An Invitation to Modal Logic: Lecture 3

Philosophy 150

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November 30, 2007

Plan

✓ Motivating Examples

✓ Formalizing the muddy children puzzle, Basic Modal Logic I

11/30: More about truth of modal formulas.

12/3: Basic Modal Logic III

12/5: Dynamics in Logic I

12/7: Dynamics in Logic II

Goal for today: Understand how the basic semantics for modal logic works.

Kripke Structures

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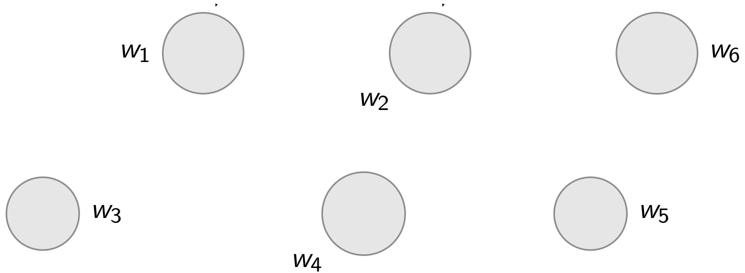

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1. A set of states, or worlds (each world specifies the truth value of all propositional variables)
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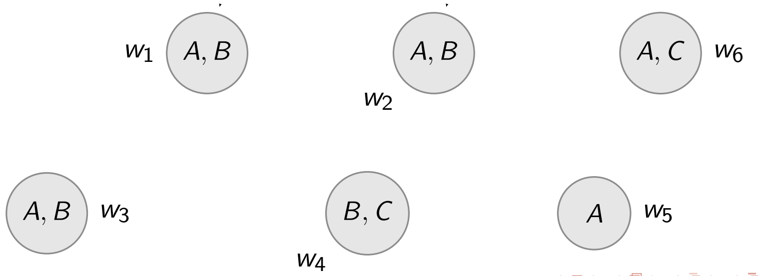
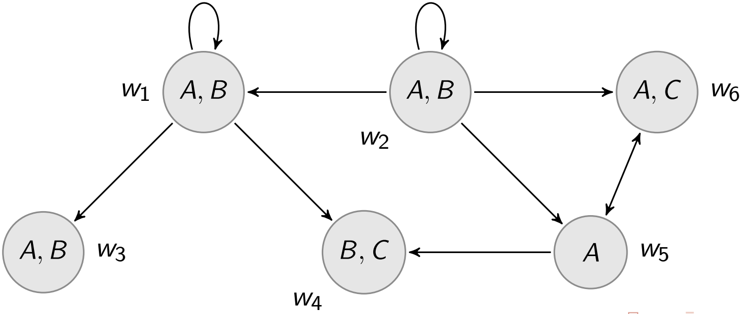
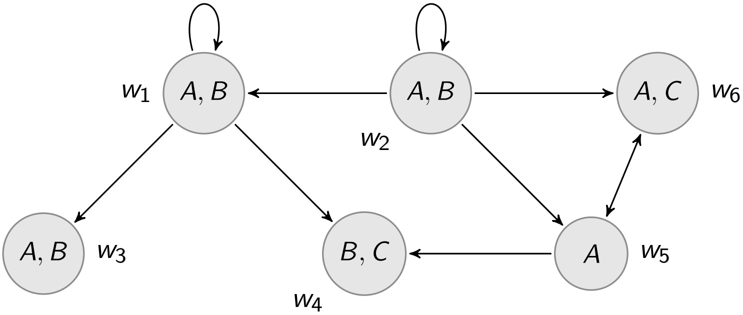


