot was a useful method for anonymity; today it would be simply a way to by-pass voters' preferences. Another kind of arbitrariness prohibited by the independence condition is utilitarian voting. Based on interpersonal comparisons of distances on scales of unknown length, utilitarian voting gives advantages to persons with finer perception and broader horizons. Furthermore, independence prohibits the arbitrariness of the Borda count (see section 5.F).

6. *Nondictatorship (Condition D).* This is the requirement that there be no person, i, such that, whenever $x\ P_i\ y$, the social choice is x, regardless of the opinions of other persons. Since the whole idea of democracy is to avoid such situations, the moral significance of this condition is obvious.

Finally, the condition of logicality is that the social choice is a weak order, by which is meant that the set, X, is connected and its members can be *socially* ordered by the relation, R, which is the transitive social analogue of preference and indifference combined. (This relation, as in $x\ R\ y$, means that x is chosen over or at least tied with y.) In contrast to the previous discussion, in which the method of amalgamation or choice, F, simply selected an element from X, it is now assumed that F selects repeatedly from pairs in X to produce, by means of successive selections, a social order analogous to the individual orders, D_i. And it is the failure to produce such an order that constitutes a violation of the condition of logicality.[2]

Since an individual weak order or the relation R_i is often spoken of as individual rationality, social transitivity, or R, is sometimes spoken of as collective rationality—Arrow himself so described it. And failure to produce social transitivity can also be regarded as a kind of social irrationality.

Arrow's theorem, then, is that every possible method of amalgamation or choice that satisfies the fairness conditions fails to ensure a social ordering. And if society cannot, with fair methods, be certain to order its outcome, then it is not clear that we can know what the outcomes of a fair method mean. This conclusion appears to be devastating, for it consigns democratic outcomes—and hence the democratic method—to the world of arbitrary nonsense, at least some of the time.

Naturally there has been a variety of attempts to interpret and sidestep this conclusion. One line of inquiry is to raise doubts about its practical importance; another is to look for some theoretical adjustment that deprives the theorem of its force. The rest of this chapter is devoted to a survey of both branches of this huge and important literature, so that in Chapter 6 it will be possible to assess fully the political significance of Arrow's theorem.

I will begin with inquiries about the practical importance of the theorem. One such inquiry is an estimate of the expected frequency of profiles, D, that do not lead to a transitive order.

5.B. The Practical Relevance of Arrow's Theorem: The Frequency of Cycles

One meaning of Arrow's theorem is that, under any system of voting or amalgamation, instances of intransitive or cyclical outcomes can occur.

Since, by definition, no one of the alternatives in a cycle can beat all the others, there is no Condorcet winner among cycled alternatives. All cycled alternatives tie with respect to their position in a social arrangement in the sense that $x\ y\ z\ x$, $y\ z\ x\ y$, and $z\ x\ y\ z$ have equal claims to being the social arrangement. Borda voting similarly produces a direct tie among cycled alternatives. Hence a social arrangement is indeterminate when a cycle exists. When the arrangement is indeterminate, the actual choice is arbitrarily made. The selection is not determined by the preference of the voters. Rather it is determined by the power of some chooser to dominate the choice or to manipulate the process to his or her advantage. Every cycle thus represents the failure of the voting process. One way to inquire into the practical significance of Arrow's theorem is, therefore, to estimate how often cycles can occur.

For this estimate, a number of simplifying assumptions are necessary. For one thing, majority voting (rather than positional voting or any other kind of amalgamation) is always assumed. This assumption of course limits the interpretation severely. For another thing, only cycles that preclude a Condorcet winner are of interest. Voting may fail to produce a weak order in several ways:

1. With all three alternatives, there may be a cycle: $x\ R\ y\ R\ z\ R\ x$ or simply $x\ y\ z\ x$.
2. With four or more alternatives, there may be
   a. A Condorcet winner followed by a cycle: $w\ x\ y\ z\ x$
   b. A cycle among all alternatives: $w\ x\ y\ z\ w$; or intersecting cycles: $s\ t\ w\ x\ y\ z\ w\ v\ s$
   c. A cycle in which all members beat some other alternative: $x\ y\ z\ x\ w$

If one is interested in social welfare judgments involving an ordering of all alternatives, then all cycles are significant no matter where they occur. But if one is interested in picking out a social choice, as in the voting mechanisms discussed here, then the significant cases are only 1, 2(b), and 2(c), where there is no unique social choice. (These are often called *top cycles*.) Attempts to estimate the significance of Arrow's theorem by some sort of calculation have all been made from the point of view of social choice rather than welfare judgments and have therefore concerned the frequency of top cycles.

For Arrow's theorem, Condition U allows individuals to have any weak ordering, R_i, of preference and indifference, as in (5.1). Calculation is simpler, however, based on strong orders—that is, individual preference orders, P_i, with indifference not allowed.

