A Dynamic Logic of Observation and Access
First Draft!

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1 Introduction and Motivation

Reasoning about rational agents interacting over time is a central topic in many areas of philosophy, computer science and economics. An important challenge for the logician is to account for the many dynamic processes that govern the agents' interaction over time. Inference, observation and communication are all examples of such processes that are the focus of current logics of informational update and belief revision (see, for example, van Benthem, 1996; van Ditmarsch et al., 2007; Parikh and Ramanujam, 2003)[1]. A recurring issue in any formal model that represents agents' informational attitudes over time is how to account for the fact that the agents are *limited* in their access to the available inference steps, possible observations and available messages. This may be because the agents are not logically omniscient and so do not have unlimited reasoning ability. But it can also be because the agents are following a predefined *protocol* that explicitly limits statements available for observation and/or communication.

Within the broad literature on epistemic logic, there are a variety of alternative accounts that making precise a notion of an agent's "limited access". An early approach of Fagin and

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[1]Of course, one may argue that (logical) *inference* is the central topic of *any* logic. What we have in mind here is reasoning *about* agents that make inferences.

Halpern (1988)[2] extends standard epistemic logic with an *awareness* operator $A\varphi$ intended to mean "the agent is aware of the sentence φ". Recent work building on Artemov's Logic of Proofs (Artemov, 2001) labels epistemic modal operators with *proof terms* that explicitly keep track of the agent's justification, or evidence, for (the truth of) a formula (see, for example, Fitting, 2005; Artemov and Nogina, 2005). Other logics focus on explicitly modeling inferential steps that individual agents can make while interacting with other agents and the environment (see, for example, Eberle, 1974; Ågotnes and Alechina, 2007; van Benthem, 2008; Velazquez-Quesada, 2009). Finally, van Benthem et al. (2008) develop logics to reason about situations where the facts that agents can observe are limited by a predefined protocol (cf. Parikh and Ramanujam, 2003). Section 4 contains a detailed comparison between the logical framework we develop in this article with the logical frameworks referenced above, so we will not go into any details here.

This main technical contribution of this paper builds on recent work formally relating the two main logical accounts of the dynamics of information in social interactive situations (van Benthem et al., 2008). The first is *epistemic temporal logic* (ETL, Fagin et al., 1995; Parikh and Ramanujam, 2003) which uses linear or branching time models with added epistemic structure induced by the agents' different capabilities for observing events. These models provide a "grand stage" where histories (i.e., sequences of events) of some social situations are constrained by a **protocol**. Here a **protocol** is intended to represent the rules or conventions that govern many of our social interactions. Imposing such rules *restricts* the legitimate sequences of possible *events* (eg., messages or observations). The other framework is *dynamic epistemic logic* (DEL, Gerbrandy, 1999; Baltag et al., 1998; van Ditmarsch et al., 2007) that describes social interactions in terms of epistemic **event models** (which may occur inside modalities of the language). Similar to the way Kripke structures[3] are used to capture the information the agents' have about a *fixed* social situation, an **event**

[2]See also (Halpern and Rego, 2005; Halpern, 2001) for a recent discussion and references to relevant literature on the notion of awareness in game theory.

[3]We assume the reader is familiar with standard epistemic logic and the various dynamic versions mentioned above. There are a number of textbooks and survey papers that have thorough introductions to these topics. For example, see (Fagin et al., 1995; van Ditmarsch et al., 2007; **?**; **?**).

model describes the agents' information about which actual events are currently taking place. The temporal evolution of the situation is then computed from some initial epistemic model through a process of successive "product updates". The logic that we develop in this paper *merges* these two perspectives on rational interaction (cf. van Benthem et al., 2008, Section 5).

The first example of such a logic that merges ideas from ETL and DEL can be found in Section 4 of (van Benthem et al., 2008). This logic reexamines *public announcement logic* (PAL, **??**) in situations where the availability of formulas for observation is constrained by a predefined protocol. More formally, a **PAL protocol** is a tree of (epistemic) formulas. ETL models are then generated from an initial epistemic model by performing a public announcement of formulas permitted by some fixed PAL protocol[4]. In this new setting the PAL formula $\langle A\rangle\top$ not only express that the current model is updated by the public announcement of A but also that A is *permitted* according to the predefined PAL protocol. Taking into account this new interpretation of the PAL language, van Benthem et al. (2008) give a sound and complete axiomatization of the class of all ETL models generated by some epistemic model and PAL protocol[5]. However, this is only one specific example of how to merge the ETL and DEL frameworks as suggested by the following questions:

1. A public announcement is one specific type of event model, can we axiomatize classes of ETL models generated by other types of event models?

