> **DUE DATE: Wednesday, October 8.** Please answer all the problems from the required section below and try at least one question from the challenging section. When quoting problems from Hodges, $n - m$ means page n, question m. Questions from the Basic section should be straightforward and need not be handed in.

Basic

1. Hodges: 9-1.

2. Let L be a language and L_0 be the language containing all the relation symbols of L, an $(n+1)$-ary relation symbol R_f for each n-ary function symbol f of L and a unary relation symbol R_c for each constant symbol c of L. Let A be an L-structure and A^* be the $L \cup L_0$ expansion of A where we interpret R_f as the graph of f^A and R_c as $\{c^A\}$. Let A_0 be the reduct of A^* to L_0.

   Show that $Y \subseteq (dom(A))^n$ is X-definable in A iff it is X-definable in A^* iff it is X-definable in A_0.

3. Consider the structure $A = (\mathbb{Z}, +)$. What is $\langle X \rangle_A$ if X is

   (a) $\{0\}$

   (b) $\{1\}$

   (c) $\{2, -2\}$

Required

1. Hodges: 10-4 (a).

> **Note**: In this problem, Hodges asks for an *extension* from one structure to another. For part (a), assume that "extension" means "injective homomorphism". Now, for part (b), note that every embedding is also an injective homomorphism, so part (b) is trivial using this definition of extension (simple take B to be C and let $f = g$ and h the identity map.). The "piece of set theory that is swept under the carpet" is the following: suppose that $f : A \to C$ is an embedding. We can think of A as a substructure of C because we can add elements to the domain of A to construct a new structure B in such a way that there is an isomorphism from B to C. We do this as follows, let $dom(B)$ be the (disjoint) union of $dom(A)$ and $\{c \in dom(C) \mid c \notin rng(f)\}$, where $rng(f)$ is the range of f. Interpret the function and relations symbols as you do in A for the A-part of the domain of B and as you do in C for the C-part of the domain. Now, this construction works because f is an embedding from A to C. Furthermore the function $h : B \to C$ such that $h|_{dom(A)} = f$ and $h|_X = \mathbf{1}$ where $X = \{c \in dom(C) \mid c \notin rng(f)\}$ is easily seen to be an isomorphism.

2. Hodges: 17-3

3. Let A be an L-structure and $X \subseteq dom(A)$. We say that $b \in dom(A)$ is definable over X if there is a formula $\varphi(v, \bar{x})$ and $\bar{a}$ from X such that

$$A \models \varphi(b, \bar{a}) \wedge \forall y(\varphi(y, \bar{a}) \to y = b)$$

   In other words, $\{b\}$ is X-definable.

   (a) Show that b is definable over X iff for some n there is a 0-definable function $f : (dom(A))^n \to dom(A)$ and $\bar{a}$ from A such that $f(\bar{a}) = b$. *Note that a function f is definable if its graph is definable.*

   (b) Suppose that b is definable over X and σ is an automorphism of A such that $\sigma(a) = a$ for all $a \in X$. Show that $\sigma(x) = x$.

   (c) Let $dcl(X) = \{a \in dom(A) \mid a$ is definable over $X\}$. Show that $dcl(dcl(X)) = dcl(X)$

Challenging (try at least one)

1. Let φ be a $L_{\omega\omega}$ sentence. The **finite spectrum** of φ is the set $\{n \in \mathbb{N} \mid$ there is $A \models \varphi$ with $|A| = n\}$.

   (a) Let $L = \{E\}$ where E is a binary relation and let φ be the sentence that asserts that E is an equivalence relation where every equivalence class has exactly two elements[1]. Show that the finite spectrum of φ is the set of positive even numbers.

   (b) For each of the following subsets of $\mathbb{N}$, show that X occurs as the finite spectrum of an L-sentence for some language L:

      i. $\{2^n 3^m \mid n, m > 0\}$
      ii. $\{p^n \mid p$ is prime and $n > 0\}$
      iii. $\{p \mid p$ is prime $\}$

2. Prove that for the first order language $L_{\omega\omega}$, there are at most $2^{|L|}$ nonequivalent models (*cf.* question 2 on page 5 in Hodges). Find necessary and sufficient conditions on L so that there will be *exactly* $2^{|L|}$ non-equivalent models. In general, find the conditions on L so that there are exactly 2^κ non-equivalent models for each infinite cardinal κ.

3. For each $n \in \mathbb{N}$, find a language with only a finite number of symbols and a model A_n for this language which has exactly n undefinable elements (*cf.* question 3 in the Required section above).

4. Recall that a successor structure is a structure $A = (N, s, 0)$ where N is a set and s^A is a unary function satisfying the following: $s^A(x) = s^A(y)$ implies $x = y$, $s^A(x) \neq 0^A$ and $x \neq 0^A$ implies there is a y such that $s^A(y) = x$. Show that any uncountable collection of countable non-isomorphic successor structures has to contain a successor structure with infinitely many cycle components.

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[1]The sentence is the conjunction of the usual sentences asserting that E is an equivalence relations with the sentence $\forall x \exists y (x \neq y \land E(x,y) \land \forall z(E(x,z) \rightarrow (z = x \lor z = y)))$.