Logical Systems

Course Notes for M682
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Contents

About These Notes

These course notes covered material taught in my Fall 1997 class on Model Theory, and some of the material in its continuation class given the following semester. In thinking about the topic of model theory, I was lead to turn away from the standard presentations for several reasons:

1. The main thrust of contemporary model theory is to return to classical mathematics with worked-out methods and results originating in mathematical logic. Although this goal is quite admirable, it would not be possible here at Indiana University to go very far in that direction. Not only are there no follow-up courses, but we also see very few talks in the classical areas of model theory. So preparing people for (say) stability theory seemed a bit odd. (I am reminded of my years as a mediocre violin student, struggling through concertos that I clearly would never play with any orchestra.)
2. On the other hand, it seems to me that there is a wealth of topics that have much the same spirit as classical model theory: the concern with classes of structures, logics, completeness theorems, correspondences between syntax and semantics. This kind of work usually gets done in computer science or philosophy classes, but the mathematics involved is significant and interesting. In addition, this general area is one that our students will be exposed to, both in classes and in seminar talks. (Later I took up several styles of folk music and enjoy fiddling.)

As a result, I decided to re-fashion the Model Theory course into something else. Exactly what wasn't so clear, and the titles that came to mind were thigns like: Old-Fashioned Model Theory, Applied Model Theory, and even Everything But Model Theory. Right now, the best title seems to be Structures of Applied Logic.

The specific goals that I have in mind for such a class are as follows:

1. To give students lots of experience with completeness theorems.
2. To introduce the basics of modal logic, dynamic logic, equational logic, and other logical systems.
3. To use the experiences with those logical systems as a way into category theory, then to teach just the basics of that subject and use it in the remainder of the course.
4. To use dynamic logic as a starting point for looking at dynamic logics in natural language semantics, as well as in logics of knowledge.
5. To teach aspects of classical model theory in the direction of finite model theory, such as game-theoretic methods and zero-one laws.
6. To introduce *applied logic* by stressing how topics in the course relate to areas of ongoing research, and also to teach something about the process of making mathematical models and how logic relates to that.

Most of these goals are not addressed in any way by the standard courses in model theory. Logic students typically see two completeness proofs: for propositional logic and for first-order logic. They also do not see most of the particular logics covered in the class, and some of the work on first-order logic itself is probably too new to have found its way into the standard curriculum.

The notes at this point cover the first semester of the course. Actually, they were revised a bit after the semester was over, and a few sections were added. Also, some of the longer chapters were split into shorter ones. My experience has been that the class covered about ten pages of text each week. Also, I assigned about half of the homework problems every week or two.

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Natural Logic

For most of its history, logic was concerned with *syllogisms*. One simple example, perhaps the most famous one, is:

$$\frac{\begin{array}{l}\textit{All men are mortal.}\\ \textit{Socrates is a man.}\end{array}}{\textit{Socrates is mortal.}}$$

The idea is that the sentences above the line should *semantically entail* the one below the line. Specifically, in every model in which *All men are mortal* and *Socrates is a man* are true, it must be the case that *Socrates is mortal* is also true. We have to say what *semantically entail* means, and this will come in due course. The matter might be clearer with another example. Suppose someone accepts as true the following sentences:

1. *All raredos are slonados.*
2. *John is a raredo.*
3. *Mary is an alphatoric.*
4. *John is Mary.*

Then they should also accept as true the conclusion *Some slonado is an alphatoric*. We have purposely used nonsense words here; the whole point is that the inference depends only on the form of the argument. In this case, the key elements of the form include the worlds *All* and *Some*, and two different uses of *is*. So rather than deal with actual words, we instead consider things schematically. Assuming

1. *All* X *are* Y.
2. J *is an* X.
3. M *is a* Z.
4. J *is* M.

We should accept *Some* Y *is a* Z.

Natural Logic is concerned with a mathematical model of these kinds of inferences. We'd like to know when a given sentence would be a good conclusion to a given argument, and when it would not. (Incidentally, the same question arises for the traditional syllogisms. But those are three-line arguments, and the question of which syllogisms are intuitively valid is a special case of the question we ask in this chapter.)

To make life simple here, we are only going to consider a few very restricted forms of English sentences. These are the ones we list in Figure 1.1 below. We are going to be fairly strict in restricting attention to just sentences of those forms. The only deviation is that we write *a* or *an* following the usual uses in English, as we did in (2) and (3) just above.

To define *validity of an argument*, we first say what the semantics of individual sentences is. This again is given in Figure 1.1. Here is an example. Let $\mathcal{M}$ be the set $\{1, 2, 3, 4, 5\}$. Let $[\![X]\!] = \{1, 3, 4\}$, $[\![Y]\!] = \{1, 5\}$, $[\![Z]\!] = \{5\}$, $[\![J]\!] = 3$, and $[\![M]\!] = 1$. Then $[\![J \textit{ is an } X]\!] = \mathsf{T}$, but for all three other assumptions R, $[\![R]\!] = \mathsf{F}$.

Let Γ be our set of four assumptions, and let S be *Some* Y *is a* Z. Then $\Gamma \models S$ means that all models $\mathcal{M}$ satisfying all sentences in Γ also satisfy S. The example in the previous paragraph did not satisfy all sentences in Γ. But if we change $[\![Y]\!]$ to $\{1, 2, 3, 4\}$, $[\![Z]\!]$ to $\{5\}$, and $[\![M]\!]$ to 3, we would satisfy all the assumptions in Γ. We would also satisfy S.

This last model is just one example, and we want to know whether *all* models of Γ are models of S. The idea is that we cannot determine this by looking at examples; there are "too many". Besides, the reason that someone would accept S on the basis of Γ does not have so much to do with examples as with *reasons*. This is what our proof system intends to model. The second part of Figure 1.1 defines proof trees. For the same Γ and S, here is a proof tree which shows that $\Gamma \vdash S$:

$$\dfrac{\dfrac{\textit{All } X \textit{ are } Y \quad J \textit{ is an } X}{J \textit{ is a } Y} \quad \dfrac{M \textit{ is a } Z \quad J \textit{ is } M}{J \textit{ is an } Z}}{\textit{Some } Y \textit{ is a } Z} \tag{1.1}$$

The idea is that what counts as a proof tree is an entirely *syntactic* matter: the meaning of the English words such *All* and *Some* is completely irrelevant. A computer, or a speaker of some other language, could *check* whether a given labeled tree obeyed the conditions in the definition.

This is probably a good place to mention the ways in which we are (and are not) strict with rules. In writing this chapter, I have tried to be completely strict about the left-right match in rules. So since one of

Syntax: We start with *variables* $X, Y, \ldots$, representing plural common nouns of English. We also also *names* J, M, ... Then we consider sentences S of the following very restricted forms:
All X are X, *Some X are X*, *No X are X*, *J is an X*, *J is M*.