2. Which formal languages are best suited to describe these DEL generated ETL models?

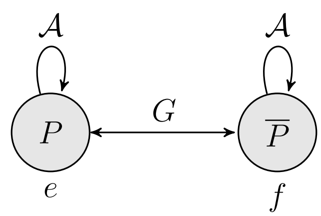
Our primary goal in this paper is to push the style of analysis from (van Benthem et al., 2008) forward by investigating answers to the above questions. We will also relate the logical framework from Sections **??** & 3.1 to the different logics of knowledge and "access" mentioned above.

[4]Actually the most general situation is where there are different PAL protocols at each state. This more general setting is not important for the discussion here and the reader can consult (van Benthem et al., 2008) for details.

[5]Note that the usual method for proving completeness via reduction axioms will not work here. In particular, since A being true does not necessarily imply that A can be announced the PAL validity $A \leftrightarrow \langle A\rangle\top$ is not valid (of course, $\langle A\rangle\top \rightarrow A$ is valid since we are working with PAL protocols.)

2 Our Framework

This section describes the specific logical system that we will investigate in this paper. We assume that at each moment different pieces of information are made publicly available. However, as opposed to *public announcements* where all agents have access to the information that is made available, we assume that different agents have access to different pieces of information. Thus we are interested in ETL models generated by protocols consisting of the following type of event model:



Where $\mathcal{A}$ is the (finite) set of agents, P is a formula (in the language defined below) and $\overline{P}$ is the "negation" of P (i.e., $\overline{P} = Q$ if $P = \neg Q$ and $\overline{P} = \neg P$ otherwise). This event model represents situations where (the truth of) P is made available but only the agents $\mathcal{A} - G$ have access to this information. Of course, the event itself is public so which agents actually have access to which pieces of information is commonly known (i.e., the agents in G know that the agents in $\mathcal{A} - G$ know whether P is true). We will discuss this and other underlying assumptions in Section 4. This section focuses on the formal details of our framework.

The Language The language includes standard epistemic and dynamic modalities plus operators intended to describe each agents' *protocol* (i.e., which formulas the agent has access to). Fix a finite set $\mathcal{A}$ of agents and a (countable) set of atomic propositions At. The language $\mathcal{L}_{DLOA}$ is defined inductively:

$$\varphi ::= \top \mid p \mid \neg\varphi \mid \varphi \wedge \varphi \mid K_i\varphi \mid \langle\varphi\rangle\varphi \mid A_i\varphi$$

where $p \in \mathsf{At}$ and $i \in \mathcal{A}$. The dual operators, $\widehat{K}_i$ and $[\varphi]$, and other boolean operators are defined as usual. The epistemic fragment of $\mathcal{L}_{DLOA}$ without the '$\langle\varphi\rangle$' and 'A_i' operators is denoted $\mathcal{L}_{EL}$. The intended meaning of the modal operators is summarized below:

- $K_i\varphi$ is intended to mean "according to i's current information φ is true" (following the standard convention we may also say "agent i knows that φ").
- $\langle\varphi\rangle\psi$ is intended to mean "after φ is made publicly available, ψ is true" (we may also say "after φ is announced ψ is true", but this should not be confused with the *public announcement* of φ from (**??**))
- $A_i\varphi$ is "agent i has access to φ" (alternatively we may say "agent i can observe φ" or "agent i has the ability to observe φ").

The Semantics As noted above, we are interested in ETL models generated by protocols of a specific type of event models: public announcements of φ with only some of the agents having access to content of the announcement. A **protocol** describes for each agent which formulas that agent can observe. Note that "being able to observe φ" is an event *type* which we take to mean that in situations where φ is true, the agent observes that φ and when φ is false the agents observes that $\neg\varphi$. This explains the closure condition placed in the definition of protocols: Let $\overline{\varphi}$ be φ if φ is of the form $\neg\psi$; $\neg\varphi$ otherwise.