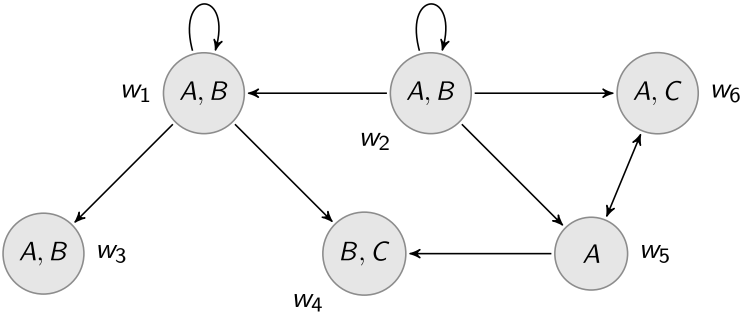
Kripke Structures

A **Kripke structure** is

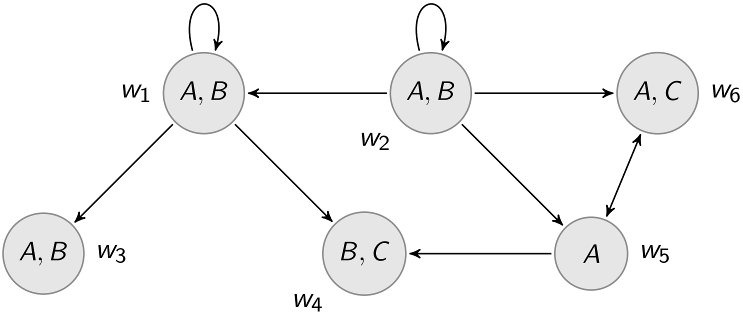
1. A set of states, or worlds (each world specifies the truth value of all propositional variables)
2. A **relation** on the set of states (specifying the "relevant situations")




w_1
A, B
A, B
A, C
w_6
w_2
A, B
w_3
B, C
A
w_5
w_4



- $\Box P$ is true at state w iff P is true in all accessible worlds.
  $w \models \Box P$ iff for all v, if wRv then $v \models P$



$w_2 \models \Box A$

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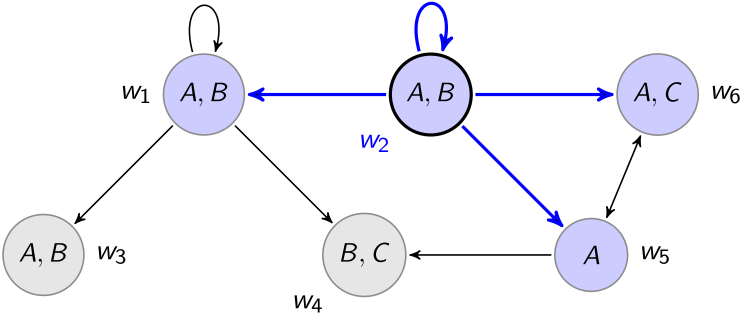

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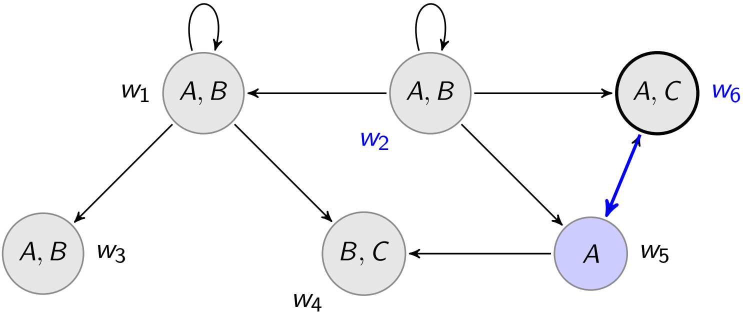

$w_2 \models \Box A$ and $w_6 \models \Box A$

- $\Box P$ is true at state w iff P is true in all accessible worlds.
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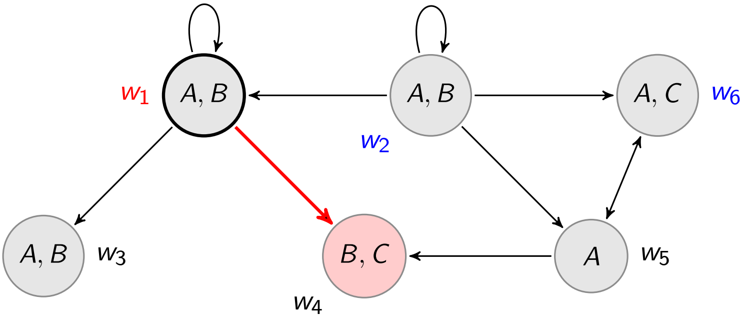
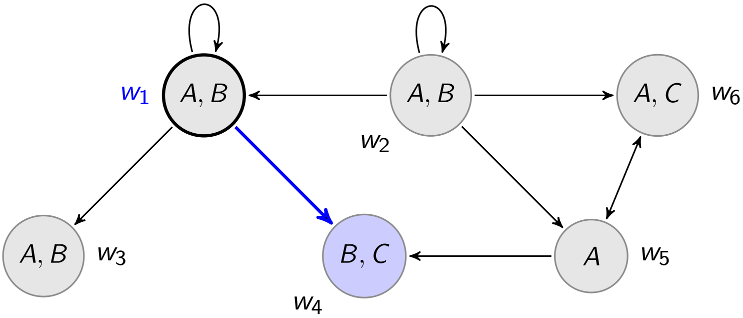


