Problem Set # 3 (Midterm Questions)

Complete all the questions below. Each question is worth 20 points for a total of 100 points. **The problem set is DUE Wednesday, November 2**.

1. Recall the muddy children puzzle discussed in Lecture 1: *A Primer on Epistemic Logic* (see slides 10 - 11). In the puzzle, after the father's announcement, the children's announcements are simultaneous. What happens if the children speak in turn? That is, suppose there are three children two of which have mud on their forehead (suppose it is the 2nd and 3rd child that have mud on their forehead), and the children announce in order (child 1 speaks first, then child 2, and finally child 3) whether they know if they have mud on their forehead. Draw the successive updated models to explain your answer. Does your answer change if it is child 1 and child 2 with mud on their forehead (but the speaking order remains the same)?

2. Three men are standing on a ladder, each wearing a hat. Each can see the colors of the hats of the people below him, but not his own or those higher up. It is common knowledge that only the colors red and white occur, and that there are more white hats than red ones. The actual order is white, red, white from top to bottom. Draw the epistemic model. The top person says: I know the color of my hat. Is that true? Draw the update. Who else knows his color now? If that person announces that he knows his color, what does the bottom person learn?

3. Recall that an **epistemic-plausibility model** is a tuple

$$\mathcal{M} = \langle W, \{\sim_i\}_{i\in\mathcal{A}}, \{\preceq_i\}_{i\in\mathcal{A}}, V\rangle$$

where W is a non-empty set of states, for each $i \in \mathcal{A}$, $\sim_i$ is an equivalence relation on W, for each $i \in \mathcal{A}$, $\preceq_i$ is reflexive, transitive, and *well-founded* (every subset $X \subseteq W$ has a $\preceq$-minimal element), and $V : \mathsf{At} \to \wp(W)$ is a valuation function. In addition, the following two properties are satisfied:

   (a) *plausibility implies possibility*: if $w \preceq_i v$ then $w \sim_i v$.

   (b) *locally-connected*: if $w \sim_i v$ then either $w \preceq_i v$ or $v \preceq_i w$.

Let $\mathcal{L}_{KB}$ be the modal language defined by the following grammar:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \psi \mid K_i\varphi \mid B^{\varphi}\psi \mid [\preceq_i]\varphi \mid B^s\varphi$$

with $p \in \mathsf{At}$. Recall that $Min_{\preceq_i}(X) = \{v \in X \mid v \preceq_i w \text{ for all } w \in X\}$ (which is always non-empty since $\preceq_i$ is well-founded). Truth for the modal operators is defined as follows:

- $\mathcal{M}, w \models K_i\varphi$ iff for all $v \in W$, if $w \sim_i v$ then $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models B_i^{\varphi}\psi$ iff for all $v \in Min_{\preceq_i}([\![\varphi]\!]_{\mathcal{M}} \cap [w]_i)$, $\mathcal{M}, v \models \psi$
- $\mathcal{M}, w \models [\preceq_i]\varphi$ iff for all $v \in W$, if $v \preceq_i w$ then $\mathcal{M}, v \models \varphi$
- $\mathcal{M}, w \models B^s\varphi$ iff $[\![\varphi]\!]_{\mathcal{M}} \cap [w]_i \neq \emptyset$ and $[\![\varphi]\!]_{\mathcal{M}} \preceq_i [\![\neg\varphi]\!]_{\mathcal{M}}$

where $[\![\varphi]\!]_{\mathcal{M}} = \{w \mid \mathcal{M}, w \models \varphi\}$.