Semantics: One starts with a set $\mathcal{M}$, a subset $[\![X]\!] \subseteq \mathcal{M}$ for each variable X, and an element $[\![J]\!] \in \mathcal{M}$ for each name J. This gives a *model* $\mathcal{M} = (\mathcal{M}, [\![\ ]\!])$.
We then assign a semantics $[\![S]\!] \in \{\mathsf{T}, \mathsf{F}\}$ to the sentence S in a model $\mathcal{M}$, as follows:

$$\begin{array}{lll} [\![\text{All } X \text{ are } Y]\!] = \mathsf{T} & \text{iff} & [\![X]\!] \subseteq [\![Y]\!] \\ [\![\text{Some } X \text{ are } Y]\!] = \mathsf{T} & \text{iff} & [\![X]\!] \cap [\![Y]\!] \neq \emptyset \\ [\![\text{No } X \text{ are } Y]\!] = \mathsf{T} & \text{iff} & [\![X]\!] \cap [\![Y]\!] = \emptyset \\ [\![J \text{ is an } X]\!] = \mathsf{T} & \text{iff} & [\![J]\!] \in [\![X]\!] \\ [\![J \text{ is } M]\!] = \mathsf{T} & \text{iff} & [\![J]\!] = [\![M]\!] \end{array}$$

We write $\mathcal{M} \models S$ if $[\![S]\!] = \mathsf{T}$. And if Γ is a set of sentences, then we write $\mathcal{M} \models \Gamma$ to mean that $\mathcal{M} \models S$ for all $S \in \Gamma$.

Main semantic definition: $\Gamma \models S$ means that every model which makes all sentences in the set Γ true also makes S true. We say Γ *semantically implies* S.

Inference rules of the logical system: The complete set of rules may be found in Figure 1.7 at the end of this chapter. Rules for various fragments are presented as needed.

Proof trees: A *proof tree over* Γ is a finite tree whose nodes are labeled with sentences in our fragment, with the additional property that each node is either an element of Γ or comes from its parent(s) by an application of one of the rules.

Formal proofs: $\Gamma \vdash S$ means that there is a proof tree over Γ whose root is labeled S. We say Γ *proves*, or *derives*, S.

FIGURE 1.1 The main definitions in this chapter.

the rules is

$$\frac{M \textit{ is an } X \quad J \textit{ is } M}{J \textit{ is an } X} \tag{1.2}$$

I would not make a tree like

$$\frac{J \textit{ is } M \quad M \textit{ is an } X}{J \textit{ is an } X}$$

This kind of strictness is not essential, however. You should feel free to loosen it. It is more important to note that the rules are to be read *schematically*: one is allowed to substitute other variables or names for the ones in the statement of the rules. We already did this in (1.1): into the actual rule in (1.2) we substituted Z for X (and kept the other variables as they are in (1.2)).

Here is another example, chosen to make some different points: Let Γ be

$$\{\textit{All } A \textit{ are } B, \textit{All } Q \textit{ are } A, \textit{All } B \textit{ are } D, \textit{All } C \textit{ are } D, \textit{All } A \textit{ are } Q\}$$

Let S be *All* Q *are* D. Here is a proof tree showing that $\Gamma \vdash S$:

$$\frac{\textit{All } Q \textit{ are } A \qquad \dfrac{\dfrac{\textit{All } A \textit{ are } B \quad \textit{All } B \textit{ are } B}{\textit{All } A \textit{ are } B} \qquad \textit{All } B \textit{ are } D}{\textit{All } A \textit{ are } D}}{\textit{All } Q \textit{ are } D}$$

Note that all of the leaves belong to Γ except for one that is *All* B *are* B. Note also that some elements of Γ are not used as leaves. This is permitted according to our definition. The proof tree above shows that $\Gamma \vdash S$. Also, there is a smaller proof tree that does this, since the use of *All* B *are* B is not really needed. (The reason why we allow leaves to be labeled like this is so that that we can have one-element trees labeled with sentences of the form *All* A *are* A.)

The main theoretical question for this chapter is: what is the relation the semantic notion $\Gamma \models S$ with the proof-theoretic notion $\Gamma \vdash S$? This kind of question will present itself for all of the logical systems in this course. Probably the first piece of work for you is to *be sure you understand the question.*

Lemma 1.1 (Soundness) *If* $\Gamma \vdash S$, *then* $\Gamma \models S$.

Proof By *strong induction on the number of nodes* of proof trees $\mathcal{T}$ over Γ. If $\mathcal{T}$ is a tree with one node, let S be the label. Either S belongs to Γ, or else S is of the form *All* A *are* A or J *is* J. In the first case, every model satisfying every sentence in Γ clearly satisfies S, as S belongs to Γ. And in the second case, every model whatsoever satisfies S.

Let's suppose that we know our result for all proof trees over Γ with fewer than n nodes, and let $\mathcal{T}$ be a proof tree over Γ with n nodes. The argument breaks into cases depending on which rule is used at the root. Suppose the root and its parents are labeled

$$\frac{\textit{All } X \textit{ are } Z \quad \textit{All } Z \textit{ are } Y}{\textit{All } X \textit{ are } Y}$$

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be the subtrees ending at *All* X *are* Y and *All* Y *are* Z. Then $\mathcal{T}_1$ and $\mathcal{T}_2$ are proof trees over Γ themselves. For some variable Y, the root of $\mathcal{T}_1$ is labeled *All* X *are* Y, and the root of $\mathcal{T}_2$ is labeled *All* Y *are* Z. Now $\mathcal{T}_1$ and $\mathcal{T}_2$ both have fewer nodes than $\mathcal{T}$. By our induction hypothesis, $\Gamma \models$ *All* X *are* Y, and also $\Gamma \models$ *All* Y *are* Z. We claim that $\Gamma \models$ *All* X *are* Z. To see this, take any model $\mathcal{M}$ in which all sentences in Γ are true. Then $[\![X]\!] \subseteq [\![Y]\!]$ by our first point above. And $[\![Y]\!] \subseteq [\![Z]\!]$ by second. So $[\![X]\!] \subseteq [\![Z]\!]$ by transitivity of the inclusion relation on sets. Since the model $\mathcal{M}$ here is arbitrary, we conclude that $\Gamma \models$ *All* X *are* Z.

