Social Choice Theory for Logicians

Lecture 3

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Plan

1. Arrow, Sen, Muller-Satterthwaite
2. Characterizing Voting Methods: Majority (May, Asan & Sanver), Scoring Rules (Young), Borda Count (Farkas and Nitzan, Saari), Approval Voting (Fishburn)
3. Voting to get things "right" (Distance-based measures, Condorcet and extensions)
4. Strategizing (Gibbard-Satterthwaite)
5. Generalizations
   5.1 Infinite Populations
   5.2 Judgement aggregation (List & Dietrich)
6. Logics
7. Applications

Muller-Satterthewaite

Linear Preferences: $\mathcal{L} = \{> \mid <\subseteq X \times X \text{ is a linear order}\}$

Social choice function: $C : \mathcal{L}^n \to X$

(weak) Pareto: C satisfies weak unanimity provided if for every preference profile $>\in \mathcal{L}^n$, if there is a pair of alternatives x and y such that $x >_i y$ for all $i \in N$, then $C(>) \neq y$.

Monotonicity: C is monotonic provided if for every preference profile $>\in \mathcal{L}^n$ such that $C(>) = x$, if $>'$ is another profile such that $x >'_i y$ whenever $x >_i y$ for every agent i and alternative y, then $C(>') = x$.

Dictator: A voter i is a dictator in a social choice function C if C always selects is top choice: for every preference profile $>$, $C(>) = a$ iff for all $y \in X$ different from x, $x >_i y$.

Proof of Muller-Satterthwaite, I

Lemma. Assuming Mon and P, a coalition S is blocking iff it is winning.

M-S Theorem. If $|X| \geq 3$ and C is Mon and P, then C is a dictator.

Proof of Muller-Satterthwaite, II

If S is blocking then S is winning.

Claim: For any $>$ with b at the top for each $i \in S$, we have $C(>) = b$

$$
\begin{array}{cccccc}
b & \cdots & \mathrm{b} & & \cdots & \\
\vdots & \vdots & \vdots & \vdots & \vdots & \\
 & \cdots & & a & \cdots & a \\
a & \cdots & a & b & \cdots & b
\end{array}
$$

If $C(>) = b$, then we are done

Proof of Muller-Satterthwaite, II

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\begin{array}{cccccc}
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\vdots & \vdots & \vdots & \vdots & \vdots & \\
 & \cdots & & a & \cdots & a \\
a & \cdots & a & b & \cdots & b
\end{array}
$$

If $C(>) = b$, then we are done

$C(>) \neq a$

Proof of Muller-Satterthwaite, II

If S is blocking then S is winning.

Claim: For any $>$ with b at the top for each $i \in S$, we have $C(>) = b$

$$\begin{array}{cccccc}
b & \cdots & \mathrm{b} & c & \cdots & c \\
c & \vdots & c & \vdots & \vdots & \\
 & \cdots & & a & \cdots & a \\
a & \cdots & a & b & \cdots & b
\end{array}$$

If $C(>) = b$, then we are done

$C(>) \neq a$

$C(>) \stackrel{?}{=} c$ for $c \notin \{a, b\}$

Proof of Muller-Satterthwaite, II

If S is blocking then S is winning.

Claim: For any $>$ with b at the top for each $i \in S$, we have $C(>) = b$

$$
\begin{array}{ccccccc}
b & \cdots & \mathrm{b} & & c & \cdots & c \\
c & \vdots & c & & \vdots & \vdots & \\
 & \cdots & & & a & \cdots & a \\
a & \cdots & a & & b & \cdots & b
\end{array}
$$

If $C(>) = b$, then we are done

$C(>) \neq a$

$C(>') = c$ for $c \notin \{a, b\}$

Proof of Muller-Satterthwaite, II

If S is blocking then S is winning.

$$\begin{array}{ccccccc} b & \cdots & b & & c & \cdots & c \\ a & \cdots & a & & & \cdots & \\ c & \vdots & c & & \vdots & \vdots & \\ & \cdots & & & b & \cdots & b \\ & \cdots & & & a & \cdots & a \end{array}$$

$C(>') = c$ for $c \not\in \{a, b\}$

$C(>'') \neq a$, , ,

Proof of Muller-Satterthwaite, II

If S is blocking then S is winning.