(a) Prove that the following two formulas are valid on every epistemic-plausibility model (recall that $L_i\varphi$ is defined to be $\neg K_i\neg\varphi$ and $B_i\varphi$ is $B^{\top}\varphi$):

- $B_i^{\varphi}\psi \;\leftrightarrow\; L_i\varphi \to L_i(\varphi \wedge [\preceq_i](\varphi \to \psi))$
- $B_i^s\varphi \;\leftrightarrow\; B_i\varphi \wedge K_i(\varphi \to [\preceq_i]\varphi)$

(b) Let $\mathcal{M}$ be an epistemic-plausibility model and define $K_w = \{\varphi \mid \mathcal{M}, w \models B_i\varphi\}$. Define a revision operator $*$ as follows: $K_w * \psi = \{\varphi \mid \mathcal{M}, w \models B^{\psi}\varphi\}$. Prove that $*$ satisfies the AGM postulates ($K*1 - K*8$ on pg. 20 of lecture 15: *Introduction to Belief Revision, II*).

4. Recall the definition of product update (slide 6 of Lecture 12: *Dynamic Logics of Information Change*). Let $\mathcal{M} = \langle W, R, V\rangle$ be a Kripke models and $\mathcal{E} = \langle E, S, \mathsf{pre}\rangle$ and event model. Prove that if R and S are both transitive, then the relation in $\mathcal{M} \otimes \mathcal{E}$ is also transitive. Is this also true if R and S are serial (a relations T is serial if for all x there is a y such that xTy)?

5. Let W be a set of states, $\mathcal{L}$ the propositional language with At as the set of atomic formulas, and $V\mathsf{At} \to \wp(W)$ a valuation function. For $\varphi \in \mathcal{L}$, define $[\![\varphi]\!]$ by recursion as follows: $[\![p]\!] = V(p)$, $[\![\neg\varphi]\!] = W - [\![\varphi]\!]$ and $[\![\varphi \wedge \psi]\!] = [\![\varphi]\!] \cap [\![\psi]\!]$. Let $X \subseteq W$. A system of spheres centered at X is a collection of sets $\mathbf{S} \subseteq \wp(W)$ satisfying the following properties

   (a) $\mathbf{S}$ is totally ordered by $\subseteq$ (for $U, V \in \mathbf{S}$, either $U \subseteq V$ or $V \subseteq U$)

   (b) X is the $\subseteq$-minimum element of $\mathbf{S}$ ($X \in \mathbf{S}$ and for all $V \in \mathbf{S}$, $X \subseteq V$)

   (c) $W \in \mathbf{S}$

   (d) For every propositional formula $\varphi \in \mathcal{L}$ and sphere $U \in \mathbf{S}$, if $U \cap [\![\varphi]\!] \neq \emptyset$ then there is a $\subseteq$-minimal sphere $U_0 \in \mathbf{S}$ such that $U_0 \cap [\![\varphi]\!] \neq \emptyset$.

   Let $X \subseteq W$ and let $\leq$ be a binary relation on W. We say that $\leq$ is X-persistent if it satisfies the following properties:

   (a) $\leq$ is a weak order ($\leq$ is reflexive, transitive and complete: for all $w, v \in W$, $w \leq v$ or $v \leq w$)

(b) [1]For every $\varphi \in \mathcal{L}$, if $[\![\varphi]\!] \neq \emptyset$, then $\{v \mid v \in [\![\varphi]\!] \text{ and } v \leq w \text{ for all } w \in [\![\varphi]\!] \} \neq \emptyset$

(c) For each $w \in W$, w is a $\leq$-minima ($w \leq v$ for all $v \in W$) if and only if $w \in X$

Let $X \subseteq W$

(i) Show that every system of spheres centered on X generates a X-persistent binary relation.

(ii) Show that every X-persistent binary relation generates a system of spheres centered on X.

[1]There was a typo in the earlier version of this midterm. The earlier version was "For every $\varphi \in \mathcal{L}$, if $[\![\varphi]\!] \neq \emptyset$, then $\{v \mid v \in [\![\varphi]\!] \text{ and } v \leq w \text{ for all } w \in W \} \neq \emptyset$" which says that for each consistent φ, $[\![\varphi]\!]$ consists of $\leq$-minimal elements, which is not what we want.