Definition 2.1 (Protocols) A **protocol** is a fuction $\mathsf{p} : \mathcal{A} \times \mathbb{N} \to \wp(\mathcal{L}_{DLOA})$ such that, for every $n \in \mathbb{N}$, $i \in \mathcal{A}$, and $\varphi \in \mathcal{L}_{DLOA}$, $\varphi \in \mathsf{p}(i,n)$ iff $\overline{\varphi} \in \mathsf{p}(i,n)$. We denote the set of protocols by $Ptcl$. For brevity, we write p_i^n for $\mathsf{p}(i,n)$ where $\mathsf{p} \in Ptcl$, $i \in \mathcal{A}$, and $n \in \mathbb{N}$. $\triangleleft$

So a protocol describes for each agent and each moment which formulas that agent can access. The closure condition states that agents have access to a formula iff they have access to its "negation". Of course, this is only one of many different properties that may be assumed about the protocol. For example, it may be natural to assume that if an agent has access to both φ and ψ then the agent must also have access to $\varphi \wedge \psi$. Using the access modality A_i such properties will be expressible in our language. This will be discussed in more detail below.

The formal definition of our models can be given using the machinery developed in (van Benthem et al., 2008, Section 3). However, in the present setting many of these definitions can be simplified.

Definition 2.2 (Models) An **epistemic model** is a triple $(W, \sim, V)$ where W is a non-empty set, $\sim: \mathcal{A} \to \wp(W \times W)$, and $V : \mathsf{At} \to \wp(W)$. By convention, when $(w, v) \in \sim (i)$, we write $w \sim_i v$. An **epistemic model with a protocol** is a quadruple $(W, \sim, V, \mathsf{p})$, where $(W, \sim, V)$ is an epistemic model and p is a protocol. Given an epistemic model $\mathcal{M} = (W, \sim, V)$ and a protocol p, we denote by $(\mathcal{M}, \mathsf{p})$ the model $(W, \sim, V, \mathsf{p})$. $\triangleleft$

We are restricting attention to one type of dynamic epistemic action: "making φ publicly available where only some of the agents have access to (the observation of) φ". Here "having access to φ" means that the agent can incorporate the observation of φ into the agent's current information. Intuitively, if an agent incorporates the observation of φ, the should consider possible only states where φ is true. But in order to do this, the agent must have access to $\overline{\varphi}$ (this also explains the closure condition placed on protocols in Definition 2.1). More formally, we define what it means to "make φ publicly available":

Definition 2.3 (Making φ publicly available) Let $\mathcal{P} = (W, \sim, V, \mathsf{p})$ be an epistemic model with a protocol. Incorporating the observation of ψ into $\mathcal{P}$ is the model $\mathcal{P} \otimes \psi = (W', \sim', V', \mathsf{p}')$ where

$$\begin{array}{lcl} W' & := & W \\ \sim'_i & := & \begin{cases} \sim_i & \text{if } \overline{\psi} \notin \mathsf{p}(i,0) \\ \{(w,v) \in \sim_i \mid \mathcal{P}, w \models \psi \text{ iff } \mathcal{P}, v \models \psi\} & \text{if } \overline{\psi} \in \mathsf{p}(i,0) \end{cases} \\ V' & := & V \\ \mathsf{p}'(i,n) & := & \mathsf{p}(i, n+1) \end{array}$$

where $\mathcal{P}, w \models \psi$ means "ψ is true at state w in $\mathcal{P}$" defined below. $\triangleleft$

Definition 2.4 (Truth) Let $\mathcal{P} = (W, \sim, V, \mathsf{p})$ be an epistemic model with a protocol. The truth of a formula φ in $\mathcal{L}_{DLOA}$ is defined inductively as follows:

$$
\begin{array}{lll}
\mathcal{P}, w \models p & \text{iff} & w \in V(p) \quad (\text{with } p \in P) \\
\mathcal{P}, w \models \neg\varphi & \text{iff} & \mathcal{P}, w \not\models \varphi \\
\mathcal{P}, w \models \varphi \wedge \psi & \text{iff} & \mathcal{P}, w \models \varphi \text{ and } \mathcal{P}, w \models \psi \\
\mathcal{P}, w \models K_i\varphi & \text{iff} & \forall v \in W: \text{ if } w \sim_i v \text{ then } \mathcal{P}, v \models \varphi \\
\mathcal{P}, w \models \langle\psi\rangle\varphi & \text{iff} & \text{(i) } \mathcal{P}, w \models \psi \text{ and (ii) } \mathcal{P} \otimes \psi, w \models \varphi \\
\mathcal{P}, w \models A_i\varphi & \text{iff} & \varphi \in \mathsf{p}_i^0
\end{array}
$$

◁

▶ Maybe add an example?