The other cases on the label of the root of $\mathcal{T}$ are similar. Of special interest might be the case for the rule

$$\frac{\textit{Some } X \textit{ are } Y \quad \textit{No } X \textit{ are } Y}{R}$$

(The intuitive point here is that *every* sentence R follows from the contradictory hypotheses.) Let $\mathcal{T}$ be a proof tree over Γ ending up with an application of this rule. We claim that there are no models of Γ. To see this, suppose toward a contradiction that $\mathcal{M} \models \Gamma$. By induction hypothesis, the sentences *Some* X *are* Y and *No* X *are* Y are true in $\mathcal{M}$. That is, $[\![X]\!] \cap [\![Y]\!] = \emptyset$, and also $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$. This is a contradiction, and from it, we see that there are no models of Γ. So vacuously, every model of Γ is a model of the root S. $\dashv$

So at this point, we know that our logic is *sound*: If we have a tree showing that $\Gamma \vdash S$, then S follows semantically from Γ. This means that the formal logical system is not going to give us any bad results. Now this is a fairly weak point. If we dropped some of the rules, it would still hold. Even if we decided to be conservative and say that $\Gamma \vdash S$ *never* holds, the soundness fact would still be true. So the more interesting question to ask is whether the logical *strong* enough to prove everything it should. We want to know if $\Gamma \models S$ implies that $\Gamma \vdash S$; if it does for all Γ and S, then we say that our system is *complete*. As it happens, our system *is* complete. We show this in Section 1.5. There are several reasons why we do not present the completeness result in one fell swoop. First of all, doing so would not give you any idea of what is going on in the proof. So we have divided things up into smaller steps. And second, considering fragments of the logic gives us additional information (that is, additional completeness theorems) that we would not be able to obtain from the overall completeness result.

Exercise 1.1 Check that

$$\{\textit{Some } X \textit{ are } Y, \textit{Some } Y \textit{ are } Z\} \not\models \textit{Some } X \textit{ are } Z$$

by building a model in which $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$ and $[\![Y]\!] \cap [\![Z]\!] \neq \emptyset$, but $[\![X]\!] \cap [\![Z]\!] = \emptyset$.

Exercise 1.2 Check that

$$\{\textit{Some } X \textit{ are } Y, \textit{Some } Y \textit{ are } Z\} \nvdash \textit{Some } X \textit{ are } Z$$

by examining proofs.

Exercise 1.3 This exercise asks you to come up with definitions and to check their properties.

1. Define the appropriate notions of *submodel* and *homomorphism of models*.
2. Which sentences S in our language have the property that if $\mathcal{M}$ is a submodel of $\mathcal{M}'$ and $\mathcal{M}' \models S$, then also $\mathcal{M} \models S$?
3. Which sentences S in our language have the property that if $\mathcal{M}$ is a surjective homomorphic image of $\mathcal{M}'$ and $\mathcal{M} \models S$, then also $\mathcal{M}' \models S$?
4. Would anything change if we changed "if" to "iff"?

1.1 *All*

To begin, we'll only deal with sentences *All* X *are* Y. We call the set of sentences of this form, or of some other restricted form, a *fragment* of our overall language. As with all our fragments, we get a soundness result immediately from Lemma 1.1. In fact, any subsystem of a sound logical system is itself sound.

Theorem 1.2 *The logic of Figure 1.2 is complete for the fragment with* All.

Proof Let $Z_1, \ldots, Z_k$ be all the variables that occur in Γ or in S. Let S be *All* X *are* Y. Define a model by $\mathcal{M} = \{*\}$,[1] and

$$[\![Z_i]\!] \quad = \quad \begin{cases} \mathcal{M} & \text{if } \Gamma \vdash \textit{All } X \textit{ are } Z_i \\ \emptyset & \text{otherwise} \end{cases} \tag{1.3}$$

It is important that in (1.3), the X is the same variable as in the sentence S from the statement of our result. We claim that if Γ contains

[1] This just means that $\mathcal{M}$ is some one-element set. It doesn't matter which one-element set. Actually, it doesn't even matter that $\mathcal{M}$ has just one element: *any* non-empty set $\mathcal{M}$ would work.

$$\frac{}{\textit{All } X \textit{ are } X} \qquad \frac{\textit{All } X \textit{ are } Z \quad \textit{All } Z \textit{ are } Y}{\textit{All } X \textit{ are } Y}$$

FIGURE 1.2 The logic of *All* X *are* Y.

All V *are* W, then $[\![V]\!] \subseteq [\![W]\!]$. For this, we may assume that $[\![V]\!] \neq \emptyset$ (otherwise the result is trivial). So $[\![V]\!] = \mathcal{M}$. Thus $\Gamma \vdash$ *All* X *are* V. So we have a proof tree[2] as on the left below:

$$\frac{\begin{array}{c}\vdots\\ \textit{All } X \textit{ are } V\end{array} \quad \textit{All } V \textit{ are } W}{\textit{All } X \textit{ are } W}$$

The tree overall has as leaves *All* V *are* W plus the leaves of the tree above *All* X *are* V. Overall, we see that all leaves are labeled by sentences in Γ. This tree shows that $\Gamma \vdash$ *All* X *are* W. From this we conclude that $[\![W]\!] = \mathcal{M}$. In particular, $[\![V]\!] \subseteq [\![W]\!]$.

Now our claim implies that the model $\mathcal{M}$ we have defined makes all sentences in Γ true. So it must make the conclusion true. Therefore $[\![X]\!] \subseteq [\![Y]\!]$. And $[\![X]\!] = \mathcal{M}$, since we have a one-point tree for *All* X *are* X. Hence $[\![Y]\!] = \mathcal{M}$ as well. But this means that $\Gamma \vdash$ *All* X *are* Y, just as desired. $\dashv$

A Stronger Result Theorem 1.2 proves the completeness of the logical system. But it doesn't give us all the information we would need to have an efficient procedure to decide whether or not $\Gamma \vdash S$ in this fragment.[3]

Lemma 1.3 *Let* Γ *be a set of sentence, and define* $\leq$ *from* Γ *by*

$$U \leq V \quad \textit{iff} \quad \Gamma \vdash \textit{All } U \textit{ are } V \tag{1.4}$$

Then $\leq$ *is a pre-order:*

1. *For all* U, $U \leq U$.
2. *If* $U \leq V$ *and* $V \leq W$, *then* $U \leq W$.

Define a relation $\preceq$ on the variables in question by: $V \preceq W$ if Γ contains as an element *All* V *are* W. Let $\preceq^*$ be the reflexive-transitive closure of $\preceq$. Once again, we note that $\preceq^*$ depends on Γ.

[2]The vertical dots $\vdots$ mean that there is some tree over Γ establishing the sentence at the bottom of the dots.

