Neighborhood Semantics for Modal Logic
An Introduction

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Welcome to *Neighborhood Semantics for Modal Logic*! The course will consist of five 90 minute lectures roughly organized as follows:

Day 1: Introduction, Motivation and Basic Concepts

Day 2: Non-normal Modal Logics: Completeness, Decidability and Complexity

Day 3: Advanced Topics I

Day 4: Advanced Topics II

Day 5: Applications and Extensions

The precise topics for the course will depend in part on the interests and background of the students attending the lectures. This document contains an introduction to the course including some pointers to relevant literature. The focus is on the basics of neighborhood semantics with only pointers to literature relating to the more advanced topics. The website for the course is

`staff.science.uva.nl/∼epacuit/nbhd_esslli.html`

On this website you will find the lecture slides (and possibly course notes) updated daily. Two other sources will also be useful:

1. Brian Chellas, *Modal Logic: an Introduction* (Cambridge University Press, 1980), Chapters 7 - 9 cover basic results about classical modal logic, and

2. Helle Hvid Hansen, *Monotonic modal logics*, Master's Thesis, Institute for Logic, Language and Information (2003). Available at

   `http://www.few.vu.nl/~hhhansen/papers/scriptie_pic.pdf`.

Although the course will be self-contained, it is recommended to have basic knowledge of modal logic. For example, it will be useful to be familiar with:

1. The first four chapters (skipping the advanced track sections) of *Modal Logic* by P. Blackburn, M. de Rijke and Y. Venema (Cambridge University Press, 2001), and/or

2. *Modal Logic: an Introduction* by Brian Chellas (Cambridge University Press, 1980).

The main goal of the corse is to understand the basic techniques, results and applications of neighborhood semantics for modal logic and to understand the exact relationship with the standard relational semantics. Enjoy the course and please remember to ask questions during the lecture, point out any mistakes and/or omitted references in this text and my lecture notes.

1 Introduction and Motivation

Neighborhood models are a generalization of the standard Kripke, or relational, models for modal logic invented by[1] Dana Scott and Richard Montague (independently in [32] and [26]). Before diving into the details of neighborhood models we take some time to review some key ideas about relational semantics for modal logic (please consult [4] for a complete discussion).

Early on (in 1971) Krister Segerberg wrote an essay [33] presenting some basic results about neighborhood models and the classical systems that correspond to them and later on Brian Chellas incorporated these and other salient results in part III of his textbook [6]. Nevertheless for more than 15 years or so after 1971, in the apparent absence of applications or in the absence of guiding intuitions concerning the role of neighborhoods, non-normal classical modal logics were studied mainly in view of their intrinsic mathematical interest. This situation has changed in important ways during the last 18 or so years.

[1] In fact, the idea for neighborhood semantics for modal logic is already implicit in the seminal work of McKinsey and Tarski [25].

1.1 Relational Semantics for Modal Logic

The **basic modal language**, denoted $\mathcal{L}(\mathsf{At})$, where At is a (finite or infinite) set of atomic sentences (we will often write $\mathcal{L}$ when At is clear from context) is generated by the following grammar[2]:

$$p \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi \mid \Diamond\phi$$

where $p \in \mathsf{At}$. The usual abbreviations will be used.

Definition 1.1 (Relational Structures) A **relational frame** is a pair $\langle W, R\rangle$ where W is a nonempty set and R is a relation on W (i.e., $R \subseteq W \times W$). We write wRv if $(w, v) \in R$. A **relational model** based on a frame $\mathbb{F} = \langle W, R\rangle$ is a tuple $\langle W, R, V\rangle$ where $V : \mathsf{At} \to 2^W$ is called a **valuation function**. The set W called the domain of $\mathbb{M}$ ($\mathbb{F}$) and may be denote $\mathcal{D}(\mathbb{M})$ ($\mathcal{D}(\mathbb{F})$). $\triangleleft$

The intuition is that each state is a propositional model (i.e., a boolean valuation[3]) with the boolean connectives interpreted *locally*. The additional structure in a relational model (i.e., the relation R) is used to give a truth value to formulas where the modal operator is the main connective. Formulas from $\mathcal{L}$ are interpreted at states (elements of W).