$$\begin{array}{ccccccc} b & \cdots & b & & c & \cdots & c \\ a & \cdots & a & & & \cdots & \\ c & \vdots & c & & \vdots & \vdots & \\ & \cdots & & & b & \cdots & b \\ & \cdots & & & a & \cdots & a \end{array}$$

$C(>') = c$ for $c \not\in \{a, b\}$

$C(>'') \neq a$, $C(>'') \neq b$, ,

Proof of Muller-Satterthwaite, II

If S is blocking then S is winning.

$$
\begin{array}{ccccccc}
b & \cdots & b & \quad & c & \cdots & c \\
a & \cdots & a & & & \cdots & \\
c & \vdots & c & & \vdots & \vdots & \\
 & \cdots & & & b & \cdots & b \\
 & \cdots & & & a & \cdots & a
\end{array}
$$

$C(>') = c$ for $c \not\in \{a, b\}$

$C(>'') \neq a$, $C(>'') \neq b$, $C(>'') \neq d$ with $d \not\in \{a, b, c\}$,

Proof of Muller-Satterthwaite, II

If S is blocking then S is winning.

$$\begin{array}{ccccccc}
b & \cdots & b & & c & \cdots & c \\
a & \cdots & a & & & \cdots & \\
c & \vdots & c & & \vdots & \vdots & \\
 & \cdots & & & b & \cdots & b \\
 & \cdots & & & a & \cdots & a
\end{array}$$

$C(>') = c$ for $c \not\in \{a, b\}$

$C(>'') \neq a$, $C(>'') \neq b$, $C(>'') \neq d$ with $d \not\in \{a, b, c\}$,

$C(>'') = c$

Proof of Muller-Satterthwaite, II

If S is blocking then S is winning.

$$\begin{array}{ccccccc}
a & \cdots & a & & c & \cdots & c \\
b & \cdots & b & & & \cdots & \\
c & \vdots & c & & \vdots & \vdots & \\
 & \cdots & & & b & \cdots & b \\
 & \cdots & & & a & \cdots & a
\end{array}$$

$C(>') = c$ for $c \not\in \{a, b\}$

$C(>'') \neq a$, $C(>'') \neq b$, $C(>'') \neq d$ with $d \not\in \{a, b, c\}$,

$C(>'') = c$ implies $C(>''') = c$, contradicts S is blocking.

Proof of Muller-Satterthwaite, II

If S is blocking then S is winning.

Claim: For any $>$ with b at the top for each $i \in S$, we have $C(>) = b$

$$\begin{array}{ccccccc}
b & \cdots & \mathrm{b} & & c & \cdots & c \\
c & \vdots & c & & \vdots & \vdots & \\
 & \cdots & & & a & \cdots & a \\
a & \cdots & a & & b & \cdots & b
\end{array}$$

If $C(>) = b$, then we are done

$C(>) \neq a$

$C(>) \neq c$ for $c \notin \{a, b\}$

Muller-Satterthwaite, III

Suppose that S is a **minimal** winning coalition.

Claim: if there exists $S_1 \neq \emptyset$ and $S_2 \neq \emptyset$ such that $S = S_1 \cup S_2$, then contradiction.

Then $|S| = 1$.

Muller-Satterthwaite, III

$$
\begin{array}{ccccccccc}
a & \cdots & a & b & \cdots & b & c & \cdots & c \\
b & \cdots & b & c & \cdots & c & a & \cdots & a \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \cdots & & & \cdots & & & \cdots & \\
c & \cdots & c & a & \cdots & a & b & \cdots & b
\end{array}
$$

Muller-Satterthwaite, III

$$\begin{array}{ccccccccc}
a & \cdots & a & b & \cdots & b & c & \cdots & c \\
b & \cdots & b & c & \cdots & c & a & \cdots & a \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \cdots & & & \cdots & & & \cdots & \\
c & \cdots & c & a & \cdots & a & b & \cdots & b
\end{array}$$

$C(>) \neq c$ (o.w. contradicts S is winning)

$C(>) \neq d$ with $d \notin \{a, b, c\}$ (o.w. contradicts S is winning)

Muller-Satterthwaite, III

$$\begin{array}{ccccccccc}
a & \cdots & a & b & \cdots & b & c & \cdots & c \\
b & \cdots & b & c & \cdots & c & a & \cdots & a \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \cdots & & & \cdots & & & \cdots & \\
c & \cdots & c & a & \cdots & a & b & \cdots & b
\end{array}$$