[3]The reason is that we still have to examine all possible models on a one element set in order to check whether $\Gamma \models S$ or not. It might look like there are very few such models. But if the sequent $\Gamma \models S$ contains k of our X variables, then there are 2^k models to consider. The question of efficient decidability is for us the question of whether a polynomial-time algorithm exist. For that, we need to do further work.

$$\frac{}{(X,Y,X)} \qquad \frac{}{(X,Y,Y)} \qquad \frac{(X,Y,U) \quad (X,Y,V) \quad (U,V,Z)}{(X,Y,Z)}$$

FIGURE 1.3 The logic of *All X which are Y are Z*, written here (X,Y,Z).

Theorem 1.4 *Let Γ be any set of sentences in this fragment, let $\preceq^*$ be defined from Γ as above. Let X and Y be any variables. Then the following are equivalent:*

1. $\Gamma \vdash$ All X are Y.
2. $\Gamma \models$ All X are Y.
3. $X \preceq^* Y$.

Proof (1)$\Longrightarrow$(2) is by soundness, and (3)$\Longrightarrow$(1) is by induction on $\preceq^*$. The most significant part (2)$\Longrightarrow$(3). Consider the model $\mathcal{M}$ whose universe is a singleton $\{*\}$, and with $[\![Z]\!] = \mathcal{M}$ iff $X \preceq^* Z$. We claim that all sentences in Γ are true in $\mathcal{M}$. Consider *All V are W*. We may assume that $[\![V]\!] = \mathcal{M}$, or else our claim is trivial. Then $X \preceq^* V$. But $V \preceq W$, so we have $X \preceq^* W$, as desired. This verifies that $\mathcal{M} \models \Gamma$. And since $[\![X]\!] = \mathcal{M}$, we have $[\![Y]\!] = \mathcal{M}$ as well. Thus $X \preceq^* Y$, as needed for (3). $\dashv$

The original definition of the entailment relation $\Gamma \models S$ involves looking at *all* models of the language. Theorem 1.4 is important because part (3) gives a criterion the entailment relation that is algorithmically sensible. To see whether $\Gamma \models$ *All X are Y* or not, we only need to construct $\preceq^*$. This is the reflexive-transitive closure of a syntactically defined relation, so it is computationally very manageable.

1.1.1 A digression: *All X which are Y are Z*

At this point, we digress from our main goal of the examination of the syllogistic system with which we began. Instead, we consider the logic of *All X which are Y are Z*. To save space, we abbreviate this by (X,Y,Z). We take this sentence to be true in a given model $\mathcal{M}$ if $[\![X]\!] \cap [\![Y]\!] \subseteq [\![Z]\!]$. Note that *All X are Y* is semantically equivalent to (X,X,Y).

Theorem 1.5 *The logic of* All X which are Y are Z *in Figure 1.3 is complete.*

Proof Suppose $\Gamma \models (X,Y,Z)$. Consider the interpretation $\mathcal{M}$ given by $\mathcal{M} = \{*\}$, and for each variable Z, $[\![Z]\!] = \{*\}$ if $\Gamma \vdash (X,Y,Z)$. We claim that for $(U,V,W) \in \Gamma$, $[\![U]\!] \cap [\![V]\!] \subseteq [\![W]\!]$. For this, we may assume that

$$\frac{\textit{Some } X \textit{ are } Y}{\textit{Some } Y \textit{ are } X} \qquad \frac{\textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } X}$$

$$\frac{\textit{All } Y \textit{ are } Z \quad \textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } Z}$$

FIGURE 1.4 The logic of *Some* and *All*, in addition to the logic of *All*.

$\mathcal{M} = [\![U]\!] \cap [\![V]\!]$. So we use the proof tree

$$\frac{\begin{array}{c}\vdots\\(X,Y,U)\end{array} \quad \begin{array}{c}\vdots\\(X,Y,V)\end{array} \quad (U,V,W)}{(X,Y,W)}$$

This shows that $[\![W]\!] = \mathcal{M}$, as desired.

Returning to our sentence (X,Y,Z), our overall assumption that $\Gamma \models (X,Y,Z)$ tells us that (X,Y,Z) is true in $\mathcal{M}$. The first two axioms show that $* \in [\![X]\!] \cap [\![Y]\!]$. Hence $* \in [\![Z]\!]$. That is, $\Gamma \vdash (X,Y,Z)$. $\dashv$

Exercise 1.4 Let Γ be a set of sentences in this fragment.

1. Show that if $\Gamma \models (X,Y,Z)$, then $\Gamma \models (Y,X,Z)$.
2. Show that if $\Gamma \vdash (X,Y,Z)$, then $\Gamma \vdash (Y,X,Z)$.
3. Suppose that we remove the axiom (X,Y,Y), and in its place take the symmetry rule
$$\frac{(Y,X,Z)}{(X,Y,Z)}$$
Show that the new system is complete.

Exercise 1.5 For each sentence $S = \textit{All } X \textit{ are } Y$, let $S^* = (X,X,Y)$. If Γ is a set of sentences of the first fragment, let $\Gamma^* = \{S^* : S \in \Gamma\}$. It is easy to check that if $\Gamma \vdash S$, then $\Gamma^* \vdash S^*$. Prove the converse. We say that the system of this section is a *conservative extension* of the system for *All*.

1.2 *All* and *Some*

We want to now enrich our language with sentences *Some X are Y*. We call these sentences *existentials*, since formalizing them in first-order logic would use the existential quantifier $\exists$. As for all the fragments of this chapter, the semantics is given in Figure 1.1.

Theorem 1.6 *The first two rules in Figure 1.4 give a complete proof system for* Some.

Proof Suppose $\Gamma \models$ *Some* X *are* Y. Let $\mathcal{M}$ be the set of sets of unordered pairs (i.e., sets with one or two elements) of variables. Let

$$[\![U]\!] \;=\; \{\{U, V\} : \Gamma \text{ contains } \textit{Some } U \textit{ are } V \text{ or } \textit{Some } V \textit{ are } U\}.$$

Observe that the elements of $[\![U]\!]$ are unordered pairs with one element being U. If Γ contains *Some* U *are* V, then $\{U, V\} \in [\![U]\!] \cap [\![V]\!]$. So $\mathcal{M}$ satisfies the sentences in Γ. By our assumption, $\mathcal{M}$ satisfies *Some* X *are* Y. Thus $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$. We have two cases, depending on whether $X \neq Y$ or $X = Y$. In the first case, by our observation above we have $[\![X]\!] \cap [\![Y]\!] = \{X, Y\}$. This means that either *Some* X *are* Y or *Some* Y *are* X belongs to Γ. Either way, we easily see that $\Gamma \vdash S$. The other case is when $X = Y$. Then to say that $[\![X]\!] \cap [\![Y]\!]$ is non-empty is to say the same of $[\![X]\!]$. So for some Y, either *Some* X *are* Y or *Some* Y *are* X belongs to Γ. And now, we again see easily that $\Gamma \vdash$ *Some* X *are* X. $\dashv$

Lemma 1.7 *Let* Γ *be any set of sentences in* Some *and* All. *Then there is a model* $\mathcal{M}$ *with the following properties:*

1. $\mathcal{M} \models \Gamma$.
2. *If* S *is any sentence in* Some *and* $\mathcal{M} \models S$, *then* $\Gamma \vdash S$.