With m alternatives, there are $m!$ (i.e., $1 \cdot 2 \cdot \ldots \cdot m$) such linear orders possible; and, when $m = 3$, these are:

$$x\,y\,z, \quad x\,z\,y, \quad y\,x\,z, \quad y\,z\,x, \quad z\,x\,y, \quad z\,y\,x$$

Each such order is a potential D_i. When each of n voters picks some (not necessarily different) D_i, a profile, D, is created. Since the first voter picks from $m!$ orders, the second from $m!, \ldots$, and the last from $m!$, the number of possible different profiles, D, is $(m!)^n$, which is the number of members of the set, $\boldsymbol{D}$, of all profiles, when voters have only strong orders.

A calculation that yields some estimate of the significance of cycles is the fraction, $p(n, m)$, of D in $\boldsymbol{D}$ without a Condorcet winner:

$$p(n, m) = \frac{\text{Number of } D \text{ without a Condorcet winner}}{(m!)^n}$$

If one assumes that each D is equally likely to occur (which implies also that, for each voter, the chance of picking some order is $1/m!$), then $p(n, m)$ is an a priori estimate of the probability of the occurrence of a top cycle. Several calculations have been made, as set forth in Display 5-1.[3] As is apparent from the Display, as the number of voters and alternatives increases, so do the number of profiles without a Condorcet winner. The calculation thereby implies that instances of the paradox of voting are very common. Most social choices are made from many alternatives (though often we do not realize this fact because the number has been winnowed down by various devices such as primary elections and committees that select alternatives for agendas) and by many people, so the calculations imply that Condorcet winners do not exist in almost all decisions.

But, of course, there are a number of reasons to believe that such calculations are meaningless. People do not choose an ordering with probability $1/m!$. Rather, at any particular moment, some orders are more likely to be chosen than others. The six strong orders over triples generate two cycles:

"Forward Cycle"	"Backward Cycle"
1. $x\,y\,z$	4. $x\,z\,y$
2. $y\,z\,x$	5. $z\,y\,x$
3. $z\,x\,y$	6. $y\,x\,z$

(5-2)

Display 5-1

Values of $p(n, m)$: Proportion of Possible Profiles Without a Condorcet Winner

	n = Number of Voters						
m = Number of Alternatives	3	5	7	9	11	. . .	Limit
3	.056	.069	.075	.078	.080		.088
4	.111	.139	.150	.156	.160		.176
5	.160	.200	.215				.251
6	.202						.315
Limit	1.000	1.000	1.000	1.000	1.000		1.000

The entry in the row for four alternatives and in the column for seven voters—namely, .150—is the ratio of the number of profiles without a Condorcet winner to the number of profiles possible when seven voters order four alternatives.

Cycles occur when voters concentrate on one or the other of these sets of three orders. But suppose voters are induced by, for example, political parties, to concentrate heavily on, say, (1), (2), and (5). Then there is no cycle. Furthermore, there is good reason to believe that debate and discussion do lead to such fundamental similarities of judgment. Calculations based on equiprobable choices very likely seriously overestimate the frequency of cycles in the natural world.

On the other hand, it is clear that one way to manipulate outcomes is to generate a cycle. Suppose that in Display 5-2 profile D exists and that person 2 realizes that his or her first choice, y, will lose to the Condorcet winner, x. Person 2 can at least prevent that outcome by generating a cycle (or a tie) by voting as if his or her preference were $y\ z\ x$ as in D'.

The tendency toward similarity may thus reduce the number $p(n, m)$, while the possibility of manipulation may increase the number. It seems to me that similarity probably reduces the number of profiles without Condorcet winners on issues that are not very important and that no one has a motive to manipulate, while the possibility of manipulation

Display 5-2

The Generation of a Cycle

D		D'	
D_1:	$x\ y\ z$	D'_1:	$x\ y\ z$
D_2:	$y\ x\ z$	D'_2:	$y\ z\ x$
D_3:	$z\ x\ y$	D'_3:	$z\ x\ y$

Note. Majoritarian ordering of D: $x\ P\ y\ P\ z$.

Note. Cycle in D' under majoritarian voting: $x\ P\ y\ P\ z\ P\ x$.

In D' person 2 has reversed z and x from D, thereby generating a cycle.

increases the number of such profiles on important issues, where the outcome is worth the time and effort of prospective losers to generate a top cycle. Neither of these influences appears in the calculations and thus renders them suspect from two opposite points of view.

5.C. The Practical Relevance of Arrow's Theorem: Conditions for Condorcet Winners

Another approach to estimating the practical significance of Arrow's theorem is to inquire into what kinds of profiles are certain to produce a Condorcet winner. As in the previous approach, only majoritarian voting is considered, which limits the relevance of the inquiry to the theorem but does say something about its practical effect on this kind of decision process. For example, as can be seen in Display 5-1, for $m = n = 3$, the number of elements of $\boldsymbol{D} = (m!)^n = 216$ and $p(n, m) = 12/216 = .056$. It is natural to look for the features that guarantee a Condorcet winner for 204 of the profiles in $\boldsymbol{D}$. If one can generalize about the sets of preference orders that produce these results, then it may be possible to estimate the practical significance of the theorem for majoritarian voting.