$C(>) \neq c$

$C(>) \neq d$ with $d \notin \{a, b, c\}$

$C(>) \stackrel{?}{=} a$

Muller-Satterthwaite, III

$$\begin{array}{ccccccccc}
a & \cdots & a & b & \cdots & b & c & \cdots & c \\
c & \cdots & c & c & \cdots & c & a & \cdots & a \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \cdots & & & \cdots & & & \cdots & \\
b & \cdots & b & a & \cdots & a & b & \cdots & b
\end{array}$$

$C(>) = a$

$C(>') = a$

Muller-Satterthwaite, III

$$\begin{array}{ccccccccc}
a & \cdots & a & b & \cdots & b & a & \cdots & a \\
c & \cdots & c & c & \cdots & c & c & \cdots & c \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \cdots & & & \cdots & & & \cdots & \\
b & \cdots & b & a & \cdots & a & b & \cdots & b
\end{array}$$

$C(>) = a$

$C(>') = a$, $C(>'') = a$, so $S_1 \cup T$ is blocking hence winning

Muller-Satterthwaite, III

$$
\begin{array}{ccccccccc}
a & \cdots & a & b & \cdots & b & c & \cdots & c \\
c & \cdots & c & c & \cdots & c & b & \cdots & b \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \cdots & & & \cdots & & & \cdots & \\
b & \cdots & b & a & \cdots & a & a & \cdots & a
\end{array}
$$

$C(>) = a, C(>') = a,\ C(>'') = a$, so $S_1 \cup T$ is winning

$C(>''') \neq a$

Muller-Satterthwaite, III

$$\begin{array}{ccccccccc}
a & \cdots & a & b & \cdots & b & c & \cdots & c \\
c & \cdots & c & c & \cdots & c & b & \cdots & b \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \cdots & & & \cdots & & & \cdots & \\
b & \cdots & b & a & \cdots & a & a & \cdots & a
\end{array}$$

$C(>) = a, C(>') = a$, $C(>'') = a$, so $S_1 \cup T$ is winning

$C(>''') \neq a$, $C(>''') \neq b$

Muller-Satterthwaite, III

$$
\begin{array}{ccccccccc}
a & \cdots & a & b & \cdots & b & c & \cdots & c \\
c & \cdots & c & c & \cdots & c & b & \cdots & b \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
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b & \cdots & b & a & \cdots & a & a & \cdots & a
\end{array}
$$

$C(>) = a, C(>') = a$, $C(>'') = a$, so $S_1 \cup T$ is winning

$C(>''') \neq a$, $C(>''') \neq b$, $C(>''') \neq c$

Muller-Satterthwaite, III

$$
\begin{array}{ccccccccc}
a & \cdots & a & b & \cdots & b & c & \cdots & c \\
c & \cdots & c & c & \cdots & c & b & \cdots & b \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & \cdots & & & \cdots & & & \cdots & \\
b & \cdots & b & a & \cdots & a & a & \cdots & a
\end{array}
$$

$C(>) = a, C(>') = a,\ C(>'') = a$, so $S_1 \cup T$ is winning

$C(>''') \neq a,\ C(>''') \neq b,\ C(>''') \neq c,\ C(>''') \neq d \notin \{a, b, c\}$

Muller-Satterthwaite, III

$$\begin{array}{ccccccccc}
a & \cdots & a & b & \cdots & b & c & \cdots & c \\
b & \cdots & b & c & \cdots & c & a & \cdots & a \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
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$C(>) \neq d$ with $d \not\in \{a, b, c\}$

$C(>) \neq a$

$C(>) \neq b$

Tacking Stock

Impossibility results: $|X| \geq 3$, Arrow (*social welfare function*, IIA, P, UD, Non-Dictator), Sen (Liberalism, Pareto), Muller-Satterthwaite (*social choice function*, Mon, P, Non-Dictator)

Phenomena: Monotonicity, Condorcet vs. Borda (cancellation), Multiple-districts paradox

Multiple Districts Paradox

Totals	Rankings	H over W	W over H
417	B H W	417	0
82	B W H	0	82
143	H B W	143	0
357	H W B	357	0
285	W B H	0	285
324	W H B	0	324
1608		**917**	**691**

B: $417 + 82 = 499$
H: $143 + 357 = 500$
W: $285 + 324 = 609$

Multiple Districts Paradox

Totals	Rankings	H over W	W over H
417	X H W	417	0
82	X W H	0	82
143	H X W	143	0
357	H W X	357	0
285	W X H	0	285
324	W H X	0	324
1608		**917**	**691**

H Wins

Multiple Districts Paradox

Totals	Rankings	East	West
417	B H W	160	257
82	B W H	0	82
143	H B W	143	0
357	H W B	0	357
285	W B H	0	285
324	W H B	285	39
1608		**588**	**1020**

B would win both districts!

