An Invitation to Modal Logic: Lecture 6

Philosophy 150

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Plan

✓ Motivating Examples

✓ Formalizing the muddy children puzzle, Introduction to Modal Logic

✓ More about truth of modal formulas

✓ Focus on Epistemic Logic.
Digression: A small experiment.

✓ Multiagent Epsitemic Logic, Dynamics in Logic

12/7: Dynamics in Logic II

Common Knowledge and Coordination

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M. Chwe. *Rational Ritual*. 2001.

Common Knowledge

The operator "everyone knows P", denoted EP, is defined as follows

$$EP \quad := \quad \bigwedge_{i \in \mathcal{A}} K_i P$$

$w \models CP$ iff every finite path starting at w ends with a state satisfying P.

$$CP \rightarrow ECP$$

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Suppose you are told "Ann and Bob are going together,"' and respond "sure, that's common knowledge." What you mean is not only that everyone knows this, but also that the announcement is pointless, occasions no surprise, reveals nothing new; in effect, that the situation after the announcement does not differ from that before. ...the event "Ann and Bob are going together" — call it P — is common knowledge if and only if some event — call it Q — happened that entails P and also entails all players' knowing Q (like all players met Ann and Bob at an intimate party). *(Robert Aumann)*

$$P \wedge C(P \rightarrow EP) \rightarrow CP$$

Another Example

Two players Ann and Bob are told that the following will happen. Some positive integer n will be chosen and *one* of n, $n+1$ will be written on Ann's forehead, the other on Bob's. Each will be able to see the other's forehead, but not his/her own.

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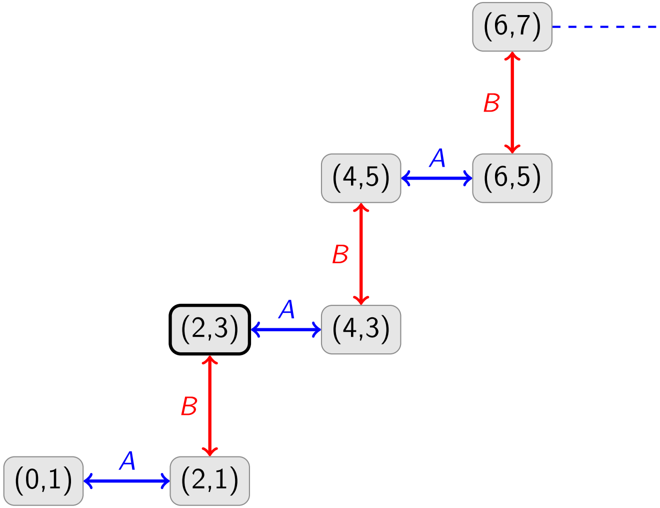
Do the agents know there numbers are less than 1000?

Is it common knowledge that their numbers are less than 1000?


(6,7)
B
(4,5)
A
(6,5)
B
(2,3)
A
(4,3)
B
(0,1)
A
(2,1)

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Logic and Games

Aggregating Judgements

P: The student fails essay 1

Q: The student fails essay 2

R: The student fails the course

	P	Q	$(P \wedge Q) \leftrightarrow R$	R
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

Aggregating Judgements

Should we accept R?

	P	Q	$(P \wedge Q) \leftrightarrow R$	R
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

Aggregating Judgements

Should we accept R? No, a simple majority votes no.

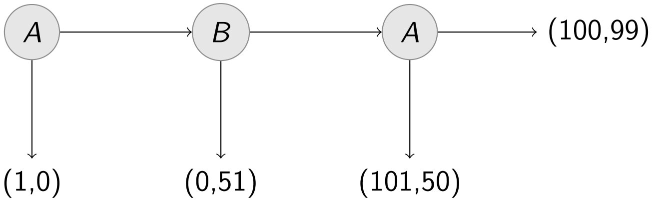
	P	Q	$(P \wedge Q) \leftrightarrow R$	R
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

Aggregating Judgements

Should we accept R? Yes, a majority votes yes for P and Q and $(P \wedge Q) \leftrightarrow R$ is a legal doctrine.

	P	Q	$(P \wedge Q) \leftrightarrow R$	R
1	yes	yes	yes	yes
2	yes	no	yes	no
3	no	yes	yes	no

Backwards Induction



Adjusted Winner

Adjusted winner (AW) is an algorithm for dividing n divisible goods among two people (invented by Steven Brams and Alan Taylor).