Definition 1.2 (Truth in Relational Models) Let $\mathbb{M} = \langle W, R, V\rangle$ be a relational model. Truth of a formula $\phi \in \mathcal{L}$ at a state $w \in W$ is defined inductively as follows:

1. $\mathbb{M}, w \models p$ iff $w \in V(p)$
2. $\mathbb{M}, w \models \neg\phi$ iff $\mathbb{M}, w \not\models \phi$
3. $\mathbb{M}, w \models \phi \wedge \psi$ iff $\mathbb{M}, w \models \phi$ and $\mathbb{M}, w \models \psi$
4. $\mathbb{M}, w \models \Box\phi$ iff for each $v \in W$, if wRv then $\mathbb{M}, v \models \phi$
5. $\mathbb{M}, w \models \Diamond\phi$ iff there is a $v \in W$ such that wRv and $\mathbb{M}, v \models \phi$

We say a formula $\phi \in \mathcal{L}$ is **valid in a model** $\mathbb{M}$, written $\mathbb{M} \models \phi$, if for each state w in W, $\mathbb{M}, w \models \phi$. The formula ϕ is **valid in a frame** $\mathbb{F}$, written $\mathbb{F} \models \phi$, if for all models $\mathbb{M}$ based on that frame, $\mathbb{M} \models \phi$. Finally, a formula ϕ is said to be **satisfiable** in a model $\mathbb{M}$ if there is a state $w \in \mathcal{D}(\mathbb{M})$ such that $\mathbb{M}, w \models \phi$. $\triangleleft$

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[2]Typically only one of $\Box$ and $\Diamond$ is taken as primitive and the other is *defined* to be the dual, for example $\Diamond\phi$ is sometimes defined to be $\neg\Box\neg\phi$. We have opted to take both $\Box$ and $\Diamond$ as primitive for reasons which will become clear later in Section 2.

[3]To make this precise, think of the valuation function V as a function from $W \times \mathsf{At}$ to $\{0, 1\}$ where $V(w, p) = 1$ iff $w \in V(p)$ and $V(w, p) = 0$ iff $w \notin V(p)$.

The above models have successfully been used to model a number of different intensional notions. For example, we can interpret $\Box\phi$ as "an agent knows ϕ". From this point of view, the accessibility relation R in a model has a clear interpretation — wRv is intended to mean "according to the agent's current information the states w and v are *indistinguishable*". Here we see the role of the relation R in the above models is to provide us with information that is not provided by the set of states W — the information the agent has *about* the states W. In other words, for every sentence $\phi \in \mathcal{L}$, the relation R provides a mechanism to determine whether or not the agent knows ϕ. So we can think of relational models as a simple and elegant way to represent ground facts (i.e., the facts about the current state of affairs), the agent's knowledge of these facts, the agent's knowledge of its knowledge about these facts, and so on *ad infinitum*.

Using relations to represent this additional information forces us to accept a number of principles. It is not hard to see (and well-known) that the following formulas are valid in any relational model:

1. $\Box(\phi \wedge \psi) \rightarrow \Box\phi \wedge \Box\psi$
2. $\Box\phi \wedge \Box\psi \rightarrow \Box(\phi \wedge \psi)$
3. $\Box\top$
4. $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$
5. $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$

We give the details that the first formula is valid in any relational model and leave the simple but illuminating proofs of the other facts to the reader.

Observation 1.3 *The formula $\Box(\phi \wedge \psi) \rightarrow \Box\phi \wedge \Box\psi$ is valid on any relational model.*

Proof. Let $\mathbb{F} = \langle W, R\rangle$ be any relational frame and $\mathbb{M} = \langle W, R, V\rangle$ any model based on $\mathbb{F}$. Suppose $w \in W$ and $\mathbb{M}, w \models \Box(\phi \wedge \psi)$. Let $v \in W$ be any state with wRv. Then $\mathbb{M}, v \models \phi \wedge \psi$ and hence $\mathbb{M}, v \models \phi$. Thus, for any $v \in W$, if wRv then $\mathbb{M}, v \models \phi$; and so, $\mathbb{M}, w \models \Box\phi$. Similarly, $\mathbb{M}, w \models \Box\psi$. Hence $\mathbb{M}, w \models \Box\phi \wedge \Box\psi$. Therefore, $\mathbb{M}, w \models \Box\phi \wedge \Box\psi$. As $\mathbb{M}$ is an arbitrary model based on $\mathbb{F}$ and w an arbitrary state, $\mathbb{F} \models \Box(\phi \wedge \psi) \rightarrow \Box\phi \wedge \Box\psi$. QED

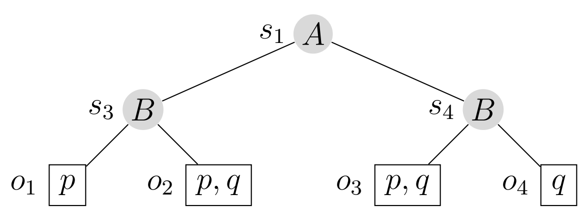
Exercise 1.4 *Prove that all of the above formulas are valid on any relational frame.*