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Characterizing Majority Rule

If there are only **two** options, then majority voting is the "best" procedure:

Characterizing Majority Rule

If there are only **two** options, then majority voting is the "best" procedure: Choosing the outcome with the most votes (allowing for ties) is the *only* group decision method satisfying the previous properties.

K. May. *A Set of Independent Necessary and Sufficient Conditions for Simple Majority Decision*. Econometrica, Vol. 20 (1952).

May's Theorem: Details

Suppose there are only two candidates A and B and n voters (let $N = \{1, \ldots, n\}$ be the set of voters).

Then the voters' preferences can be represented by elements of $\{-1, 0, 1\}$ (where 1 means A is preferred to B, -1 means B is preferred to A and 0 means indifference between A and B).

A **social decision method** is a function $F : \{-1, 0, 1\}^n \rightarrow \{-1, 0, 1\}$.

May's Theorem: Details

- **Unanimity**: unanimously supported alternatives must be the social outcome.
- **Anonymity**: all voters should be treated equally.
- **Neutrality**: all candidates should be treated equally.
- **Pos. response**: unidirectional shift in voters' opinions should not harm the alternative toward which this shift occurs

May's Theorem: Details

- **Unanimity**: unanimously supported alternatives must be the social outcome.

  If for all $i \in N$, $v_i = x$ then $F(v) = x$ (for $x \in \{-1, 0, 1\}$).

- **Anonymity**: all voters should be treated equally.

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- **Anonymity**: all voters should be treated equally.

  $F(v_1, v_2, \ldots, v_n) = F(v_{\pi(1)}, v_{\pi(2)}, \ldots, v_{\pi(n)})$ where π is a permutation of the voters.

- **Neutrality**: all candidates should be treated equally.

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- **Neutrality**: all candidates should be treated equally.

  $F(-v) = -F(v)$ where $-v = (-v_1, \ldots, -v_n)$.

- **Pos. response**: unidirectional shift in voters' opinions should not harm the alternative toward which this shift occurs

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- **Unanimity**: unanimously supported alternatives must be the social outcome.

  If for all $i \in N$, $v_i = x$ then $F(v) = x$ (for $x \in \{-1, 0, 1\}$).

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- **Pos. response**: unidirectional shift in voters' opinions should not harm the alternative toward which this shift occurs

  If $F(v) = 0$ or $F(v) = 1$ and $v \prec v'$, then $F(v') = 1$ (where $v \prec v'$ means for all $i \in N$ $v_i \leq v'_i$ and there is some $i \in N$ with $v_i < v'_i$) then $F(v') = 1$.

May's Theorem: Details

May's Theorem (1952) A social decision method F satisfies unaniminity, neutrality, anonminity and positive responsiveness iff F is majority rule.

Proof Idea

If $(1, 1, -1)$ is assigned 0 or -1 then

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Proof Idea

If $(1, 1, -1)$ is assigned 0 or -1 then

- Neutrality implies $(-1, -1, 1)$ is assigned 0 or 1
- Anonymity implies $(1, -1, -1)$ is also assigned 0 or 1

Proof Idea

If $(1, 1, -1)$ is assigned 0 or -1 then

- Neutrality implies $(-1, -1, 1)$ is assigned 0 or 1
- Anonymity implies $(1, -1, -1)$ is also assigned 0 or 1
- Positive Responsiveness implies $(1, 0, -1)$ is assigned 1

Proof Idea

If $(1, 1, -1)$ is assigned 0 or -1 then

- Neutrality implies $(-1, -1, 1)$ is assigned 0 or 1
- Anonymity implies $(1, -1, -1)$ is also assigned 0 or 1
- Positive Responsiveness implies $(1, 0, -1)$ is assigned 1
- Positive Responsiveness implies $(1, 1, -1)$ is assigned 1, Contradiction.