3 Main Results

The main technical result is a sound and complete axiomatization of the class of all epistemic models with protocols (cf. Definition 2.2) in the language $\mathcal{L}_{DLOA}$. There are two main categories of axiom schemes. The first are essentially reduction axioms describing the effect epistemic event on an epistemic model:

R1 $\langle\theta\rangle p \leftrightarrow \theta \wedge p$ where $p \in \mathsf{At}$

R2 $\langle\theta\rangle\neg\varphi \leftrightarrow \theta \wedge \neg\langle\theta\rangle\varphi$

R3 $\langle\theta\rangle(\varphi \wedge \psi) \leftrightarrow \langle\theta\rangle\varphi \wedge \langle\theta\rangle\psi$

R4 $\langle\theta\rangle K_i\varphi \leftrightarrow \theta \wedge (A_i\theta \rightarrow K_i[\theta]\varphi) \wedge (\neg A_i\theta \rightarrow K_i([\theta]\varphi \wedge [\overline{\theta}]\varphi))$

Note that $\langle\theta\rangle\top \leftrightarrow \theta$ is a consequence of R1. This means that any true formula can always be made publicly available. Of course, it may be that none of the agents actually have access to this information (i.e., θ is no on any of the agents protocol). Thus, unlike the framework from (van Benthem et al., 2008, Seciton 4), $\langle\theta\rangle\top$ does not mean that θ is permitted according to the protocol. The only other axiom schema requiring some discussion is R4. Following the usual reduction axiom methodology, the right-hand side describes what agents know after θ is made publicly available in terms of the agent's current information. The second type of axiom schemes highlight the assumptions we make about the protocols:

P-neg $A_i\varphi \leftrightarrow A_i\overline{\varphi}$

Ptcl $A_j\varphi \rightarrow K_iA_j\varphi$

Uni $\langle\alpha\rangle A_i\varphi \rightarrow [\beta]A_i\varphi$

The first axiom scheme encodes the closure condition on protocols that agents have access to φ iff they have access to $\overline{\varphi}$. The last two axiom schemes highlight the assumption that the protocols are not only common knowledge, but also are *uniform*. The last assumption means that what formulas the agent can access does not depend on the earlier observations the agent has made. Putting everything together, we have:

Definition 3.1 (Axiomatization) The logic DLOA is the smallest set containing all instances of the following axiom schemes[6] **Axiom Schema**

K $K_i(\varphi \rightarrow \psi) \rightarrow (K_i\varphi \rightarrow K_i\psi)$

R1 $\langle\theta\rangle p \leftrightarrow \theta \wedge p$ where $p \in \mathsf{At}$

R2 $\langle\theta\rangle\neg\varphi \leftrightarrow \theta \wedge \neg\langle\theta\rangle\varphi$

R3 $\langle\theta\rangle(\varphi \wedge \psi) \leftrightarrow \langle\theta\rangle\varphi \wedge \langle\theta\rangle\psi$

R4 $\langle\theta\rangle K_i\varphi \leftrightarrow \theta \wedge (A_i\theta \rightarrow K_i[\theta]\varphi) \wedge (\neg A_i\theta \rightarrow K_i([\theta]\varphi \wedge [\overline{\theta}]\varphi))$

P-neg $A_i\varphi \leftrightarrow A_i\overline{\varphi}$

Ptcl $A_j\varphi \rightarrow K_iA_j\varphi$

[6]For concreteness, we only include the axiom schema $K_i(\varphi \rightarrow \psi) \rightarrow (K_i\varphi \rightarrow K_i\psi)$, but other axiomatizations will work as well such as $\mathsf{S5}$.

Uni $\langle\alpha\rangle A_i\varphi \to [\beta]A_i\varphi$

and is closed under necessitation for K_i and $[\theta]$. We write $\vdash_{\mathsf{DLOA}} \varphi$ if $\varphi \in \mathsf{DLOA}$. $\lhd$

Our goal in this section is to prove the following result

Theorem 3.2 DLOA *is sound and strongly complete with respect to the class of epistemic models with protocols.*

The proof is a variant of the proof found in (van Benthem et al., 2008, Section 4) which itself is a variant of the standard Henkin construction. We construct the canonical model from the set of DLOA maximal consistent sets (mcs). The key observation is that each mcs contains a description of a protocol.

