Problem Set 4

1. Let $\Gamma = \{\neg\forall v_1 Pv_1, Pv_2, Pv_3, \ldots, Pv_n, \ldots\}$. Is Γ consistent? Is Γ satisfiable?

2. Assume that $\Gamma \vdash \varphi$ and that P is a predicate symbol which occurs neither in Γ nor in φ. Is there a deduction of φ from Γ in which P nowhere occurs? *Hint*: There are two different approaches to this problem. One approach is to consider two languages one containing P and one not containing P. The other is to consider the question whether P can be systematically eliminated from a given deduction of φ from Γ.

3. We have already noticed an analogy between the modal operator '$\Box$' and the universal quantifier '$\forall$'. In this question, we will make this connection more formal. There is a precise sense in which the modal language is a *fragment* of the first order language. Let $\mathcal{L}$ be the basic modal language built with $\mathsf{At} = \{p, q, r, \ldots\}$ the set of atomic propositional letters. The first order language $\mathcal{L}_1$ corresponding to $\mathcal{L}$ contains a binary relation symbol $\mathbf{R}$ and for each atomic propositional letter $p \in \mathsf{At}$, a unary predicate symbol $\mathbf{P}$. Any modal model $\mathcal{M} = \langle W, R, V\rangle$ can be viewed as a (first-order) structure for $\mathcal{L}_1$ where the domain of $\mathcal{M}$ is W, the interpretation of $\mathbf{R}$ is the relation R (i.e., $\mathbf{R}^{\mathcal{M}} = R$) and for each unary predicate symbol $\mathbf{P}$ (corresponding to atomic proposition $p \in \mathsf{At}$), $\mathbf{P}^{\mathcal{M}} = \{w \mid V(w, p) = 1\}$.

   Now we can (faithfully) translate the modal language into the first-order language. Formally, we define a map $ST : \mathcal{L} \to \mathcal{L}_1$ sending modal formulas to first-order ($\mathcal{L}_1$) formulas with one free variable as follows:

$$\begin{aligned}
ST_x(p) &= \mathbf{P}x \\
ST_x(\neg\varphi) &= \neg ST_x(\varphi) \\
ST_x(\varphi \wedge \psi) &= ST_x(\varphi) \wedge ST_x(\psi) \\
ST_x(\Box\varphi) &= \forall y(\mathbf{R}xy \to ST_y(\varphi)) \text{ (where } y \text{ is a new variable)} \\
ST_x(\Diamond\varphi) &= \exists y(\mathbf{R}xy \wedge ST_y(\varphi)) \text{ (where } y \text{ is a new variable)}
\end{aligned}$$

   For example,

$$ST_x(\Box\Diamond p \wedge \Box\neg q) = \forall x_1(Rxx_1 \to \exists x_2(Rx_1x_2 \wedge Px_2)) \wedge \forall x_3(Rxx_3 \to \neg Qx_3)$$

   We first check that this translation preserves truth:

   (a) Show that for all modal formulas $\varphi \in \mathcal{L}$ and modal models $\mathcal{M} = \langle W, R, V\rangle$, for all states $w \in W$,
$$\mathcal{M}, w \models \varphi \text{ iff } \mathcal{M}, \mathbf{s}[x/w] \models ST_x(\varphi)$$

   Now, it is clear that not all formulas of $\mathcal{L}_1$ are translation of modal formulas (i.e., of the form $ST_x(\varphi)$ for some modal formula φ). For example, $\forall y(\mathbf{P}x \to \mathbf{R}xy)$ is not of the right syntactic form. However, for some $\mathcal{L}_1$ formulas φ, even if φ is not of the form $ST_x(\psi)$ for some modal formula ψ, φ may be logically equivalent to such a formula.

(b) Is $\exists y(\mathbf{R}xy \land \neg\mathbf{R}yx \land \mathbf{P}y)$ logically equivalent to the standard translation of some modal formula? (i.e., does there exists a modal formuls ψ such that $\exists y(\mathbf{R}xy \land \neg\mathbf{R}yx \land \mathbf{P}y)$ is logically equivalent to $ST_x(\psi)$?) If yes, provide the modal formula. If the answer is no, explain why.

A natural question is *which formulas of $\mathcal{L}_1$ are equivalent to translations of modal formulas*? The answer to this is beyond the scope of the course (the theorem that answers this question is called the Van Benthem Characterization Theorem).