An Invitation to Modal Logic: Lecture 1

Philosophy 150

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Setting the Stage

Much of this course has focused on techniques to evaluate arguments.

Arguments have been analyzed from both syntactic and semantic perspectives.

Setting the Stage

list of *sentences* (premises followed by a conclusion)

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Setting the Stage

Is the argument *valid*?

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Setting the Stage

formal proofs

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Truth-tables, first-order structures

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Setting the Stage: Two Logics

Boolean Logic (BL)

- Language: $P \wedge Q$, $P \rightarrow (Q \vee \neg R)$, etc.
- Proof-Theory: $\wedge$-elim, $\wedge$-intro, $\vee$-elim, etc.
- Semantics: Truth-tables

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First-Order Logic (FOL)

- Language: $x = y$, $\forall x \exists y (F(x) \rightarrow (G(x, y) \wedge \neg R(y)))$, etc.
- Proof-Theory: $\forall$-elim, $\forall$-intro, etc.
- Semantics: First-order structures

Setting the Stage

Do we need the quantifiers?

Setting the Stage

Do we need the quantifiers? Yes!

All men are mortal
Socrates is a man

Socrates is mortal

Setting the Stage

There are some valid arguments that cannot be formalized using either boolean or first-order logic.

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There are some valid arguments that cannot be formalized using either boolean or first-order logic.

Plan for today: highlight a number of such arguments.

Plan for the next 6 classes

11/26: Motivating Examples

11/28: Motivating Examples, Basic Modal Logic I

11/30: Basic Modal Logic II

12/3: Basic Modal Logic III

12/5: Dynamics in Logic I

12/7: Dynamics in Logic II

Modern Modal Logic began with C.I. Lewis' dissatisfaction with the material conditional ($\rightarrow$).

The Material Conditional

X	Y	$X \to Y$

The Material Conditional

X	Y	$X \rightarrow Y$
T	T	T

The Material Conditional

X	Y	$X \to Y$
T	T	T
T	F	F

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Monotonicity

X	Y	$X \to Y$
T	T	T
T	F	F
F	T	T
F	F	T

$$\frac{S \to E}{(S \wedge G) \to E}$$

Monotonicity

S	E	$S \to E$
T	T	T
T	F	F
F	T	T
F	F	T

$$\frac{\overbrace{S \to E}^{T}}{(S \wedge G) \to E}$$

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$$\frac{S \to E}{(S \wedge G) \to E}$$

$$\frac{\text{If I put sugar in my coffee, then it will taste excellent}}{\text{If I put sugar and gasoline in my coffee then it will taste excellent}}$$

Dorothy Edgington's Proof of the Existence of God

X	Y	$X \to Y$
T	T	T
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$$\neg G \to \neg(P \to A)$$

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If God does not exist, then it's not the case that if I pray, my prayers will be answered

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$$\neg P$$

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I don' pray

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T	T	T
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$$\neg G \rightarrow \overbrace{\neg(P \rightarrow A)}^{F}$$

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God exists!

What do we want?

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Many subtle issues!

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Prosecutor: $G \rightarrow A$
Defense: $\neg(G \rightarrow A)$
Judge: $\neg(G \rightarrow A)$

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Defense: $\neg(G \rightarrow A)$
Judge: $\neg(G \rightarrow A)$ $\Leftrightarrow G \wedge \neg A$, therefore G!

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Prosecutor: "If Eric is guilty then he had an accomplice."
Defense: "I disagree!"
Judge: "I agree with the defense."

Prosecutor: $\Box(G \rightarrow A)$ (It is necessarily the case that ...)
Defense: $\neg\Box(G \rightarrow A)$
Judge: $\neg\Box(G \rightarrow A)$ (*What can the Judge conclude?*)

The Basic Modal Language

A wff of Modal Logic is defined *inductively*:

1. Any atomic propositional variable is a wff
2. If P and Q are wff, then so are $\neg P$, $P \wedge Q$, $P \vee Q$ and $P \rightarrow Q$
3. If P is a wff, then so is $\Box P$ and $\Diamond P$

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Eg., $\Box(P \rightarrow \Diamond Q) \vee \Box\Diamond\neg R$

What is a modal?

