Midterm Exam

(100 points)

Directions: In contrast to the homework assignments, you may **not** collaborate on this midterm exam. You may not discuss the exam with anybody but the the instructor, who will only answer clarification questions.

1. (15 pts) Exercise 1.10 on pg. 16 of the Moss notes on Natural Logic.

2. (20 pts) Let $+$ be the exclusive or connective: $\alpha + \beta$ is true if ether α is true or β is true, but not both. So,

$$v(\alpha + \beta) = \begin{cases} 1 & v(\alpha) = 1 \text{ or } v(\beta) = 1, \text{ and } v(\alpha) \neq v(\beta) \\ 0 & \text{otherwise} \end{cases}$$

   Prove that the set of connectives $\{\wedge, \leftrightarrow, +\}$ is truth-functionally complete, but no proper subset of these connectives is truth-functionally complete.

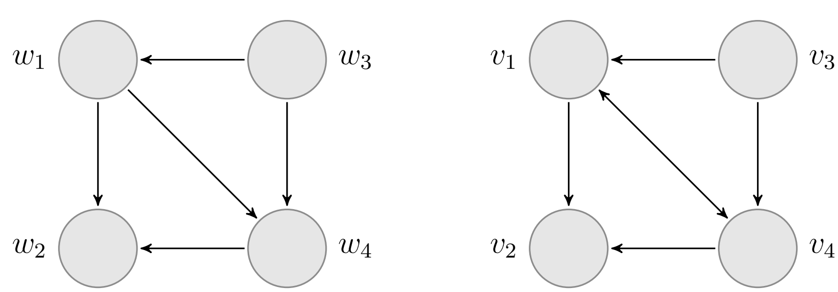
3. (15 pts) This question concerns the axiom system for propositional logic in the Fitting Notes.

   (a) Prove that $\{\neg s \vee r, r \supset p, s\} \vdash p$

   (b) Using the Deduction theorem, prove that $\{p \supset q, \neg q\} \vdash \neg p$ (recall that $\neg p$ is defined to be $p \supset \bot$). Following the algorithm discussed in the proof of the deduction theorem, find the derivation of $\neg p$ from $\{p \supset q, \neg q\}$.

4. (10 pts) Let $\mathcal{M} = \langle W, R, V\rangle$ and $\mathcal{M}' = \langle W, R, V'\rangle$ be two models that differ only in their valuation functions. Suppose that φ is a modal formula where $V(w, p) = V'(w, p)$ for all states and sentence letters p occurring in φ. Prove that $\mathcal{M} \models \varphi$ iff $\mathcal{M}' \models \varphi$. [Hint: the proof is by induction on the structure of φ.]

5. (10 pts) Consider the following two relational structures:



$\mathcal{M}$ $\mathcal{N}$

(a) Show that the following sets are definable in $\mathcal{M}$: $\emptyset$, $\{w_1\}$, $\{w_2\}$, $\{w_3\}$, $\{w_4\}$, $\{w_1, w_2, w_3, w_4\}$.

(b) Which of the following sets are definable in $\mathcal{N}$: $\{v_1\}$, $\{v_1, v_3\}$, $\{v_1, v_2, v_4\}$? (Note: you must explain your answers.)

6. (10 pts)

   (a) Prove that $\Diamond(P \to Q) \to (\Box P \to \Diamond Q)$ is valid.

   (b) Show that $\Box(\Box P \to Q) \vee \Box(\Box Q \to P)$ is not valid.

7. (20 pts)

   (a) Consider the following new operator $\Box^{\leftarrow}$ defined on relational structures $\mathcal{M} = \langle W, R, V \rangle$ as follows:

$$\mathcal{M}, w \models \Box^{\leftarrow}\varphi \text{ iff for all } v \in W, \text{ if } vRw \text{ then } \mathcal{M}, v \models \varphi$$

   Prove that $\Box^{\leftarrow}$ is not definable in the modal language.

   (b) We say $\mathcal{F} = \langle W, R \rangle$ is **Euclidean** if R is a Euclidean relation (for all $x, y, z \in W$ if xRy and xRz then yRz). Prove that $\mathcal{F}$ is Euclidean iff $\mathcal{F} \models \Diamond\varphi \to \Box\Diamond\varphi$.

The midterm is DUE Thursday, November 1 in class.