1.2 Motivating Examples

Thus, when one uses relational structures to interpret modal formulas, one is forced to accept the above principles. For many interpretations this is relatively uncontroversial. However, there are interpretations in which the validity of some of the above axioms can be called into question. We now run through some of the most convincing examples. We do not go into full details, but only give the main ideas. The relevant references are provided in Section **??**

Logics of High Probability: Suppose that the intended interpretation of $\Box\phi$ is "ϕ is assigned 'high' probability", where "high" probability means above a certain threshold $r \in [0, 1]$. Under this interpretation, the second formula is not longer valid. To see this, suppose that p and q represent *independent events* (i.e., there are no states that satisfy both p and q) and suppose that the threshold $r = \frac{2}{3}$. Suppose that both p and q are assigned probability $\frac{3}{4}$. Then $\Box p$ and $\Box q$ holds (i.e., their probability is higher than the threshold). However, the probability of 'p and q' is $\frac{9}{16}$, which is less than the threshold. Thus $\Box(p \wedge q)$ does not hold. Therefore, $\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)$ is not valid under this interpretation. Note that this argument does not work if we take the threshold to be 1. What happens if $r < \frac{1}{2}$? $\triangleleft$

Game Logics: Consider the following game between Ann (A) and Bob (B):



Here Ann has the first move followed by a choice for Bob. It is also indicated which of the propositional variables p and q are true at the outcome nodes (labeled o_1, o_2, o_3 and o_4).

We say that a player can *force* a set of outcome states X if that player has a *strategy* that guarantees the game will end in one of the states in X. For example, in the above game, Ann can force the set $X_1 = \{o_1, o_2\}$. This follows because Ann has a strategy (move left) such that no matter what action Bob chooses, the outcome of the game will be a state in X_1. Similarly, Ann can also force the set $X_2 = \{o_3, o_4\}$. However, Ann cannot force the set $X_3 = \{o_2, o_3\}$ since Bob has the freedom to select either o_1 or o_3 depending on Ann's choice. We can make this observation more formal: given an internal state s, we write $s \models \Box\phi$ if Ann[4] has a strategy to force the outcome of the game to be a state that satisfies ϕ. Then in the above example, $s_1 \models \Box p \wedge \Box q$ but $s_1 \not\models \Box(p \wedge q)$. $\triangleleft$

The preceding frameworks provide natural counterexamples to the validity of the second formula $(\Box\phi \wedge \Box\psi \rightarrow \Box(\phi \wedge \psi)$. What about the other formulas listed above?

Exercise 1.5 *Which of the formulas 1-4 are valid under the probabilistic interpretation? What about the game-theoretic forcing interpretation?*

One difficulty in answering this question about formula 5 $(\Box\phi \leftrightarrow \neg\Diamond\neg\phi)$ is that we must provide an interpretation for the $\Diamond$ operator. Let us consider the game-theoretic forcing example. A natural interpretation of $\Diamond\phi$ is "*Bob* has a starategy to ensure that ϕ is true". A natural assumption about the game is consistency: it cannot be the case that can force ϕ to be true and Bob can force $\neg\phi$ to be true. In other words, $\neg(\Box\phi \wedge \Diamond\neg\phi)$ is valid under this consistency assumption. Using propositional reasoning, this formula is equivalent to $\Box\phi \rightarrow \neg\Diamond\neg\phi$. The validity of the converse requires a stronger game-theoretic assumption. Rewriting $\neg\Diamond\neg\phi \rightarrow \Box\phi$ as $\Diamond\neg\phi \vee \Box\phi$, this formulas says "either Bob has a strategy to force $\neg\phi$ to be true or Ann has a strategy to force ϕ to be true." If we think of the formula ϕ as stating that Ann has won the game, then the validity of this formula amounts to assuming the games are *determined* (either Ann has a winning strategy or Bob has a winning strategy). Thus $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$ is valid with respect to all consistent, two-person determined games.

We now consider the validity of the first formula $(\Box(\phi \wedge \psi) \rightarrow \Box\phi \wedge \Box\psi)$. It is more convenient to work with a rule (called the monotonic rule) which is equivalent to assuming that this formula is valid: from $\phi \rightarrow \psi$ infer $\Box\phi \rightarrow \Box\psi$. For now, we will not go into details, but will return to this fact in Chapter **??**.