Proof List all of the existential sentences in Γ in a list:

$$\textit{Some } V_1 \textit{ are } W_1, \textit{Some } V_2 \textit{ are } W_2, \ldots, \textit{Some } V_n \textit{ are } W_n \tag{1.5}$$

Note that we might have repeats among the V's and Ws, and that some of these might well coincide with the X and Y that we are dealing with in this proof. For the universe of $\mathcal{M}$ we take $\{1, \ldots, n\}$, where n is the number in (1.5). For each variable Z, we define

$$[\![Z]\!] \quad = \quad \{i : \text{either } V_i \leq Z \text{ or } W_i \leq Z\}. \tag{1.6}$$

(Again, $\leq$ is defined in (1.4).) This defines the model $\mathcal{M}$.

Consider a sentence *All* P *are* Q in Γ. Then $P \leq Q$. It follows from (1.5) and Lemma 1.3 that $[\![P]\!] \subseteq [\![Q]\!]$. Second, take an existential sentence *Some* V_i *are* W_i on our list in (1.5) above. Then i itself belongs to $[\![V_i]\!] \cap [\![W_i]\!]$, so this intersection is not empty.

These facts imply point (1) of our lemma: $\mathcal{M} \models \Gamma$. For (2), let S be *Some* X *are* Y, and assume that $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$. Let i belong to this set. We have four cases, depending on whether $V_i \leq X$ or $V_i \leq Y$, and whether $W_i \leq X$ or $W_i \leq Y$. One case is when $V_i \leq X$ and $W_i \leq Y$. Recalling that *Some* V_i *are* W_i belongs to Γ, we have a proof tree as follows:

$$\dfrac{\begin{array}{c}\vdots\\ \textit{All } W_i \textit{ are } Y\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \textit{All } V_i \textit{ are } X\end{array} \quad \dfrac{\textit{Some } V_i \textit{ are } W_i}{\textit{Some } W_i \textit{ are } V_i}}{\textit{Some } W_i \textit{ are } X}}{\textit{Some } X \textit{ are } W_i}}{\textit{Some } X \textit{ are } Y}$$

The other cases are similar. You might like to check the details to see where the second rule of *Some* gets used. $\dashv$

Theorem 1.8 *The system in Figures 1.2 and 1.4 is complete for the logic of* Some *and* All.

Proof Suppose that $\Gamma \models S$. There are two cases, depending on whether S is of the form *All* X *are* Y or of the form *Some* X *are* Y. The cases are handled differently. We leave the first to you as Exercise 1.6. The second follows immediately from Lemma 1.7. $\dashv$

Exercise 1.6 Complete the proof of Theorem 1.8 by showing that if Γ is a set of sentences in *All* and *Some*, and if $\Gamma \models$ *All* X *are* Y, then also $\Gamma \vdash$ *All* X *are* Y.

Exercise 1.7 Let Γ be a set of sentences in *All* and *Some*, and let S be a sentence in *Some*. As we know from Lemma 1.7, if $\Gamma \not\vdash S$, there is a $\mathcal{M} \models \Gamma$ which makes S false. The proof gets a model $\mathcal{M}$ whose size of the $\mathcal{M}$ will be the number of existential sentences in Γ. Can we do better?

1. Show that there is a model as desired whose size is at most 2.
2. Show that 2 is the smallest we can get by showing that if we only look at one-element models, then

$$\{\textit{Some } X \textit{ are } Y, \textit{Some } Y \textit{ are } Z\} \models \textit{Some } X \textit{ are } Z$$

Exercise 1.8 Let Γ be the set $\{$*All* X *are* $Y\}$. Prove that there is no model $\mathcal{M}$ such that for all sentences S in the fragment of this section, $\mathcal{M} \models S$ iff $\Gamma \vdash S$. The point of this problem is that there *is* some $\mathcal{M}'$ with the property that for all S of the form *Some* U *are* V, $\mathcal{M} \models S$ iff $\Gamma \vdash S$. But it is not possible to extend this result to sentences with *All*. So we cannot hope to avoid the split in the proof of Theorem 1.8 due to the syntax of S.

Exercise 1.9 Give an algorithm which takes finite sets Γ in the fragment of this section and also single sentences S and tells whether $\Gamma \models S$ or not. [You may be sketchy, as we were in our discussion of this matter at the end of Section 1.1.]

$$\frac{}{J \textit{ is } J} \qquad \frac{M \textit{ is } J}{J \textit{ is } M}$$

$$\frac{J \textit{ is } M \quad M \textit{ is } F}{J \textit{ is } F} \qquad \frac{J \textit{ is an } X \quad J \textit{ is a } Y}{\textit{Some } X \textit{ are } Y}$$

$$\frac{\textit{All } X \textit{ are } Y \quad J \textit{ is an } X}{J \textit{ is a } Y} \qquad \frac{M \textit{ is an } X \quad J \textit{ is } M}{J \textit{ is an } X}$$

FIGURE 1.5 The logic of names, on top of the logic of *All* and *Some*.

Exercise 1.10 Suppose that one wants to say that *All X are Y* is true when $[\![X]\!] \subseteq [\![Y]\!]$ *and also* $[\![X]\!] \neq \emptyset$. Then the following rule becomes sound:

$$\frac{\textit{All } X \textit{ are } Y}{\textit{Some } X \textit{ are } Y}$$

Show that if we add this rule to the proof system for this section, then we get a complete system for the modified semantics. [Hint: Given Γ, let $\overline{\Gamma}$ be Γ together with all sentences *Some X are Y* such that *All X are Y* belongs to Γ. Show that $\Gamma \vdash S$ in the modified system iff $\overline{\Gamma} \vdash S$ in the old system.]

Exercise 1.11 What would you do to the system to add sentences of the form *Some X exists*?

1.3 Adding Names

We continue by adding names so that we can deal with sentences like *John is a secretary* and *John is Mary.* To our formal language we add *names* J, M, $\ldots$, J_1, $\ldots$, J_n, $\ldots$, etc. The sentences we add to the fragment are J *is an* X and J *is* M, where J and M are names and X is a variable. The semantics may be found in Figure 1.1. To get a proof system, we add the remaining rules in Figure 1.5.