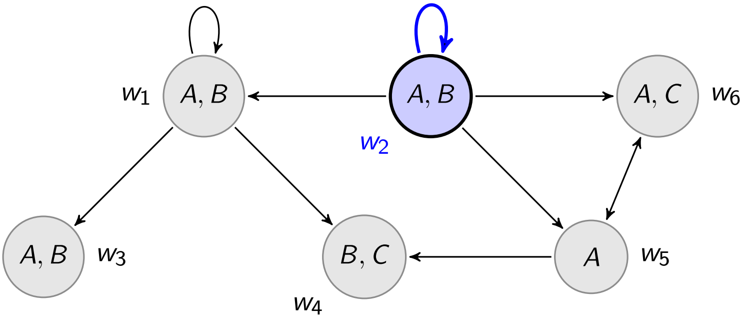
$w_2 \models \Box A$ and $w_6 \models \Box A$ and $w_1 \not\models \Box A$

- $\Box P$ is true at state w iff P is true in all accessible worlds.
  $w \models \Box P$ iff for all v, if wRv then $v \models P$



$w_1 \models \Diamond C$

- $\Diamond P$ is true at state w iff P is true at some accessible world.
  $w \models \Diamond P$ iff there exists v such that wRv and $v \models P$.



$w_1 \models \Diamond C$ and $w_2 \models \Diamond B$

- $\Diamond P$ is true at state w iff P is true at some accessible world.
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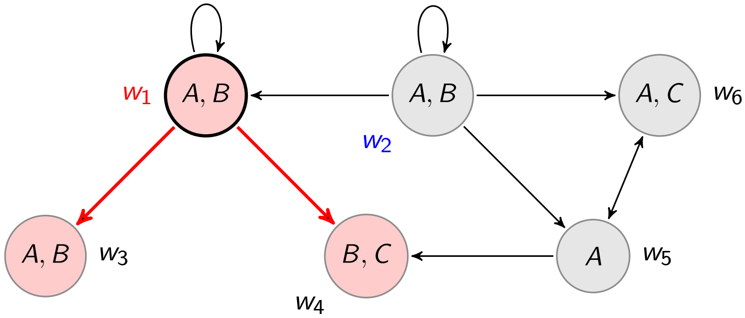
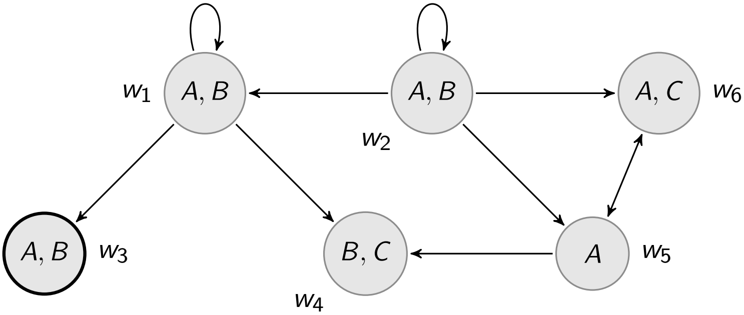


$w_1 \models \Diamond C$ and $w_2 \models \Diamond B$ and $w_1 \not\models \Diamond(\neg A \wedge \neg C)$

- $\Diamond P$ is true at state w iff P is true at some accessible world.
  $w \models \Diamond P$ iff there exists v such that wRv and $v \models P$.