To give a simple example: If each voter chooses the same preference order, D_i, then under majoritarian rules the social order for the profile D will be identical with the chosen D_i, and the unique social choice will be the first alternative in that social order.

The goal of this approach is to identify kinds of preference orders, D_i, such that when the whole profile, D, is composed of such orders, then D will lead by majoritarian methods to a weak order and a Condorcet winner as a social outcome.

Even before Arrow's theorem was uttered, Duncan Black observed one such pattern of orders in D—namely, that the profile can be expressible as a set of single-peaked curves.[4] A preference order can be graphed as in Figure 5-1. On the vertical axis is measured the degree of preference from lowest at the origin to highest at the top. On the horizontal axis is placed some ordering of the alternatives in X, an ordering appropriately chosen to depict one particular D_i as a single-peaked curve. This is always possible if D_i is a strong order (with indifference not allowed). The general definition of single-peaked curves (with indifference permitted at the top) is, as displayed in Figure 5-2, reading from left to right: (1) always downward sloping, (2) always upward sloping, (3) sloping upward to a particular point and then sloping downward, (4) sloping upward to a plateau and then sloping downward, (5) horizontal and then downward sloping, (6) upward sloping and then horizontal. Curves that are *not* single-peaked are shown in Figure 5-3.

A profile, D, is single-peaked if some ordering of alternatives on the horizontal axis allows every D_i in D to be drawn as a single-peaked curve. As already observed, for a single D_i, it is always possible to find such an ordering. But with three or more D_i, an ordering that renders D_j single-peaked may preclude that D_k be single-peaked. Indeed, it is exactly when cycles exist that single-peakedness cannot be attained for D. In Figure 5-4 assume there are three persons who have chosen different preference orders in the forward cycle (5-2). Then *all* possible orderings of $X = (x, y, z)$ on the horizontal axis result in a set of curves that fail to be single-peaked, as in Figure 5-4a–4f, where the axes are *all* the $m!$ permutations of $\{x, y, z\}$. The same is true of the backward cycle. So to say a profile, D, is single-peaked is to say it does not admit of cycles. In general, if D is single-peaked, then:

1. If all D_i are strong orders and n is odd, the social ordering is strong.
2. If all D_i are weak orders, n is odd, and no D_i involves complete indifference over a triple, the social ordering is a weak order.[5]

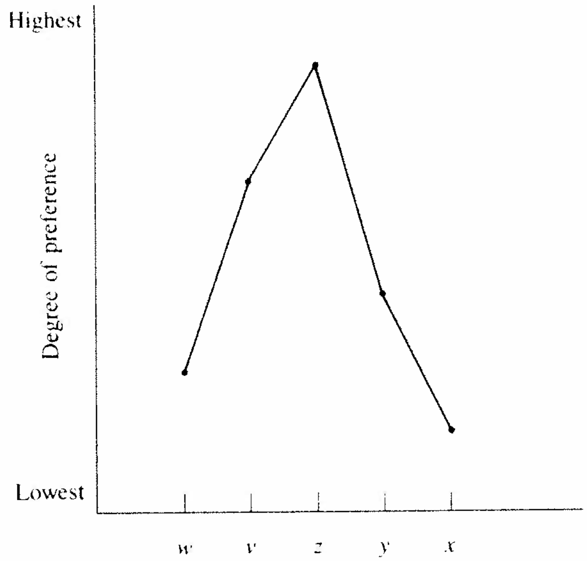

Figure 5-1 A single-peaked curve with the linear order *z v y w x*.

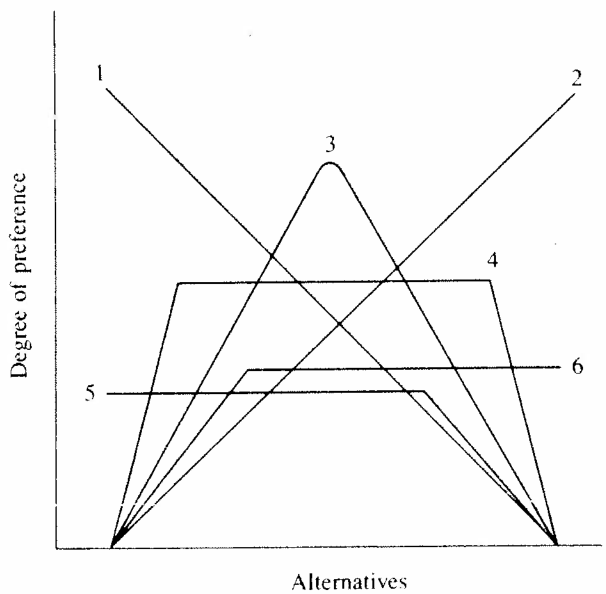

Figure 5-2 Single-peaked curves.