For more information see

- *Fair Division: From cake-cutting to dispute resolution* by Brams and Taylor, 1998
- *The Win-Win Solution* by Brams and Taylor, 2000
- `www.nyu.edu/projects/adjustedwinner`

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Item	Ann	Bob
A	5	4
B	65	46
C	30	50
Total	100	100

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Item	Ann	Bob
A	5	0
B	65	0
C	0	50
Total	70	50

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Step 3. Equitability adjustment:
Notice that $65/46 \geq 5/4 \geq 1 \geq 30/50$

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Item	Ann	Bob
A	0	4
B	65	0
C	0	50
Total	65	54

Adjusted Winner: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 3. Equitability adjustment:

Still not equal, so give (some of) B to Bob: $65p = 100 - 46p$.

Item	Ann	Bob
A	0	4
B	65	0
C	0	50
Total	65	54

Adjusted Winner: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

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yielding $p = 100/111 = 0.9009$

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Adjusted Winner: Example

Suppose Ann and Bob are dividing three goods: A, B, and C.

Step 3. Equitability adjustment:

yielding $p = 100/111 = 0.9009$

Item	Ann	Bob
A	0	4
B	58.559	4.559
C	0	50
Total	58.559	58.559

Fairness

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- **Envy-Free** if no party is willing to give up its allocation in exchange for the other player's allocation
- **Equitable** if both players receive the same total number of points
- **Efficient** if there is no other allocation that is strictly better for one party without being worse for another party

Easy Observations

- For two-party disputes, proportionality and envy-freeness are equivalent.
- *AW* only produces equitable allocations (equitability is essentially built in to the procedure).
- *AW* produces splits at most one good is split.

Adjusted Winner is Fair

Theorem (Brams and Taylor) *AW produces allocations that are efficient, equitable and envy-free (with respect to the announced valuations)*

S. Brams and A. Taylor. Fair Division. Cambridge University Press.

Adjusted Winner: Strategizing

Item	Ann	Bob
Matisse	75	25
Picasso	25	75

Ann will get the Matisse and Bob will get the Picasso and each gets 75 of his or her points.

Adjusted Winner: Strategizing

Suppose Ann knows Bob's preferences, but Bob does not know Ann's.

Item	Ann	Bob
M	75	25
P	25	75

Item	Ann	Bob
M	26	25
P	74	75

So Ann will get M plus a portion of P.

According to Ann's announced allocation, she receives 50 points

According to Ann's actual allocation, she receives $75 + 0.33 * 25 = 83.33$ points.

Strategizing: A Theorem

Theorem (Brams and Taylor) *Assume there are two goods, G_1 and G_2, all true and announced values are restricted to integers, and suppose Bob's announced valuation of G_1 is x, where $x \geq 50$. Assume Ann's true valuation of G_1 is b. Then her optimal announced valuation of G_1 is:*

$$\begin{cases} x+1 & \text{if } b > x \\ x & \text{if } b = x \\ x-1 & \text{if } b < x \end{cases}$$

Strategizing: Example

Suppose *both* players know each other's preferences but neither knows that the other knows their own preference.

Item	Ann	Bob
M	75	25
P	25	75

Item	Ann	Bob
M	26	74
P	74	26

Each will get 74 of his or her announced points, but each one is really getting only 25 of his or her *true* points.

Strategizing: Example

Suppose *both* players know each other's preferences. Moreover, Ann knows that Bob knows her preference and Bob doesn't know that Ann knows.

Item	Ann	Bob
M	26	74
P	74	26

Item	Ann	Bob
M	73	74
P	27	26

What happens as the level of knowledge increases?

Conclusions

- Logic can be used to
  - formalize arguments
  - study social situations (multiagent systems)
- Logic and games have a common interest in *interaction*
- Modal logic is language for reasoning about relational structures
- Modal logic has many applications in philosophy, computer science, game theory, linguistics, etc.

Interested?

- Phil 151
- Phil and CS course by Johan van Benthem and Yoav Shoham
- Stanford Encyclopedia of Philosophy article
- Books
  - "Modal Logic" entry in the Stanford Encyclopedia of Philosophy: http://plato.stanford.edu/entries/logic-modal/
  - *A Manual of Intensional Logic* by Johan van Benthem
  - *Modal Logics and Philosophy* by Rod Girle
  - *First-Order Modal Logic* by Melvin Fitting and Richard Mendelsohn

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