Other characterizations

Weak path independence: If $|F(R_1) - F(R_2)| \neq 2$ then $F(R_1 \oplus R_2) = F(F(R_1) \oplus F(R_2))$

G. Asan and R. Sanver. *Another Characterization of the Majority Rule*. Economics Letters, 75 (3), 409-413, 2002.

E. Maskin. *Majority rule, social welfare functions and game forms*. in *Choice, Welfare and Development*, The Clarendon Press, pgs. 100 - 109, 1995.

G. Woeginger. *A new characterization of the majority rule*. Economic Letters, 81, pgs. 89 - 94, 2003.

Scoring Rules: Young's Theorem

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Reinforcement: Suppose that X and Y are disjoint sets of voters. Let W_X be the set of winners for X and W_Y the set of winners for Y. If there is at least one candidate that wins both elections, then the winner(s) for the entire population is $W_X \cap W_Y$.

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Continuity: Suppose that a group of voters X elects a candidate A and a disjoint group of voters Y elects a different candidate B. Then there must be some number m such that the population consisting of the subgroup Y together with m copies of X will elect A.

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Theorem (Young 1975).A social decision method satisfies anonymity, neutrality, reinforcement and continuity if and only if the method is a scoring rule.

Borda Count

Cancellation: For a profile R, Suppose that $N_{a\ b} = \{i \mid aP_ib\}$. If $N_{a\ b} = N_{b\ a}$ then $aI_{F(R)}b$.

H. P. Young. *An axiomatization of Borda's rule*. Journal of Economic Theory, 9, pgs. 43 - 52, 1974.

S. Nitzan and A. Rubinstein. *A further characterization of Borda ranking method*. Public Choice, 36, pgs. 153 - 158, 1981.

Approval Voting

Fact There is no fixed rule that always elects a unique Condorcet winner.

# voters	2	2	1
	a	b	c
	d	d	a
	b	a	b
	c	c	d

Approval Voting

Fact There is no fixed rule that always elects a unique Condorcet winner.

# voters	2	2	1
	a	b	c
	d	d	a
	b	a	b
	c	c	d

The unique Condorcet winner is *a*.

Approval Voting

Fact There is no fixed rule that always elects a unique Condorcet winner.

# voters	2	2	1
	a	b	c
	d	d	a
	b	a	b
	c	c	d

Vote-for-1 elects $\{a, b\}$, vote-for-2 elects $\{d\}$, vote-for-3 elects $\{a, b\}$.

Approval Voting

Fact There is no fixed rule that always elects a unique Condorcet winner.

# voters	2	2	1
	a	b	c
	d	d	a
	b	a	b
	c	c	d

$(\{a\}, \{b\}, \{c, a\})$ elects a under AV.

Approval Voting

Fact Condorcet winners are always AV outcomes, but a Condorcet looser may or may not be an AV outcome.

Fishburn's Theroem

Theorem (Fishburn 1978). A social decision method is approval voting if and only if the method satisfies anonymity, neutrality, reinforcement and the following technical property:

- If there are exactly two voters who approve of disjoint sets of candidates, then the methods selects as winners all the candidates chosen by the two voters (i.e., the union of the ballots chosen by the voters).

Distance

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Why should we buy the idea, though, that there really is such a thing as an objectively "best" choice? Aren't values relative, and isn't the point of voting to strike a balance between conflicting opinions, not to determine a correct one?"

H. P. Young. *Optimal Voting Rules*. The Journal of Economic Perspectives, 9:1, pgs. 51 - 64, 1995.

Kemeny

"...in many situations , differences of opinion arise from differences in values, not erroneous judgments. In this case it seems better to adopt the view that group choice is an exercise in finding a compromise between conflicting opinions." (Young, p. 60)

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$d(a > b > c > d, d > a > b > c) = 3$

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Kemeny metric: Suppose that R and R' are two rankings

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Examples:

$d(a > b > c > d, d > a > b > c) = 3$

$d(a > b > c > d, c > d > a > b) = 4$

mean ranking: the ordering that minimizes the sum of squares of distances from a given set of n rankings

median ranking: the ordering that minimizes the sum of distances from the set of n rankings

# voters	21	5	4	11
	A	B	C	C
	B	C	A	B
	C	A	B	A

$A > B > C$ is the median ranking

$B > A > C$ is the mean ranking

S. Nitzan. *Some Measures of Closeness to Unanimity and Their Implications*. Theory and Decision, 13, 129 - 138, 1981.

Reaching Consensus

Let $P = (P_1, \ldots, P_n)$ be a sequence of linear orders on X.