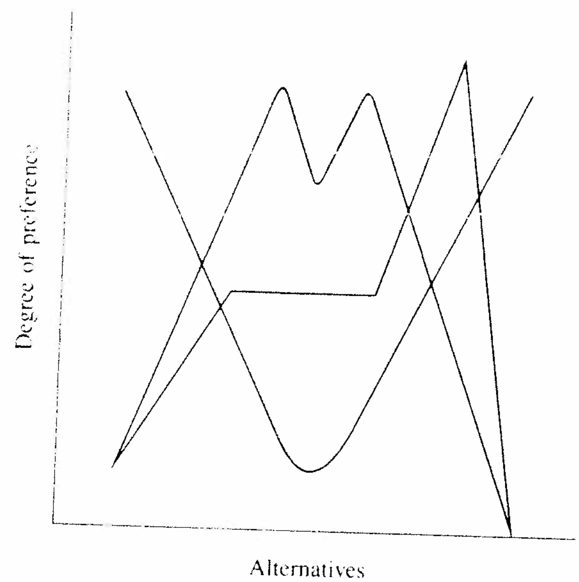

Figure 5-3 Non-single-peaked curves.

So single-peakedness implies transitivity and hence ensures the existence of a Condorcet winner.

It is furthermore a remarkable fact that, if D is single-peaked and n is odd, the Condorcet winner is immediately identifiable as the alternative on the horizontal axis beneath the median peak.[6] (If n is even, the winner is some alternative between the $n/2^{\text{th}}$ peak and the $(n/2) + 1^{\text{th}}$ peak, if such an alternative exists. And, if none exists, the alternatives at these peaks tie.) In Figure 5-5, with five peaks, the alternative beneath the median peak (3) is identified as x_{med}. If x_{med} is put against some alternative to its left, say x_1, then x_{med} wins because a majority consisting of voters 3, 4, and 5 prefer x_{med} to x_1 (that is, their curves are upward sloping from x_1 to x_{med}). Similarly, x_{med} can beat any alternative to its right, say x_4, with a majority consisting of voters 1, 2, and 3, whose curves are downward sloping from x_{med} to x_4, which means they prefer x_{med} to x_4. Hence x_{med} can beat anything to its right or left and is a Condorcet winner.

Single-peakedness is important because it has an obvious political interpretation. Assuming a single political dimension, the fact that a profile, D, is single-peaked means the voters have a common view of the political situation, although they may differ widely on their judgments. Person i may choose $D_i = x\ y\ z$, and person j may choose $D_j = z\ y\ x$; yet

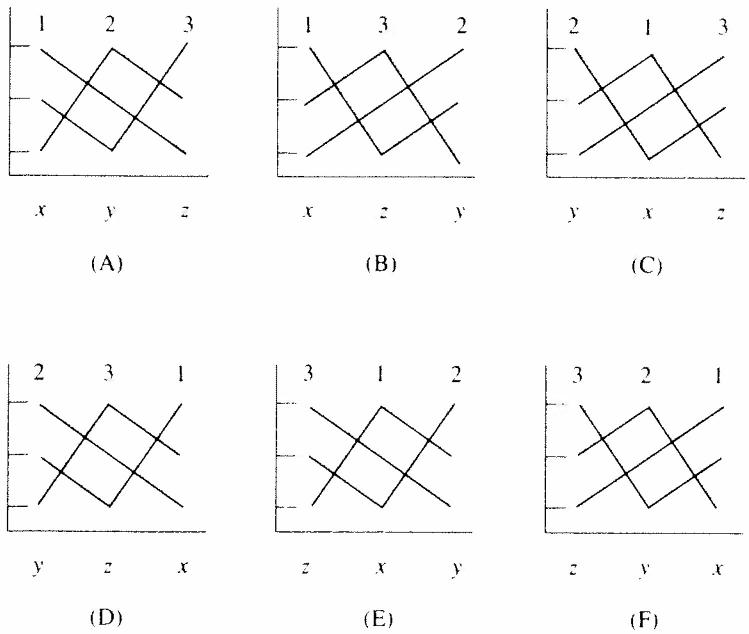

Figure 5-4 Non-single-peakedness for the forward cycle.