Fix a set Γ of sentences in this fragment. Let $\equiv$ be the relation on names defined by

$$J \equiv M \quad \text{iff} \quad \Gamma \vdash J \textit{ is } M. \tag{1.7}$$

Lemma 1.9 *$\equiv$ is an equivalence relation.*

Lemma 1.10 *Let Γ be any set of sentences in* Some, All, *and names. Then there is a model $\mathcal{M}$ with the following properties:*

1. $\mathcal{M} \models \Gamma$.
2. *If S is any sentence in* Some *or names and $\mathcal{M} \models S$, then $\Gamma \vdash S$.*

Proof As before, we define $\leq$ to be from (1.4). We also have the equivalence relation $\equiv$ from (1.7). Let the existential sentences in Γ be listed as in (1.5). Let the set of equivalence classes of $\equiv$ be $[J_1], \ldots, [J_m]$.

We take $\mathcal{M}$ to be $\{1, \ldots, n\} \cup \{[J_1], \ldots, [J_m]\}$. We assume these sets are disjoint. We define

$$\begin{array}{rcl} \llbracket Z \rrbracket & = & \{i : \text{either } V_i \leq Z \text{ or } W_i \leq Z\} \\ & & \cup\ \{[J] : \text{ for some } M \in [J], \Gamma \vdash M \text{ is a } Z\} \end{array} \tag{1.8}$$

To finish defining our model, we take $\llbracket J \rrbracket = [J]$. That is, the semantics of J is the equivalence class $[J]$.

It is easy to see that the semantics is monotone in the sense that if $V \leq W$, then $\llbracket V \rrbracket \subseteq \llbracket W \rrbracket$. This implies that all of the universal assertions of Γ are true in our model $\mathcal{M}$. The existential assertions in Γ are *Some* V_i *is* W_i for $i \leq n$, and for each i, the number i belongs to $\llbracket V_i \rrbracket \cap \llbracket W_i \rrbracket$. The identity sentences J *is* M from Γ are clearly true in $\mathcal{M}$. Finally, consider a sentence J *is a* Z in Γ. Then $\Gamma \vdash J$ *is a* Z. So $\llbracket J \rrbracket = [J] \in \llbracket Z \rrbracket$. This means that our sentence J *is a* Z is true in $\mathcal{M}$.

Let $\mathcal{M} \models$ *Some* X *are* Y. If there is some number i in $\llbracket X \rrbracket \cap \llbracket Y \rrbracket$, then the proof of Theorem 1.8 shows that $\Gamma \vdash$ *Some* X *are* Y. The only alternative is when for some J, $[J] \in \llbracket X \rrbracket \cap \llbracket Y \rrbracket$. By the definition in (1.8), there are $M \in [J]$ and $N \in [J]$ such that $\Gamma \vdash M$ *is an* X and $\Gamma \vdash N$ *is a* Y. We thus have a proof tree over Γ:

$$\dfrac{\begin{matrix}\vdots\\ M\ \textit{is an}\ X\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ N\ \textit{is a}\ Y\end{matrix} \qquad \dfrac{\begin{matrix}\vdots\\ M\ \textit{is}\ J\end{matrix} \quad \begin{matrix}\vdots\\ J\ \textit{is}\ N\end{matrix}}{M\ \textit{is}\ N}}{M\ \textit{is a}\ Y}}{\textit{Some}\ X\ \textit{are}\ Y}$$

So $\Gamma \vdash$ *Some* X *are* Y, as desired.

Continuing, let $\mathcal{M} \models J$ *is* M. Then $[J] = [M]$. So by Lemma 1.9, $\Gamma \vdash J$ is M.

Finally, suppose $\mathcal{M} \models J$ *is an* X. Then for some M, $M \in [J]$ and $\Gamma \vdash M$ is an X. So we see that $\Gamma \vdash J$ is an X using the last rule in Figure 1.5. $\dashv$

Exercise 1.12 Prove that the logic of Figures 1.2, 1.4, and 1.5 is complete for *All*, *Some*, and names.

1.4 *All* and *No*

In this section, we consider the fragment with *No* X *are* Y on top of *All* X *are* Y. In addition to the rules of Figure 1.2, we take the rules in Figure 1.6.

Lemma 1.11 *Let Γ be any set of sentences in* All *and* No*. Then there is a model $\mathcal{M}$ with the following properties:*

1. $\mathcal{M} \models \Gamma$.
2. *If S is any sentence in* All *or* No*, and $\mathcal{M} \models S$, then $\Gamma \vdash S$.*

Proof We take for $\mathcal{M}$ the set of sets a of variables satisfying the following conditions:

(a) If $V \in a$ and $V \leq W$, then $W \in a$.
(b) If $V, W \in a$, then $\Gamma \not\vdash$ *No V are W*.

(Note as a special case of (b) that if $V \in a$, then $\Gamma \not\vdash$ *No V are V*.) We set

$$[\![V]\!] \quad = \quad \{a \in \mathcal{M} : V \in a\}. \tag{1.9}$$

We claim that $\mathcal{M} \models \Gamma$. Condition (a) implies that if *All V are W* belongs to Γ, then $[\![V]\!] \subseteq [\![W]\!]$. Suppose that *No V are W* belongs to Γ. Let $a \in [\![V]\!]$. Then $V \in a$. By condition (b), $W \notin a$. So $a \notin [\![W]\!]$. This argument shows that $[\![V]\!] \cap [\![W]\!] = \emptyset$.

With (1) proved, we turn to (2). Let $\mathcal{M} \models S$. We first deal with the case that S is the of the form *All X are Y*. Let

$$a \quad = \quad \{Z : X \leq Z\}.$$

Case I: $a \notin \mathcal{M}$. Then there are $V, W \in a$ such that $\Gamma \vdash$ *No V are W*. In this case,

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ \textit{All } X \textit{ are } W\end{array} \qquad \dfrac{\dfrac{\begin{array}{c}\vdots\\ \textit{All } X \textit{ are } V\end{array} \quad \begin{array}{c}\vdots\\ \textit{No } V \textit{ are } W\end{array}}{\textit{No } X \textit{ are } W}}{\textit{No } W \textit{ are } X}}{\textit{No } X \textit{ are } X}}{\textit{All } X \textit{ are } Y} \tag{1.10}$$

Case II: $a \in \mathcal{M}$. Then since $a \in [\![X]\!]$, we have $a \in [\![Y]\!]$. (1.9) tells us that $Y \in a$, and so $\Gamma \vdash$ *All X are Y*, as desired.