A modal qualifies the truth of a judgement.

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- is permitted to be
- is obliged to be
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- will be
- has a strategy to become
- ...

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  $\Box P \leftrightarrow \neg\Diamond\neg P$

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- *A is permitted iff it is not the case that* $\neg A$ *is obligatory.*
  $PA \leftrightarrow \neg O \neg A$

There are many interesting arguments involving modalities!

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4. Hence, either it is necessary that there is a sea battle tomorrow or it is necessary that none occurs.

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Two readings:

$$\frac{\begin{array}{l} A \rightarrow \Box B \\ A \rightarrow \Box\neg B \\ A \vee \neg A \end{array}}{\Box B \vee \Box\neg B}$$

$$\frac{\begin{array}{l} \Box(A \rightarrow B) \\ \Box(A \rightarrow \neg B) \\ A \vee \neg A \end{array}}{\Box B \vee \Box\neg B}$$

Deontic Logic

OA means A is obligatory
PA means A is permitted

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Is the following argument valid?

$$\frac{\text{If } A \text{ then } B \ (A \rightarrow B)}{\text{If } A \text{ is obligatory then so is } B \ (OA \rightarrow OB)}$$

Deontic Logic

1. Jones murders Smith. (M)
2. If Jones murders Smith, then Jones ought to murder Smith gently. ($M \rightarrow OG$)

(first discussed by J. Forrester in 1984)

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5. If Jones ought to murder Smith gently, then Jones ought to murder Smith. ($OG \rightarrow OM$)
6. Jones ought to murder Smith. (OM)

(first discussed by J. Forrester in 1984)

Three children are outside playing. Two of them get mud on their forehead. They cannot see or feel the mud on their own foreheads, but can see who is dirty.

Their mother enters the room and says "At least one of you have mud on your forehead".

Then the children are repeatedly asked "do you know if you have mud on your forehead?"

What happens?

Claim: After first question, the children answer "I don't know",

[1] Corrected from the lecture

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Claim: After first question, the children answer "I don't know", after the second question the muddy children answer "I have mud on my forehead!" (but the clean child is still in the dark[1]). Then the clean child says, "Oh, I must be clean."

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Summary

- We now have (at least) three formal languages: boolean, first-order and modal (alethic, deontic, epistemic, ...)
- Modern modal logic was developed to study (strict) implications. Gradually, the study of '$\Diamond$' and '$\Box$' themselves became dominant, with the study of "implication" developing into a separate topic.
- There are many interesting arguments involving modalities.

A few questions to keep you up at night...

- Two integers x and y are chosen with $1 < x < y$ and $x + y \leq 100$. Mr. S is informed only of $s = x + y$ and Mr. P is informed only of $P = xy$. The following conversation takes place:

  1. Mr. P says: "I do not know the pair."
  2. Mr. S says: "I knew you didn't."
  3. Mr. P says: "I now know the pair."
  4. Mr. S says: "I now know too."

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- Can we give a truth-table semantics for the basic modal language? (**Hint**: there are only 4 truth-table definable functions for a single operator. Suppose we want $\Box A \rightarrow A$ to be valid, but not $A \rightarrow \Box A$ and $\neg\Box A$.)

Some Reading Material

"Modal Logic" entry in the Stanford Encyclopedia of Philosophy:
http://plato.stanford.edu/entries/logic-modal/

A Manual of Intensional Logic by Johan van Benthem

Modal Logics and Philosophy by Rod Girle

First-Order Modal Logic by Melvin Fitting and Richard Mendelsohn

Next time: basic modal logic plus more examples (a more formal analysis of the muddy children puzzle).

Questions?
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