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$P \in U(x)$, then it is unanimous that x should be the winner.

x is a **relative unanimous winner** provided the *distance* between P and $U(x)$ is no larger than the distance between P and $U(y)$ for all other alternatives y.

Distance

$$\delta(P_i, Q_i) = \frac{1}{2}|\{(x,y) \in X \times X \mid \text{the relative ranking of } (x,y) \text{ in } P_i \text{ differs from the relative ranking in } Q_i\}|$$

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$$d(P,Q) = \sum_{i=1}^{n} \delta(P_i, Q_i)$$

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$$d(P,Y) = \min_{Q \in Y} d(P,Q)$$

$$U^*(x) = \{P \in \mathcal{P}^n \mid d(P, U(x)) \leq d(P, U(y)) \text{ for all } x \in X\}$$

Fact. An alternative x has the highest Borda score iff it is a relative unanimous winner.

$$\delta_2(P_i, Q_i) = \begin{cases} 0 & \text{if } top(P_i) = top(Q_i) \\ 1 & \text{otherwise} \end{cases}$$

Fact An alternative is the plurality winner iff it is closest to the unanimous profile using the δ_2 measure.

Manipulation

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It has long been noted that a voter can achieve a preferred election outcome by misrepresenting his or her actual preferences.

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(Taken from A. Taylor *Social Choice and the Mathematics of Manipulation* who took it from D. Black *A Theory of Committees and Elections* who took it from Dodgson.)

Manipulation: setting the agenda

# voters	1	1	1
	B	A	C
	D	B	A
	C	D	B
	A	C	D

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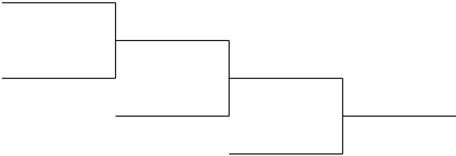
# voters	1	1	1
	B	A	C
	D	B	A
	C	D	B
	A	C	D



Manipulation: setting the agenda

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	B	A	C
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	C	D	B
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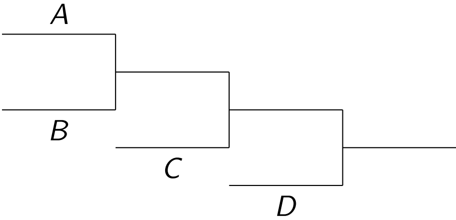

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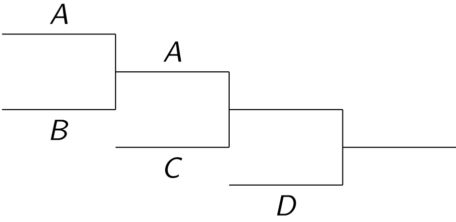

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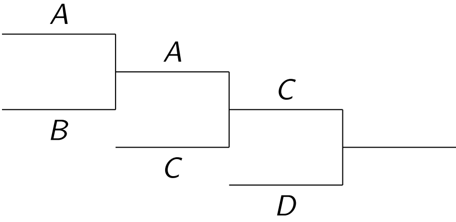

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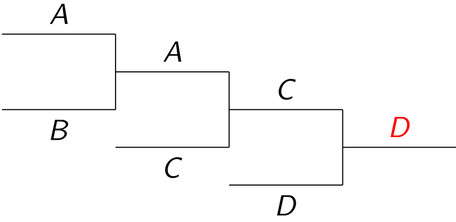
	C	D	B
	A	C	D



Manipulation: misrepresenting preferences

# voters	3	3	1
	A	B	C
	B	A	A
	C	C	B

Borda Winner: A

Manipulation: misrepresenting preferences

# voters	3	3	1
	A	B	C
	B	A	A
	C	C	B

Borda Winner: A

Manipulation: misrepresenting preferences

# voters	3	3	1
	A	B	C
	B	A	A
	C	C	B

Borda Winner: *A*

# voters	3	3	1
	A	B	C
	B	C	A
	C	A	B

Manipulation: misrepresenting preferences

# voters	3	3	1
	A	B	C
	B	A	A
	C	C	B

Borda Winner: A

# voters	3	3	1
	A	B	C
	B	C	A
	C	A	B

Borda Winner: B

Two Issues

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1. What does it *mean* to vote strategically?