This concludes our work when S is *All X are Y*. Suppose that S is *No X are Y*. Let

$$a \quad = \quad \{Z : X \leq Z \text{ or } Y \leq Z\}.$$

Note that $X, Y \in a$. We claim that $s \notin \mathcal{M}$. For if $a \in \mathcal{M}$, we would have $a \in [\![X]\!] \cap [\![Y]\!]$. And then $[\![X]\!] \cap [\![Y]\!] \neq \emptyset$. But this contradicts $\mathcal{M} \models S$. So indeed, $a \notin \mathcal{M}$. So there are $V, W \in a$ such that $\Gamma \vdash$ *No V are W*. There are four cases, depending on whether $\Gamma \vdash$ *All X are V* or $\Gamma \vdash$ *All Y are V*, and similarly for W.

$$\frac{\textit{All } X \textit{ are } Z \quad \textit{No } Z \textit{ are } Y}{\textit{No } X \textit{ are } Y} \qquad \frac{\textit{No } X \textit{ are } Y}{\textit{No } Y \textit{ are } X}$$

$$\frac{\textit{No } X \textit{ are } X}{\textit{No } X \textit{ are } Y} \qquad \frac{\textit{No } X \textit{ are } X}{\textit{All } X \textit{ are } Y}$$

FIGURE 1.6 The logic of *No* X *are* Y on top of *All* X *are* Y.

Case I: $\Gamma \vdash$ *All* X *are* V, and $\Gamma \vdash$ *All* Y *are* W. We have the tree:

$$\dfrac{\begin{array}{c}\vdots\\ \textit{All } Y \textit{ are } W\end{array} \quad \dfrac{\begin{array}{c}\vdots\\ \textit{All } X \textit{ are } V\end{array} \quad \begin{array}{c}\vdots\\ \textit{No } V \textit{ are } W\end{array}}{\textit{No } X \textit{ are } W}}{\textit{No } X \textit{ are } Y} \tag{1.11}$$

Case II: $\Gamma \vdash$ *All* X *are* V, and $\Gamma \vdash$ *All* X *are* W. In this case, take the proof tree in (1.10) and change the root to *No* X *are* Y.

The remaining two cases are similar. $\dashv$

Theorem 1.12 *The logic of Figures 1.2 and 1.6 is complete for* All *and* No.

1.5 The Full Logic

We now consider the full logic of Figure 1.1. We take all the rules in Figures 1.2, 1.4, 1.5, and 1.6. We also must add a principle relating *Some* and *No*. For the first time, we face the problem of potential inconsistency: to say *Some* X *are* Y is to deny that *No* X *are* Y. There are no models of *Some* X *are* Y and *No* X *are* Y. We see that according to our semantics, *any sentence* S *whatsoever follows* from these two. We thus add the following rule to our proof system:

$$\frac{\textit{Some } X \textit{ are } Y \quad \textit{No } X \textit{ are } Y}{S} \tag{1.12}$$

So we have all the rules in Figure 1.7.

Definition A set Γ is *inconsistent* if $\Gamma \vdash S$ for all S. Otherwise, Γ is *consistent*.

Theorem 1.13 *The logic in Figure 1.7 is complete for the set of sentences considered in this chapter.*

Proof Let Γ be a set of sentences. Suppose that $\Gamma \models S$. We show that $\Gamma \vdash S$. We may assume that Γ is consistent, or else our result is trivial.

Divide Γ into four parts in the obvious way: Γ_{All}, Γ_{Some}, Γ_{No}, Γ_{names}. There are a number of cases.

First suppose that S is a sentence in *Some* or *Names*. Let $\mathcal{M}$ be from Lemma 1.10 for $\Gamma_{All} \cup \Gamma_{Some} \cup \Gamma_{names}$.

Case I: $\mathcal{M} \models \Gamma_{No}$. Then by hypothesis, $\mathcal{M} \models S$. Then Lemma 1.10 shows that $\Gamma \vdash S$, as desired.

Case II: There is some *No A are B* in Γ_{No} such that $[\![A]\!] \cap [\![B]\!] \neq \emptyset$. And again, Lemma 1.10 shows that $\Gamma_{All} \cup \Gamma_{Some} \cup \Gamma_{names} \vdash$ *Some A are B*. So Γ is inconsistent.

We also consider the case when S is a sentence in *All* or *No*. Let $\mathcal{M}$ be from Lemma 1.11 for $\Gamma_{All} \cup \Gamma_{No}$.

Here is how we interpret a name J in $\mathcal{M}$. Let

$$a_J \quad = \quad \{X : \Gamma \vdash J \textit{ is an } X\}.$$

We claim that $a_J \in \mathcal{M}$. Clearly a is closed upwards in $\leq$. If $a \notin \mathcal{M}$, then there are U, $V \in a_J$ such that $\Gamma \vdash$ *No U are V*. But now we see that Γ is inconsistent:

$$\dfrac{\dfrac{\begin{array}{c}\vdots\\ J \textit{ is a } U\end{array} \quad \begin{array}{c}\vdots\\ J \textit{ is a } V\end{array}}{\textit{Some } U \textit{ are } V} \qquad \textit{No } U \textit{ are } V}{T}$$

This contradiction shows that $a_J \in \mathcal{M}$. We take $[\![J]\!] = a_J$. It is easy to check that now $\mathcal{M}$ satisfies all sentences in Γ_{names}.

Case I: $\mathcal{M}$ satisfies all sentences in Γ_{Some}. Then by hypothesis $\mathcal{M} \models S$. By Lemma 1.11, $\Gamma \vdash S$.

Case II: there is some sentence *Some A are B* in Γ_{Some} such that $\mathcal{M} \not\models$ *Some A are B*. But then $\mathcal{M} \models$ *No A are B*. By Lemma 1.11, $\Gamma \vdash$ *No A are B*. So again using the rule in (1.12), we see that Γ is inconsistent and hence proves S. $\dashv$

1.6 Additional Exercises

In these exercises, we are concerned with the full proof system.

Exercise 1.13 Prove that if $\Gamma \vdash S$, then there are infinitely many proof trees which establish that $\Gamma \vdash S$.

Exercise 1.14 Let $\mathcal{T}$ be a proof tree over Γ with more than one node. Prove that either $\mathcal{T}$ is a proof tree over $\emptyset$, or else every variable and name which occurs in $\mathcal{T}$ also occurs in some sentence in Γ.

Exercise 1.15 Show that Γ has a model iff Γ is consistent.

Exercise 1.16 Let $\Gamma \cup \{$*Some X are Y*$\}$ be inconsistent. Show that $\Gamma \vdash$ *No X are Y*. Similarly, let $\Gamma \cup \{$*No X are Y*$\}$ be inconsistent. Show that $\Gamma \vdash$ *Some X are Y*. [Here it is important to give a proof-theoretic argument. Prove the first part by induction on the length of the shortest path in a proof tree from a leaf labeled *Some* X *are* Y to a node labeled with an instance of the rule in (1.12).]