$w_3 \models A$ and $w_3 \not\models \Diamond A$

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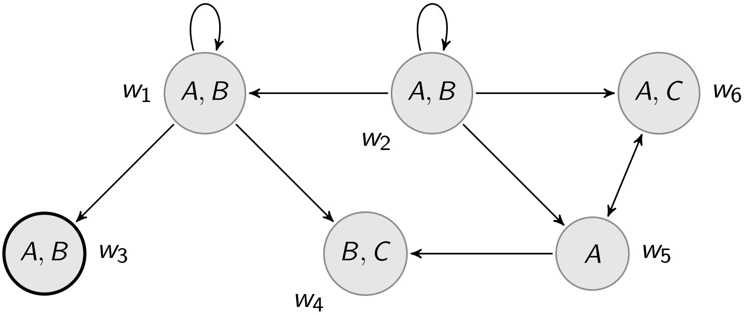
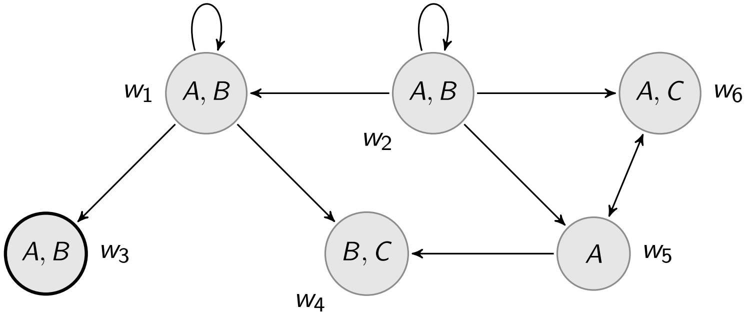


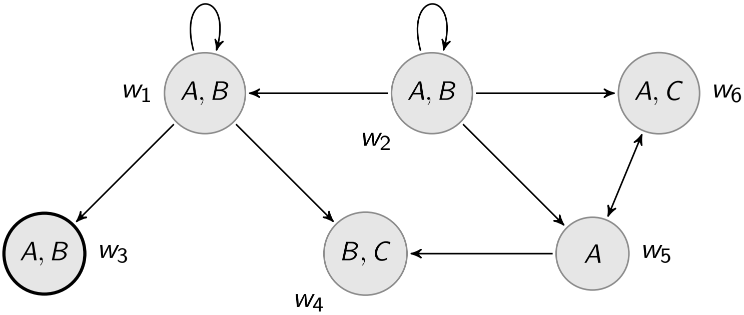
$w_3 \models A$ and $w_3 \not\models \Diamond A$ and $w_3 \not\models \Diamond(A \vee \neg A)$

- $\Diamond P$ is true at state w iff P is true at some accessible world.
  $w \models \Diamond P$ iff there exists v such that wRv and $v \models P$.



$w_3 \not\models C$ and $w_3 \models \Box C$

- $\Box P$ is true at state w iff P is true in all accessible worlds.
  $w \models \Box P$ iff for all v, if wRv then $v \models P$



$w_3 \not\models C$ and $w_3 \models \Box C$ and $w_3 \models (C \wedge \neg C)$

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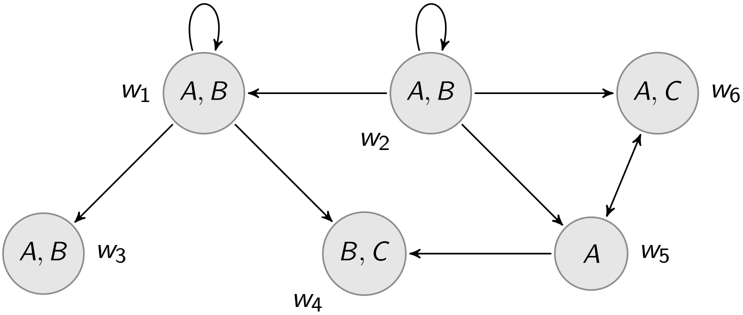

Where is $\Box A \rightarrow A$ true?