▶ The details here are still a bit sketchy, but this is the right idea

Let σ be a (possibly empty) finite sequence of $\mathcal{L}_{DLOA}$-formulas. We denote the length of σ by $\mathsf{len}(\sigma)$. Also we denote by σ_n and $\sigma_{(n)}$ the n-th element of σ and the initial segment of σ of the length n respectively. When n is greater than the length of σ, they denote the empty sequence λ. Also we write $\langle\sigma\rangle$ and $[\sigma]$ for $\langle\sigma_1\rangle\ldots\langle\sigma_{\mathsf{len}(\sigma)}\rangle$ and $[\sigma_0]\ldots[\sigma_{\mathsf{len}(\sigma)-1}]$ respectively.

Definition 3.3 (Γ-Protocol) Given a maximally consistent set Γ, we define a Γ-protocol p_Γ so that

$$\mathsf{p}_\Gamma(i,n) = \begin{cases} \{\varphi \mid A_i\varphi \in \Gamma\} & \text{if} \quad n = 0 \\ \{\varphi \mid \exists\sigma_1\ldots\sigma_n : \langle\sigma_1\rangle\ldots\langle\sigma_n\rangle A_i\varphi \in \Gamma\} & \text{if} \quad n \geq 1 \end{cases}$$

$\lhd$

Definition 3.4 (Base Canonical Model) Let Γ be a maximally consistent set. We define the Γ-generated canonical model $\mathcal{P}_{-} = (W_\Gamma, \sim_\Gamma, V_\Gamma, \mathsf{p}_\Gamma)$ by:

- $W_\Gamma = \{\Delta \mid \mathsf{p}_\Delta = \mathsf{p}_\Gamma\}$
- For all $\Delta, \Delta \in W_\Gamma$, $(\Delta, \Delta') \in \sim_\Gamma (i)$ iff $\{\varphi \mid K_i\varphi \in \Delta\} \subseteq \Delta'$

- $V_\Gamma(p) = \{\Delta \mid p \in \Delta\}$

When there is no confusion, we will omit the subscript Γ. $\triangleleft$

Given a maximlally consistent set Δ and a $\mathcal{L}_{DLOA}$-formula φ, we denote by $\Delta\varphi$ the set $\{\theta \mid \langle\varphi\rangle\theta \in \Delta\}$. It is straightforward to show the following.

Lemma 3.5 *For every maximally consistent set Δ and a $\mathcal{L}_{DLOA}$-formula φ, $\Delta\varphi$ is a maximally consistent set.*

Proof. The claim follows easily from R2. QED

Given a finite sequence σ of $\mathcal{L}_{DLOA}$-formulas, we denote $(\dots(\Delta\sigma_1)\sigma_2)\dots)$ and $(\dots(\mathcal{P}_\Gamma \otimes \sigma_1) \otimes \sigma_2)\dots) \otimes \sigma_n)$ by by $\Delta\sigma$ and $\mathcal{P}_\Gamma \otimes \sigma$ respectively.

Definition 3.6 (Canonical Model after σ) Given a (possibly empty) sequence σ of $\mathcal{L}_{DLOA}$-formulas and the canonical model $\mathcal{P}$, we define $\mathcal{P}^\sigma = (W^\sigma, \sim^\sigma, V^\sigma, \mathsf{p}^\sigma)$ inductively as follows:

- $\mathcal{P}^\lambda = \mathcal{P}$
- $W^{\sigma(n)} = \{\Delta\sigma_n \mid \Delta \text{ in } \mathcal{P}^{\sigma(n-1)}\} \cup \{\Delta\overline{\sigma}_n \mid \Delta \text{ in } \mathcal{P}^{\sigma(n-1)}\}$
- For all $\Delta\chi, \Delta'\chi' \in W^{\sigma(n)}$, $\Delta\chi \sim_i^{\sigma(n)} \Delta'\chi'$ iff
  1. $\Delta \sim_i^{\sigma(n-1)} \Delta'$ and $\chi, \chi' \notin \mathsf{p}^{\sigma(n-1)}(i, 0)$ or
  2. $\chi = \chi'$ and $\chi, \chi' \in \mathsf{p}^{\sigma(n-1)}(i, 0)$
- $V^{\sigma(n)}(p) = V^{\sigma(n-1)}(p)$
- $\mathsf{p}^{\sigma(n)}(i, n) := \mathsf{p}^{\sigma(n-1)}(i, n+1)$