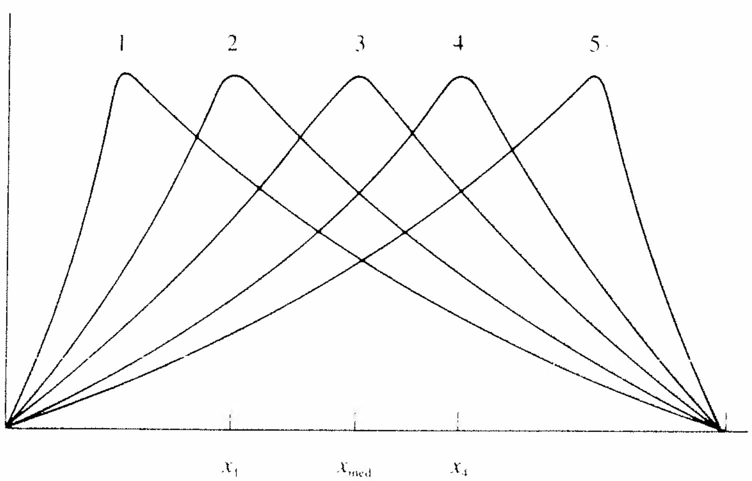

Figure 5-5 Single-peaked curves with Condorcet winner.

they agree that x is at one end of the scale, z at the other, and y in the middle, which means they agree entirely on how the political spectrum is arranged. This kind of agreement is precisely what is lacking in a cycle, where voters disagree not only about the merits of alternatives but even about where alternatives are on the political dimension.

If, by reason of discussion, debate, civic education, and political socialization, voters have a common view of the political dimension (as evidenced by single-peakedness), then a transitive outcome is guaranteed. So if a society is homogeneous in this sense, there will typically be Condorcet winners, at least on issues of *minor* importance. This fact will not prevent civil war, but it will at least ensure that the civil war makes sense.

A number of other kinds of restrictions on preference orders, D_i, that guarantee that D will produce a transitive outcome have been identified. Like single-peakedness they minimize disagreement over the dimensions of judgment. Consider "value-restrictedness," which is an obvious development from the forward and backward cycles of (5-2). One property of those cycles (observable by inspection) is that each alternative in X appears in first place in some D_i, in second place in another, and in third place in a third. So, if, for strong orders in D_i, some alternative is never first in a D_i, or never second, or never last—if, in short, an alternative is "value-restricted"—then no cycle can occur and transitivity is guaranteed.

A number of other such provisions for transitivity have been identified. They have been exhaustively analyzed by Peter Fishburn.[7] They are important because they indicate that quite a wide variety of rather mild agreement about the issue dimension guarantees a Condorcet winner. Furthermore, not all voters need display the agreement to obtain the guarantee. Richard Niemi has shown that the probabilities of the occurrence of top cycles, by calculations similar to those set forth in Display 5-1, reduce to tiny proportions (e.g., .02 to .04) when as few as three-fourths of 45 or 95 voters agree on the issue dimension while disagreeing on orders.[8] This result implies that agreement about dimensions probably renders uncontrived cyclical outcomes quite rare. So I conclude that, because of agreement on an issue dimension, intransitivities only occasionally render decisions by majoritarian methods meaningless, at least for somewhat homogeneous groups and at least when the subjects for decision are *not* politically important. When, on the other hand, subjects are politically important enough to justify the energy and expense of contriving cycles, Arrow's result is of great practical significance. It suggests that, on the very most important subjects, cycles may render social outcomes meaningless.

5.D. The Theoretical Invulnerability of Arrow's Theorem: Independence

Assuming that the practical significance of Arrow's theorem increases with the political importance of the subject for decision, it is then reasonable to inquire whether the theorem is too demanding. Does it overstate the case by stressing the possibility of intransitivity and its consequent incoherence when perhaps this is too extreme an interpretation?

To weaken the force of Arrow's theorem, it is necessary to question the conditions of either fairness or logicality. Most of the fairness conditions seem intuitively reasonable—at least to people in Western culture—so most of the attack has been focused on logicality. One fairness condition, independence, has, however, often been regarded as too strong.

The independence condition has at least three consequences:

1. It prohibits utilitarian methods of choice (for reasons discussed in section 4.I).
2. It prohibits arbitrariness in vote-counting, such as lotteries or methods that work in different ways at different times.
3. It prohibits, when choosing among alternatives in a set S, which is included in X, reference to judgments on alternatives in $X - S$.

It seems to me that one can defend the independence condition for each of these consequences. As for consequence 1, since interpersonal comparisons of utility have no clear meaning, the prohibition of utilitarian methods seems quite defensible, although a weaker form of Condition I might accomplish the same result. With respect to consequence 2, earlier in this chapter it was shown that arbitrary counting is just as unfair as violations of Conditions U, P, and D. It is difficult to imagine that any weaker form of Condition I would accomplish what I does, because the arbitrariness must be prohibited for any set.

Most attention has been given to consequence 3, because many people believe that judgments on alternatives in $X - S$ are germane to judgments on S itself.[9] In a presidential preference primary, for example, choice among several candidates may depend on judgments of still other candidates. For example, in the 1976 Democratic primaries, in thinking about a decision between Carter and Udall as if they covered the whole spectrum of party ideology, a mildly left-of-center voter might prefer Udall. But if the voter thought about Jackson also, so that Udall appeared as an extremist, that same voter might have preferred Carter to Udall. So "irrelevant" alternatives (here, Jackson) may really be "relevant."