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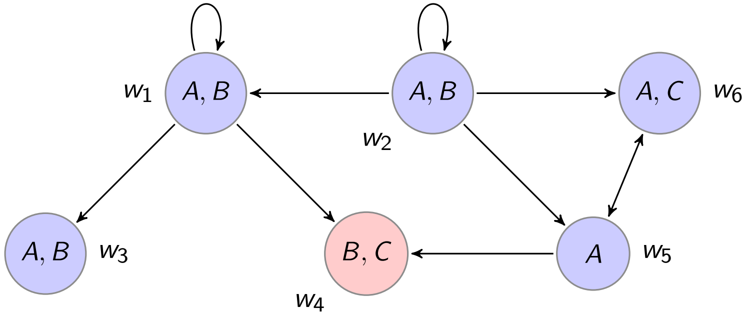

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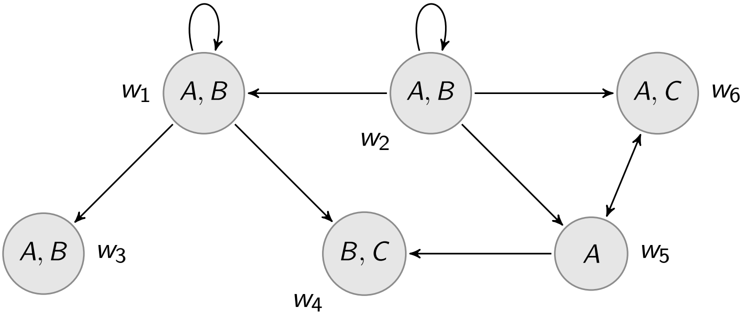

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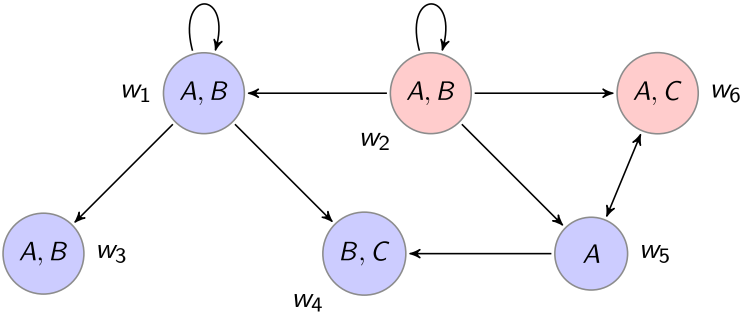

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Some Facts

- $\Box P \vee \neg\Box P$ is always true (i.e., true at any state in any Kripke structure), but what about $\Box P \vee \Box\neg P$?

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- $\Box P \leftrightarrow \neg\Diamond\neg P$ is true at any state in any Kripke structure.

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- $\Box P \wedge \Box Q \rightarrow \Box(P \wedge Q)$ is true at any state in any Kripke structure. What about $\Box(P \vee Q) \rightarrow (\Box P \vee \Box Q)$?
- $\Box P \leftrightarrow \neg\Diamond\neg P$ is true at any state in any Kripke structure.
- It is not true that $\Diamond P \rightarrow \Box P$ is true at any state in any Kripke structure.

More Facts

Determine which of the following formulas are true at any state in any Kripke structure:

1. $\Box P \rightarrow \Diamond P$
2. $\Box(P \vee \neg P)$
3. $\Box P \rightarrow P$
4. $P \rightarrow \Box\Diamond P$
5. $\Diamond(P \vee Q) \rightarrow \Diamond P \vee \Diamond Q$

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Some Facts

- $\Box P \rightarrow P$ is true at any state in any Kripke structure where each state is accessible from itself.
- $\Box P \rightarrow \Diamond P$ is true at any state in any Kripke structure where each state has at least one accessible world.

Can you think of properties that force each of the following formulas to be true at any state in any appropriate Kripke structure?

1. $\Diamond P \rightarrow \Box P$
2. $\Box P \rightarrow \Box\Box P$

Modal logic is a good formal language for "talking about" Kripke structures!