$\triangleleft$

Lemma 3.7 (Truth Lemma) *Let $\mathcal{P}$ be the canonical model. Let σ be a finite sequence of LO-formula. For every $\varphi \in \mathcal{L}_{LO}$ and Δ in $\mathcal{P}_\Gamma$,*

$$\langle\sigma\rangle\varphi \in \Delta \quad \textit{iff} \quad \mathcal{P}^\sigma, \Delta\sigma \models \varphi.$$

Proof. The proof is by induction on the complexity of φ. The base case is clear by Definition 3.4 and 3.6. The boolean cases are straightforward.

▶ The argument for the knowledge modality case can be carried out by using the axiom R3 in a way similar to the completeness proof of TPAL in (van Benthem et al., 2008)

Here we only do the cases for $\langle\chi\rangle$ and A_i.

Suppose φ is of the form $\langle\chi\rangle\psi$. The right-to-left direction is clear by Definition 3.4 and 3.6. Thus assume $\langle\sigma\rangle\langle\chi\rangle\psi \in \Delta$. Then we have $\mathcal{P}^{\sigma\chi}, \Delta\sigma\chi \models \psi$ by IH. By Definition **??**, clearly $\mathcal{P}^{\sigma\chi}$ and $\mathcal{P}^{\sigma} \otimes \chi$ is isomorphic (map $\Delta\sigma\chi$ in $\mathcal{P}^{\sigma\chi}$ to $\Delta\sigma$ in $\mathcal{P}^{\sigma} \otimes \chi$). Therefore we have $\mathcal{P}^{\sigma} \otimes \chi, \Delta\sigma \models \psi$. Also $\langle\sigma\rangle\langle\chi\rangle\psi \in \Delta$ implies $\langle\sigma\rangle\langle\chi\rangle\top \in \Delta$ by standard modal reasoning. From this, it follows by A1 that $\langle\sigma\rangle\chi \in \Delta$. This gives us $\mathcal{P}^{\sigma}, \Delta\sigma \models \chi$. Therefore, $\mathcal{P}^{\sigma}, \Delta\sigma \models \langle\chi\rangle\psi$, as desired.

Suppose φ is of the form $A_i\psi$. First assume that $\varphi \in \Delta\Theta$. By definition, we have $\langle\theta_{n-1}\rangle \dots \langle\theta_0\rangle A_i\psi \in \Delta$. Since $\mathsf{p}_\Gamma = \mathsf{p}_\Delta$ by construction, $\psi \in \mathsf{p}_\Gamma(i, n)$. Thus $\mathcal{P}_\Gamma, \Delta \models A_i\psi$. For the other direction, assume that $\mathcal{P}_\Gamma, \Delta \models A_i\psi$. By construction, this implies that $\psi \in \mathsf{p}_\Gamma(i, n)$ and thus that there is some sequence $\Theta' = \theta'_0 \dots \theta'_{n-1}$ such that $\langle\Theta'\rangle A_i\psi \in \Delta$. Repeated applications of Uni yield $\langle\Theta\rangle\psi \in \Delta$. QED

The standard argument shows, using the above Truth Lemma, that every DLOA-consistent set of formulas is satisfiable. This proves Theorem 3.2.

Other results:

- Completeness with additional closure properties on the protocol. This is easy if the property can be expressed using A_i.
- General method for finding a sound and complete axiomatization for various types of (fixed) DEL protocols

4 Comparisons

Public Announcement Logic The following schemas are valid in PAL but not in DLOA:

1. $\langle\alpha\rangle\langle\beta\rangle\varphi \to \langle\langle\alpha\rangle\beta\rangle\varphi$

2. $[p]K_i p$ where $p \in \mathsf{At}$

3. $\langle\theta\rangle K_i\varphi \leftrightarrow \langle\theta\rangle\top \wedge K_i(\langle\theta\rangle\top \to \langle\theta\rangle\varphi)$

1. For instance, take $\varphi := A_i p$. Let $\mathcal{P} = (W, \sim, V, \mathsf{p})$. $\mathcal{P}, w \models \langle\alpha\rangle\langle\beta\rangle\varphi$ requires that $\alpha \in \mathsf{p}(i, 1)$ and $\beta \in \mathsf{p}(i, 2)$, but this does not guarantee that $\langle\alpha\rangle\beta \in \mathsf{p}(i, 1)$.