The question is whether there should be some formal way to allow judgments on the "irrelevant" alternatives to enter into the choice. And the difficulty in answering is: How can one decide which nonentered candidates are relevant? Why not allow consideration of still other, even a hundred, irrelevant alternatives? But if no irrelevant alternatives are considered, then y might beat x; but with such consideration, if there is no Condorcet winner, x might beat y. Thus meaning and coherence depend on variability in the voting situations (on the size, that is, of X and S) as much as on voters' judgment.

There seems, unfortunately, no wholly defensible method to decide on degrees of irrelevance.[10] In the absence of such a method, Condition I seems at least moderately defensible. Furthermore, while some might argue about the desirability of consequence 3, Condition I seems necessary because consequence 2 is indispensable for fair decision.

5.E. The Theoretical Invulnerability of Arrow's Theorem: Transitivity

If the fairness conditions survive, then the only condition left to attack is transitivity. The sharpest attack is to assert that transitivity is a property of humans, not of groups. Hence the individual relation, R_i, should be transitive, but it is simple anthropomorphism to ask that the social relation, R, be transitive also.[11] Still, there is some reason to seek transitivity for outcomes.

Without transitivity, there is no order; and without order, there is no coherence. Social outcomes may in fact be meaningless, but one would like to obtain as much meaning as possible from social decisions. So the obvious question is: Can one, by modifying the definition of coherence, obtain some lesser coherence compatible with fairness? Unfortunately, the answer is mainly negative.

The social relation, R, which generates a weak order in Arrow's logicality condition, combines social preference, P, and social indifference, I. And R is useful for the purpose Arrow had in mind—namely, social judgments involving comparisons and ordering of all feasible social policies, such as distributions of income. Suppose, however, that one does not require quite so general a result. For purposes of making a social choice, which is the interest in this book, one does not need to impose a complete order on the whole set X merely to find a best alternative in X.

We can think of a best element in X as one that is chosen over or tied with every other alternative.[12] The best alternative is then the choice from X or $C(X)$.[13]

A requirement, weaker than transitivity, that nevertheless ensures the existence of one best alternative is *quasi-transitivity*—that is, the transitivity of P, but not of R or I. This means that, if $x\,P\,y$ and $y\,P\,z$, then $x\,P\,z$; but if the antecedent does not hold (e.g., if $x\,I\,y$), then the consequent need not hold either. For example, quasi-transitivity allows (as in note 12) $y\,P\,z$, $z\,I\,x$, and $x\,I\,y$, which is clearly intransitive in both R and I, although it is enough to establish that the choice from $X = (x, y, z)$ is $C(X) = (x, y)$.

Another, even weaker requirement for a choice, is *acyclicity,* which is the requirement that alternatives in X can be arranged so that there is no cycle.[14] It turns out that, by using the logical requirement of acyclicity rather than transitivity, it is possible to find social choice that satisfies all of Arrow's fairness conditions as well as the revised condition of logicality. A. K. Sen offers an example of such a method: For a set $X = (a, b, \ldots,)$, let a be chosen for $C(X)$ over b if everybody prefers a to b and let a and b both be chosen if not everybody prefers a to b or b to a.[15] This rule satisfies Condition U because all individual orders are allowed. It satisfies Condition P because it is based on the principle of unanimity. It satisfies Condition I because the choice between any pair depends only on individual preferences on that pair, and it satisfies Condition D because the only way a can be better socially than b is for everyone to prefer a to b. Finally, it is always acyclic. So even if one cannot guarantee an order with fair procedure, it appears that one can at least guarantee a best choice.

Unfortunately, however, something very much like dictatorship is required to guarantee quasi-transitivity or acyclicity. Quasi-transitive social outcomes can be guaranteed only if there is an oligarchy.[16] (An *oligarchy* is a subset of choosers who, if the members agree, can impose a choice, or, if they do not agree, enables all members individually to veto the choice.) If one modifies Condition D from no dictator to no vetoer, then even a quasi-transitive social outcome cannot be guaranteed.[17] As for acyclicity, Donald Brown has shown that acyclic choice requires a "collegium" such that alternative a is chosen over b if and only if the whole collegium and some other persons prefer a to b. Thus, although a collegium cannot unilaterally impose a choice, unlike an oligarchy it can always at least veto.[18]

Furthermore, if one strengthens Arrow's conditions just a little bit by requiring not just the monotonicity that enters into Condition P, but a condition of positive responsiveness (Condition PR), then quasi-transitivity again involves dictatorship. (Monotonicity requires merely that, if a voter raises her or his valuation of an alternative, the social valuation does not go down. In contrast, positive responsiveness requires that, if a voter raises her or his valuation, society does so as well, if that is possible.) It is then the case that any quasi-transitive social result that satisfies Conditions U, P, I, and PR must violate Condition D if there are three alternatives; and, furthermore, someone must have a veto if there are four or more alternatives.[19]

Weakening transitivity into some logical condition that requires only a social choice but not a full ordering does not gain very much. This brief survey indicates there is a *family* of possibility theorems of which Arrow's theorem is a special case. And in the whole family there is still some kind of serious conflict between conditions of fairness and a condition of logicality. In general, the only effective way to guarantee consistency in social outcomes is to require some kind of concentration of power in society—a dictator, an oligarchy, or a collegium. So fairness and social rationality seem jointly impossible, which implies that fairness and meaning in the content of social decisions are sometimes incompatible.