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Kripke structures, or more generally relational structures, are important in

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- Linguistics
- Theoretical Computer Science
- Game Theory
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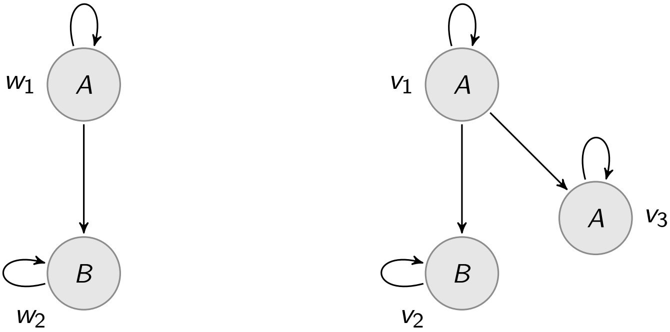
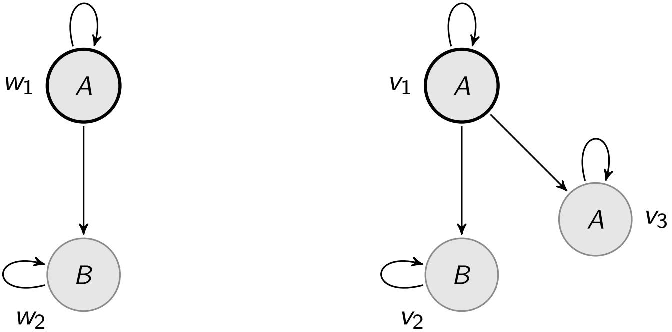
What "good" means will be discussed in Philosophy 151.



What is the difference between states w_1 and v_1?



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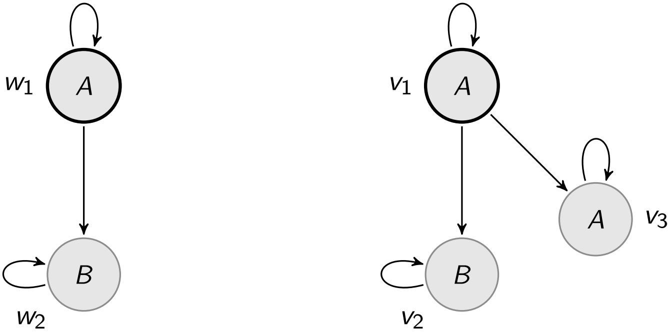
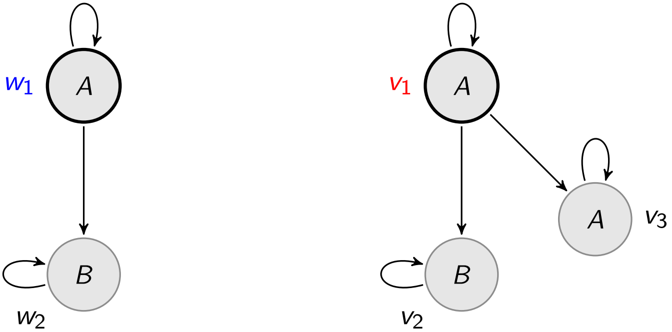
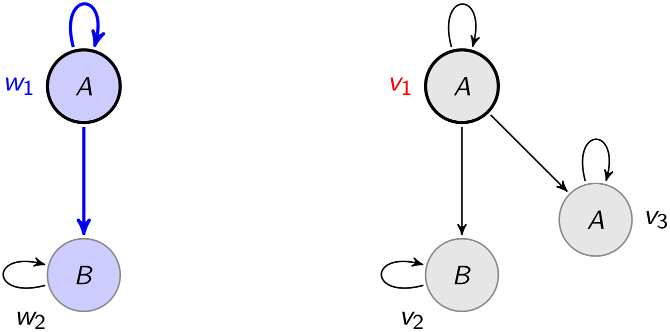


Is there a modal formula true at w_1 but not at v_1?



$w_1 \models \Box\Diamond\neg A$ but $v_1 \not\models \Box\Diamond\neg A$.