Exercise 1.17 Let Γ be a consistent set. Using Exercise 1.16, prove that there is a consistent $\Gamma' \supseteq \Gamma$ such that for all X and Y, Γ' contains either *Some* X *are* Y or *No* X *are* Y, but Γ' does not contain more sentences than Γ in *All* or *names*. We call such a Γ' a *strong extension* of Γ. [You may work with the case of Γ a *finite* set (even though the result holds in general), since the details in the finite case contain all the real work of the general case.]

The rest of the exercises outline a different proof of Theorem 1.13. It is important *not to use completeness* in working those exercises.

Exercise 1.18 Let Γ be a consistent set of sentences, and write Γ as

$$\Gamma_{All} \cup \Gamma_{Some} \cup \Gamma_{No} \cup \Gamma_{names}.$$

Let S be a sentence in *All* or *No*. Assume that $\Gamma \models S$, and prove *semantically* that $\Gamma_{All} \cup \Gamma_{No} \models S$ as well. [That is, take $\mathcal{M} \models \Gamma_{All} \cup \Gamma_{No}$. Find a model $\mathcal{M}^+$ so that $\mathcal{M}$ is a submodel of $\mathcal{M}^+$, and $\mathcal{M}^+ \models \Gamma$. Our assumption on Γ tells us that $\mathcal{M}^+ \models S$. And the submodel condition implies that $\mathcal{M} \models S$.]

Exercise 1.19 Let Γ be consistent. Let $\mathcal{M}$ be the model from the proof of Lemma 1.10 for $\Gamma_{All} \cup \Gamma_{Some} \cup \Gamma_{names}$. Show that $\mathcal{M} \models \Gamma$.

It follows from this fact that *every consistent set* Γ *has a model.*

Exercise 1.20 Use Exercises 1.16, 1.18, and 1.19 to give a different proof of the Completeness Theorem 1.13. [You'll need to show that if Γ is consistent and $\Gamma \models S$, then $\Gamma \vdash S$. For this, we again need a split into cases. The cases of *All* and *No* use Theorem 1.12.]

Exercise 1.21 The classical syllogisms also considered sentences *Some* X *is not a* Y. In our setting, it makes sense also to add other sentences with negative verb phrases: J *is not an* X, and J *is not* M. Give some sound proof rules for these sentences (on top of the system we already have).

Exercise 1.22 Adding your rules to those in Figure 1.7, prove the completeness of your system. [It will probably be easiest to use the method of Exercise 1.20. Having the extra sentences around adds a balance to the system and often makes it easier to prove theoretical properties like completeness, despite the additional cases that come from a bigger syntax.]

Exercise 1.23 Consider the language $\mathcal{L}$ with *All* X *are* Y, *Some* X *are* Y, *No* X *are* Y,*Some* X *are not* Y, and sentence involving names as we've seen them. But

$$R \quad = \quad \textit{All } X \textit{ which are } Y \textit{ are } Z$$

is *not* part of the language $\mathcal{L}$. The problem here is to show that it cannot be expressed $\mathcal{L}$. That is, there is no set Γ of sentences in $\mathcal{L}$ such that for all $\mathcal{M}$, $\mathcal{M} \models \Gamma$ iff $\mathcal{M} \models R$. Here is an outline of the proof.

1. Consider the model $\mathcal{M}$ with universe $\{x, y, a\}$ with $[\![X]\!] = \{x, a\}$, $[\![Y]\!] = \{y, a\}$, $[\![Z]\!] = \{a\}$, and also $[\![U]\!] = \emptyset$ for other variables U, and $[\![J]\!] = x$ for all names J. Consider also a model $\mathcal{N}$ with universe $\{x, y, a, b\}$ with $[\![X]\!] = \{x, a, b\}$, $[\![Y]\!] = \{y, a, b\}$, $[\![Z]\!] = \{a\}$, and the rest of the structure the same as in $\mathcal{M}$. Show that for all sentences S in $\mathcal{L}$, $\mathcal{M} \models S$ iff $\mathcal{N} \models S$.
2. Suppose towards a contradiction that we could express R, say by the set Γ. Then since $\mathcal{M}$ and $\mathcal{N}$ agree on all sentences of $\mathcal{L}$, they agree on Γ. But $\mathcal{M} \models R$ and $\mathcal{N} \not\models R$, a contradiction.

Exercise 1.24 As a continuation of Exercise 1.23, show that in $\mathcal{L}$ we cannot express *No* X *which are* Y *are* Z.

Exercise 1.25 For any sentence S, let $S[J/M]$ be the same as S except that all J's are replaced by M's. Suppose that the set Γ has the property that if $S \in \Gamma$, then $S[J/M] \in \Gamma$. Show that for all S, if $\Gamma \models S$, then $\Gamma \models S[J/M]$.

$$\frac{}{\textit{All } X \textit{ are } X} \qquad \frac{\textit{All } X \textit{ are } Z \quad \textit{All } Z \textit{ are } Y}{\textit{All } X \textit{ are } Y}$$

$$\frac{\textit{Some } X \textit{ are } Y}{\textit{Some } Y \textit{ are } X} \qquad \frac{\textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } X}$$

$$\frac{\textit{All } Y \textit{ are } Z \quad \textit{Some } X \textit{ are } Y}{\textit{Some } X \textit{ are } Z} \qquad \frac{}{J \textit{ is } J}$$

$$\frac{M \textit{ is } J}{J \textit{ is } M} \qquad \frac{J \textit{ is } M \quad M \textit{ is } F}{J \textit{ is } F}$$

$$\frac{J \textit{ is an } X \quad J \textit{ is a } Y}{\textit{Some } X \textit{ are } Y} \qquad \frac{\textit{All } X \textit{ are } Y \quad J \textit{ is an } X}{J \textit{ is a } Y}$$

$$\frac{M \textit{ is an } X \quad J \textit{ is } M}{J \textit{ is an } X} \qquad \frac{\textit{All } X \textit{ are } Z \quad \textit{No } Z \textit{ are } Y}{\textit{No } X \textit{ are } Y}$$

$$\frac{\textit{No } X \textit{ are } Y}{\textit{No } Y \textit{ are } X} \qquad \frac{\textit{No } X \textit{ are } X}{\textit{No } X \textit{ are } Y}$$

$$\frac{\textit{No } X \textit{ are } X}{\textit{All } X \textit{ are } Y} \qquad \frac{\textit{Some } X \textit{ are } Y \quad \textit{No } X \textit{ are } Y}{S}$$

FIGURE 1.7 The rules of the system in this chapter.