5.F. The Theoretical Invulnerability of Arrow's Theorem: Conditions on Social Choice

Of course, one can abandon entirely the effort to guarantee some kind of ordering for social "rationality," whether it be transitivity or merely acyclicity. One can simply provide that a social choice is made and impose no kind of ordering condition. The reason, however, that transitivity or even less restrictive ordering conditions are attractive is that they often forestall manipulation by some participants either of agenda or of sets of alternatives to obtain outcomes advantageous to the manipulator. As Arrow remarked at the conclusion of the revised edition of *Social Choice and Individual Values,* "the importance of the transitivity condition" involves "the independence of the final choice from the path to it."[20] "Transitivity," he said, "will ensure this independence," thereby ensuring also that the preferences of the participants (rather than the form of or manipulation of the social choice mechanism) determine the outcome. He went on to point out that both Robert Dahl and I had described ways in which intransitive social mechanisms had produced "unsatisfactory" results. So Arrow concluded that "collective rationality" was not merely an "illegitimate" anthropomorphism, "but an important attribute of a genuine democratic system." Consequently, if one gives up on social transitivity or some weaker form of ordering, one is in effect abandoning the effort to ensure socially satisfactory outcomes.

To ensure satisfactory outcomes without imposing an anthropomorphic collective rationality, one might impose consistency conditions on the social choice mechanism—conditions that could have the same effect of forestalling manipulation that transitivity does, but that would not attribute to society the ability to order possessed only by persons. Hopefully, one would thereby avoid all the problems of the possibility theorems put forth by Arrow and his successors. Unfortunately, however, it turns out that these consistency conditions also cannot be satisfied by social choice mechanisms that satisfy the fairness conditions. Consequently, although the problem can be elegantly restated in terms of choice rather than ordering, the main defect of the methods of amalgamation is unaffected by the new language. Just to say, for example, that $x\,P_1\,y$ and $x\,P_2\,y$ lead to $C(x, y) = x$ rather than to say that they lead to $x\,P\,y$ does not solve the problem of amalgamation. Some kind of inconsistency is ineradicable.

Consistency requirements on choice have been discussed in two quite different ways, which, however, turn out to be substantially equivalent in this context. I will discuss both ways here, despite their equivalence, because their verbal rationales are complementary.

A. K. Sen and subsequently many others have imposed on social choice conditions of logicality that were originally devised as standards for individual choice behavior. This procedure has the advantage of relating consistency in groups to consistency in persons, but it is subject to the same charge of anthropomorphism that was leveled against the use of ordering conditions. Charles Plott, however, has devised a consistency condition for social choice itself, one that could not easily be applied to persons but captures the spirit of Arrow's insistence that the final choice ought to be independent of the path to it. It is interesting and remarkable that Sen's and Plott's conditions turn out to be closely related and almost equivalent.[21]

Looking first at Sen's conditions, let S and T be sets of alternatives in $X = (x_1, x_2, \ldots, x_m)$ and let S be a subset of T. Sen's conditions are restrictions on the choice sets from these two sets of alternatives, $C(S)$ and $C(T)$:

1. *Property* α: For sets S and T, with S a subset of T, if x is in both $C(T)$ and S, then x is in $C(S)$.
2. *Property* $\beta+$: For sets S and T, with S a subset of T, if x is in $C(S)$ and y is in S, then, if y is in $C(T)$, so also is x in $C(T)$.

The meaning of these conditions is easily explained: Property α requires that, if the choice from the larger set is in the smaller set, then it is in the choice from the smaller set as well.

To see the rationale of α, consider a violation of it: A diner chooses among three items on a menu, beef (B), chicken (C), and fish (F), which are the set $\{B, C, F\}$. The diner chooses beef (B); then the restaurant runs out of fish (F). The new menu is the set $\{B, C\}$, whereupon the diner chooses chicken (C) in violation both of property α and of apparent good sense.[22]

Property α guarantees consistency in choices as the number of alternatives is *contracted* because in going from T to S the choice does not change if it is in both sets. Property $\beta+$, on the other hand, guarantees consistency in choices as the number of alternatives is *expanded*. It requires that, if any element in the smaller set is the choice from the larger set, then all choices from the smaller set are choices from the larger set. Thus, in going from S to T, if any choices from S continue to be chosen from the larger set, all such choices continue to be chosen.