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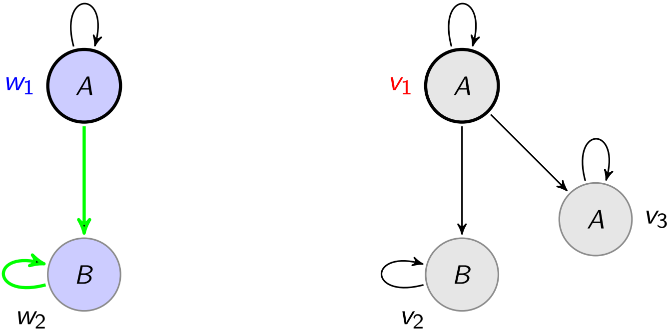

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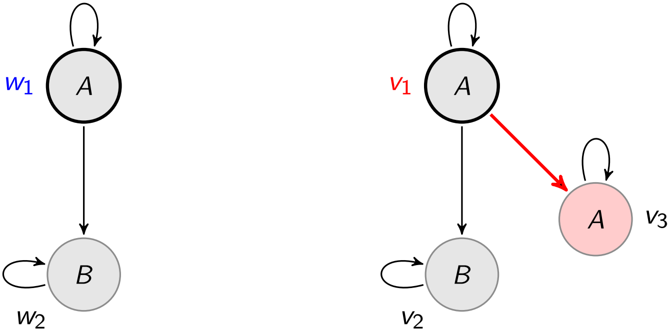

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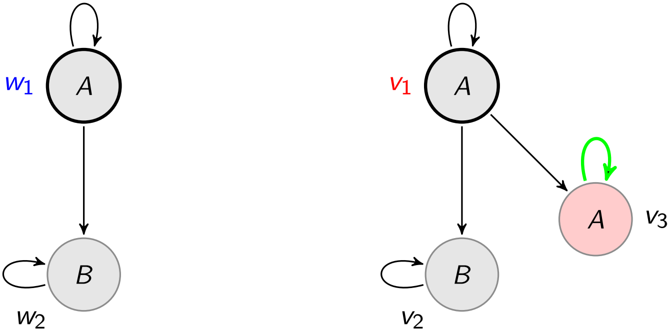

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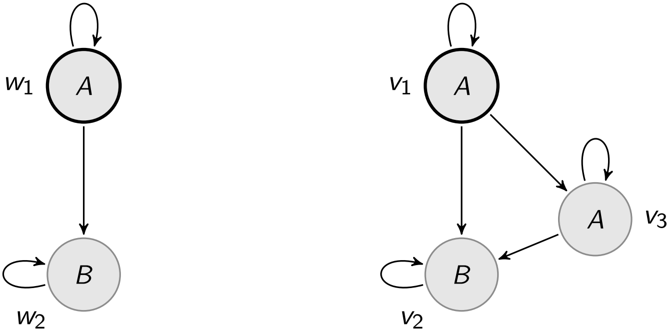
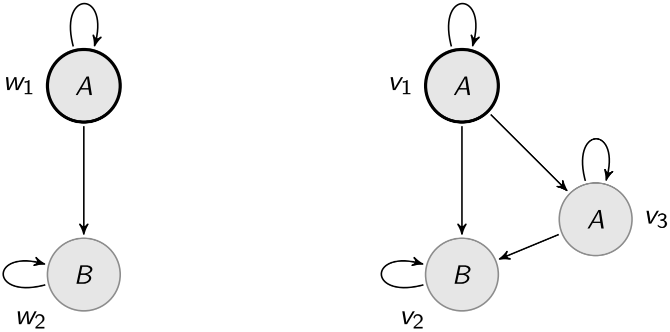


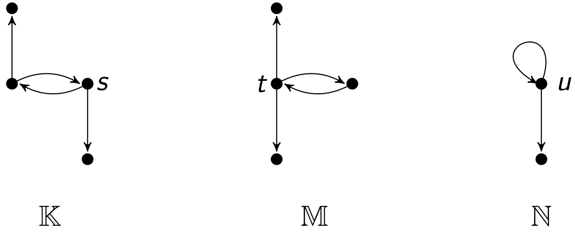
What about now? Is there a modal formula true at w_1 but not v_1?



No modal formula can distinguish w_1 and v_1!

A More Complicated Example

Which pair of states cannot be distinguished by a modal formula?