   - Voting as a game vs. voting as an act of communication

K. Dowding and M. van Hees. *In Praise of Manipulation*. British Journal of Political Science, 38 : pp 1-15, 2008.

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1. What does it *mean* to vote strategically?
   - Voting as a game vs. voting as an act of communication

   K. Dowding and M. van Hees. *In Praise of Manipulation*. British Journal of Political Science, 38 : pp 1-15, 2008.

2. The decision to strategize depends on the agents' *information* (eg. poll information).

S. Chopra, E. Pacuit and R. Parikh. *Knowledge-theoretic Properties of Strategic Voting*. JELIA 2004.

Strategizing Functions

Fix the voters' true preferences: $\mathcal{P}^* = (P^*_1, \ldots, P^*_n)$

Given a vote profile $\vec{v}$ of *actual* votes, we ask whether voter i will change its vote if given another chance to vote.

Example I

The following example is due to [Brams & Fishburn]

$$P_A^* = o_1 > o_3 > o_2$$
$$P_B^* = o_2 > o_3 > o_1$$
$$P_C^* = o_3 > o_1 > o_2$$

Size	Group	I	II
4	A	$\mathbf{o_1}$	$\mathbf{o_1}$
3	B	o_2	o_2
2	C	o_3	$\mathbf{o_1}$

If the current winner is o, then agent i will switch its vote to some candidate o' provided

1. o' is one of the top two candidates as indicated by a poll
2. o' is preferred to the other top candidate

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Example II

$$
\begin{aligned}
P^*_A &= (o_1, o_4, o_2, o_3) \\
P^*_B &= (o_2, o_1, o_3, o_4) \\
P^*_C &= (o_3, o_2, o_4, o_1) \\
P^*_D &= (o_4, o_1, o_2, o_3) \\
P^*_E &= (o_3, o_1, o_2, o_4)
\end{aligned}
$$

Size	Group	I	II	III	IV
40	A	$\mathbf{o_1}$	o_1	o_4	$\mathbf{o_1}$
30	B	o_2	$\mathbf{o_2}$	$\mathbf{o_2}$	o_2
15	C	o_3	$\mathbf{o_2}$	$\mathbf{o_2}$	o_2
8	D	o_4	o_4	o_1	o_4
7	E	o_3	o_3	o_1	$\mathbf{o_1}$

If the current winner is o, then agent i will switch its vote to some candidate o' provided

1. i prefers o' to o, and
2. the current total for o' plus agent i's votes for o' is greater than the current total for o.

Example II

$$P^*_A = (o_1, o_4, o_2, o_3)$$
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Example III

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\end{aligned}
$$

Size	Group	I	II	III	IV	V	VI	VII	$\cdots$
40	A	o_1	o_1	o_2	o_2	o_2	o_1	o_1	o_1
30	B	o_2	o_3	o_3	o_2	o_2	o_2	o_3	o_3
30	C	o_3	o_3	o_3	o_3	o_1	o_1	o_1	o_3

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$$P_A^* = (o_1, o_2, o_3)$$
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30	B	o_2	o_3	o_3	o_2	o_2	o_2	o_3	o_3
30	C	o_3	o_3	o_3	o_3	o_1	o_1	o_1	o_3

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30	C	o_3	o_3	o_3	o_3	o_1	o_1	o_1	o_3

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30	C	o_3	o_3	o_3	o_3	o_1	o_1	o_1	o_3

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30	C	o_3	o_3	o_3	o_3	o_1	o_1	o_1	o_3

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30	C	o_3	o_3	o_3	o_3	o_1	o_1	o_1	o_3

Summary

Agents, knowing an aggregation function, will strategize if they know

a. enough about other agents' preferences and
b. that the output of the aggregation function of a changed preference will provide them with a more favorable result.

The Gibbard-Satterthwaite Theorem

A social choice function is **strategy-proof** if for no individual i there exists a profile $\vec{R}$ and a linear order R_i' such that $C(\vec{R}_{-i}, R_i')$ is ranked above $F(\vec{R})$ according to i.

Theorem. Any social choice function for three or more alternatives that is Pareto and strategy-proof must be a dictatorship.

M. A. Satterthwaite. *Strategy-proofness and Arrow's conditions: Existence and correspon- dence theorems for voting procedures and social welfare functions*. Journal of Economic Theory, 10(2):187-217, 1975.

A. Gibbard. *Manipulation of voting schemes: A general result*. Econometrica, 41(4):587-601, 1973.