The rationale of $\beta+$ can be appreciated from a violation of it: For a seminar with students $S = (a, b, c, d)$, a teacher ranks d best. Then another student enrolls making $T = (a, b, c, d, e)$, whereupon the teacher ranks c best. Doubtless student d discerns an inconsistency and believes that if he is the best or among the best in S and if some other member of S is best in T, then he (d) ought to be among the best in T also.

As I have already noted, property α and property $\beta+$ apply as well to individuals as to society. Plott, however, attempted to embody Arrow's notion of "independence of the final result from the path to it" directly in a condition on *social* choice. Plott justified his condition, which, appropriately, he called "path independence," thus:

> *the process of choosing, from a dynamic point of view, frequently proceeds in a type of "divide and conquer" manner. The alternatives are "split up" into smaller sets, a choice is made over each of these sets, the chosen elements are collected, and then a choice is made from them. Path independence, in this case, would mean that the final result would be independent of the way the alternatives were initially divided up for consideration.*[23]

The definition of *path independence* is that, for any pair of sets S and T, the choice from the union of the sets is the same as the choice from the union of the separate choices from each set.[24] Manifestly, if S and T are any ways of breaking up the set of alternatives, X, then to equate the choices from their union with the choice from the union of their choice sets is to say that it makes no difference to the final outcome how X is divided up for choosing.

Path independence (*PI*) can be broken up into two parts—*PI** and **PI*:

1. *PI** is the condition that the choice from the union of S and T be included in or equivalent to the choice from the union of their choice sets.
2. **PI* is the converse of *PI**. Specifically, **PI* is the condition that the choice from the union of the choice sets of S and T be included in or equivalent to the choice from the union of S and T.[25]

It is a remarkable and important fact that *PI** is exactly equivalent to property α.[26] Furthermore, a choice function satisfying property $\beta+$ satisfies **PI*, so that, though not equivalent, **PI* is implied by $\beta+$.[27]

These standards of consistency in choice turn out to be quite similar in effect to ordering principles.[28] Although property α does not guarantee transitivity, it does guarantee acyclicity in choices from X. So also, therefore, do *PI* and *PI**. Consequently, social choice methods satisfying these conditions are dictatorial or oligarchic, just as are those satisfying ordering principles.

On the other hand, property $\beta+$ does not guarantee even acyclicity when choices from X are made in a series of pairwise comparisons. Consequently, methods satisfying $\beta+$ and **PI* do not imply dictatorship or oligarchy or any other kind of concentration of power. If one is willing to give up consistency in contracting alternatives—and this is quite a bit to give up—then reliance on simple consistency in expanding alternatives might be a way around all the difficulties discovered by Arrow. Unfortunately, however, methods of choice satisfying $\beta+$ and **PI* violate another fairness condition—namely, unanimity or Pareto optimality.[29]

Suppose a choice is to be made by three people with these preference orders: (1) $x\ y\ z\ w$, (2) $y\ z\ w\ x$, (3) $z\ w\ x\ y$. This leads to a cycle in simple majority rule, $x\ P\ y\ P\ z\ P\ w\ P\ x$, so that the choice set is all the alternatives: $C(w, x, y, z) = (w, x, y, z)$. But everyone prefers z to w, although there is a path by which w can be chosen. Let $S_1 = (y, z)$ and $C(S_1) = y$; $S_2 = (x, y)$ and $C(S_2) = x$; $S_3 = (x, w)$ and $C(S_3) = w$. Using S_1 at step 1, S_2 at step 2, and S_3 at step 3, w is selected even though z, eliminated at step 1, is unanimously preferred to w. This result is generalized by Ferejohn and Grether.[30] It tells us that, even if we rely solely on an expansion consistency condition and thus avoid concentrations of power, we still do not achieve fairness. So, in a quite different way, we are back where we began. Nothing has been gained except an elegant formalism that avoids anthropomorphizing society.

5.G. The Absence of Meaning

The main thrust of Arrow's theorem and all the associated literature is that there is an unresolvable tension between logicality and fairness. To guarantee an ordering or a consistent path, independent choice requires that there be some sort of concentration of power (dictators, oligarchies, or collegia of vetoers) in sharp conflict with democratic ideals. Even the weakest sort of consistency ($\beta+$ or $*PI$) involves a conflict with unanimity, which is also an elementary condition of fairness.

These conflicts have been investigated in great detail, especially in the last decade; but no adequate resolution of the tension has been discovered, and it appears quite unlikely that any will be. The unavoidable inference is, therefore, that, so long as a society preserves democratic institutions, its members can expect that some of their social choices will be unordered or inconsistent. And when this is true, no meaningful choice can be made. If y is in fact chosen—given the mechanism of choice and the profile of individual valuations—then to say that x is best or right or more desired is probably false. But it would also be equally false to say that y is best or right or most desired. And in that sense, the choice lacks meaning. The consequence of this defect will be explored in the ensuing chapters.