2. Take a model where $\neg K_i p$ and $p \notin \mathsf{p}(i, 0)$.

3. Let $\theta = \varphi := p$. Take a model where p is true but $p \notin \mathsf{p}(i, 0)$. Then LHS is false but RHS is true.

However there is an embedding from the language of PAL to $\mathcal{L}_{DLOA}$ that preserves validity: Let $sub(\varphi)$ be the set of subformulas of φ.

Proposition 4.1 *For every formula φ of PAL,*

$$\varphi \textit{ is valid in PAL iff } \bigwedge_{i\in\mathcal{A}} \bigwedge_{\psi\in sub(\varphi)} A_i\varphi \textit{ is valid in } \mathsf{DLOA}$$

Proof. Straightforward induction. QED

TDEL

▶ We still need to specify what exactly TDEL is, especially because we allow occurrences of DEL operators inside DEL operators. Although it can be worked out, this may need a lot of space to give all the details. However, below gives the main idea.

Let p be a protocol and φ a formula in $\mathcal{L}_{DLOA}$. Define $\mathcal{A}^n(\varphi, \mathsf{p})$ be the set $\{i \in \mathcal{A} \mid \varphi \in \mathsf{p}(i, n)\}$. Define an event model $\mathcal{E}(\varphi, \mathsf{p}, n) = (E, \to, \mathsf{pre})$ by

1. $E = \{1, 2\}$

2. $\to(i) = \begin{cases} \{(1,1),(1,2),(2,1),(2,2)\} & \text{if } \varphi \in \mathsf{p}(i,n) \\ \{(1,1),(2,2)\} & \text{if } \varphi \notin \mathsf{p}(i,n). \end{cases}$

3. $\mathsf{pre}(1) = \varphi, \mathsf{pre}(2) = \neg\varphi$

Definition 4.2 (Translation from Models in LO to Models in TDEL) Given an LO-model $\mathcal{P} = (W, \sim, V, \mathsf{p})$, we define a TDEL-model $t(\mathcal{P})$ by $\mathsf{Forest}(\mathcal{M}^{\mathcal{P}}, \mathsf{p}^{\mathcal{P}})$ where $\mathcal{M}^{\mathcal{P}} = (W, \sim, V)$ and $\mathsf{p}^{\mathcal{P}}$ is a state-dependent protocol on $\mathcal{M}$ such that, for all $w \in W$, $\mathsf{p}^{\mathcal{P}}(w)$ consists of sequences of the form $\sigma = \sigma_0 \dots \sigma_n$ where σ_k is of the form $(\mathcal{E}(\varphi, \mathsf{p}, k), e)$ with e in $\mathcal{E}(\varphi, \mathsf{p}, k)$ for all k $(0 \leq k \leq n)$. ◁

Definition 4.3 (Translation from LO to TDEL) Given a model $\mathcal{P} = (W, \sim, V, \mathsf{p})$ in LO, the translation function $t : \mathbb{N} \times \mathcal{L}_{LO} \to \mathcal{L}_{TDEL}$ is defined inductively as follows:

$$t_k(\top) = \top \qquad t_k(p) = p \ \ (p \in \mathsf{At})$$
$$t_k(\varphi \wedge \psi) = t_k(\varphi) \wedge t_k(\psi) \qquad t_k(\neg\varphi) = \neg t_k(\varphi)$$
$$t_k(\hat{K}\varphi) = \hat{K} t_k(\varphi) \qquad t_k(\langle\varphi\rangle\psi) = \langle\mathcal{E}(\varphi, \mathsf{p}, k)\rangle t_{k+1}(\psi)$$
$$t_k(A_i\varphi) = \langle\mathcal{E}(t_k(\varphi), \mathsf{p}, k), 1\rangle\top \vee \langle\mathcal{E}(t_k(\varphi), \mathsf{p}, k), 2\rangle\top$$

◁

Proposition 4.4 *Truth-Preserving Translation Given a formula $\varphi \in \mathcal{L}_{LO}$,*

$$\mathcal{P}, w \models \varphi \quad \text{iff} \quad t(\mathcal{P}), w \models t_o(\varphi).$$

Proof. Straightforward induction. QED

▶ Still need to add discussion of Awareness Logics and Justification Logics

5 Conclusion and Discussion

▶ Discussion of future work: different notions of group knowledge.

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