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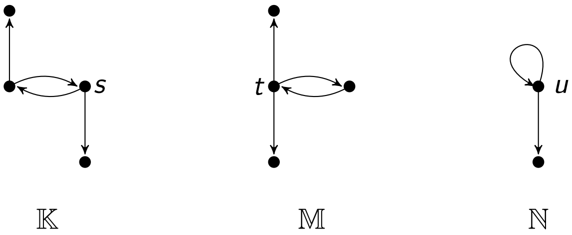

$\Box(\Box\bot \vee \Diamond\Box\bot)$

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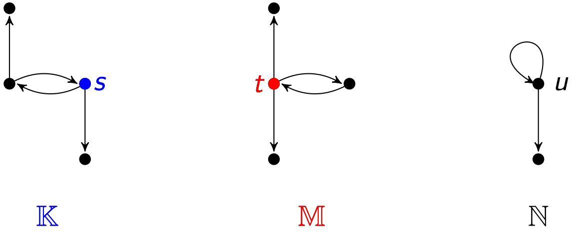

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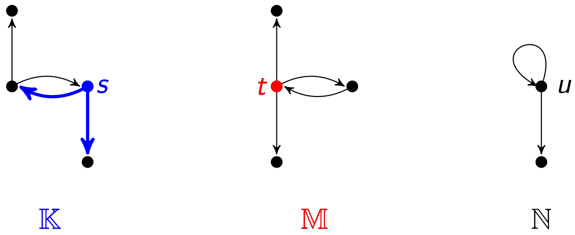

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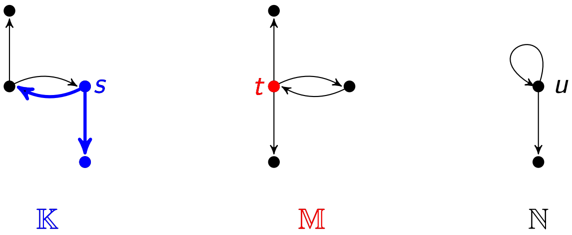

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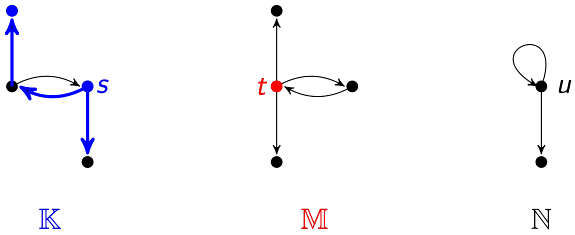

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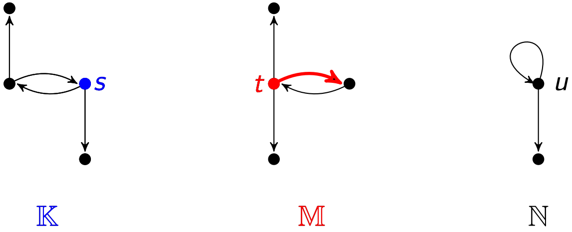

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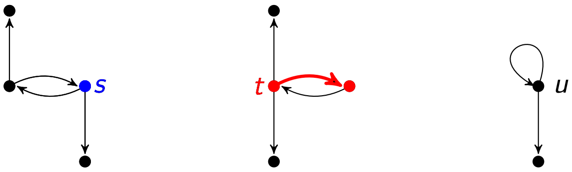

$\mathbb{K}$ $\mathbb{M}$ $\mathbb{N}$

$\Box(\Box\bot \vee \Diamond\Box\bot)$

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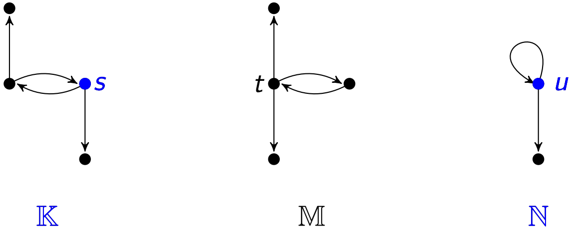

More about this in Philosophy 151!

Next week: Focus on epistemic logic.

Homework: available on the course website.

Questions?
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Office: Gates 258