Deontic Logics: Deontic logicians interpret $\Box\phi$ as "it is obliged that ϕ". We claim that under this interpretation, the monotonicity rule described above should *not* be valid. The following "Paradox of Gentle Murder" of J. Forrester illustrates the problem (we follow the discussion from L. Goble's "Murder Most Gentle: The Paradox Deepens"[14]). See also [15]. Consider the following three statements:

1. Jones murders Smith,

[4] In fact, there are many choices here. We could have taken $\Box\phi$ to mean *Bob has a strategy for force ϕ to be true* or even label the modalities with each player. See [37] for a complete discussion.

2. Jones ought not to murder Smith,

3. If Jones murders Smith, then Jones ought to murder Smith gently.

Intuitively, these sentences appear to be consistent. However, 1 and 3 together imply that

4. Jones ought to murder Smith gently

Also we have the following fact:

5. If Jones murders Smith gently, then Jones murders Smith.

Of course, this *not* a logical validity, but rather a fact about the world we live in. Now, if we assume that the monotonicity rule is valid, then 5. entails

6. If Jones ought to murder Smith gently, then Jones ought to murder Smith.

And so, 4. and 6. together imply

7. Jones ought to murder Smith.

But this contradicts statement 2. This argument strongly suggests that deontic logics should *not* validate the monotonicity rule. Of course, this is only strong evidence that the monotonicity rule *should* not be valid under the deontic interpretation rather than a specific counterexample to the rule (as in the previous two examples). But this is enough for our purposes: it is at least plausible that one may want to consider logics in which the the monotonicity rule (hence formula 1) is not valid. $\triangleleft$

There are a number of other motivations for studying weak systems of modal logic. For example, [27] looks that the modal logic of *derivability* which turns out to be a non-normal modality. Other applications include logics for reasoning about strategic abilities of agents [1, 31], weak epistemic logics without "logical omniscience" [29, 3] and certain logics for knowledge representation [28].

2 Basic Concepts

2.1 Preliminaries

Sets paired with a distinguished collections of subsets are ubiquitous in many areas of mathematics. They show up as *topologies* or *simple games* to name two of the most usual suspects. More formally, let W be a non-empty set and $\wp W$

the collection of all subsets of W. We are interested in pairs $\langle W, \mathcal{F}\rangle$ where W is a non-emtpy set and $\mathcal{F} \in \wp\wp W$. Typically, the sets $\mathcal{F} \subseteq \wp W$ satisfy certain algebraic properties. We list some relevant properties below:

1. $\mathcal{F}$ is **closed under intersections** if for any collections of sets $\{X_i\}_{i\in I}$ such that for each $i \in I$, $X_i \in \mathcal{F}$, then $\cap_{i\in I} X_i \in \mathcal{F}$. If we assume $|I| = 2$ then we say $\mathcal{F}$ is closed under **binary intersections**. If I is finite, then we say $\mathcal{F}$ is closed under **finite intersections**. Generally, for any cardinal κ, we say that $\mathcal{F}$ is **closed under $\leq \kappa$-intersections** if for each collections of sets $\{X_i\}_{i\in I}$ from $\mathcal{F}$ with $|I| \leq \kappa$, $\cap_{i\in I} X_i \in \mathcal{F}$.

2. $\mathcal{F}$ is **closed under unions** if for any collections of sets $\{X_i\}_{i\in I}$ such that for each $i \in I$, $X_i \in \mathcal{F}$, then $\cup_{i\in I} X_i \in \mathcal{F}$. (The same comments as above about binary and finite collections apply here as well: in particular, for any cardinal κ, we say that $\mathcal{F}$ is **closed under $\leq \kappa$-unions** if for each collections of sets $\{X_i\}_{i\in I}$ from $\mathcal{F}$ with $|I| \leq \kappa$, $\cup_{i\in I} X_i \in \mathcal{F}$.)

3. $\mathcal{F}$ is **closed under complements** if for each $X \subseteq W$, if $X \in \mathcal{F}$, then $X^C \in \mathcal{F}$.

4. $\mathcal{F}$ is **supplemented**, or **closed under supersets** provided for each $X \subseteq W$, if $X \in \mathcal{F}$ and $X \subseteq Y \subseteq W$, then $Y \in \mathcal{F}$.

5. $\mathcal{F}$ **contains the unit** provided $W \in \mathcal{F}$; and $\mathcal{F}$ **contains the empty set** if $\emptyset \in \mathcal{F}$.

6. Call the set $\cap_{X\in\mathcal{F}} X$ the **core of** $\mathcal{F}$. $\mathcal{F}$ **contains its core** provided $\cap_{X\in\mathcal{F}} X \in \mathcal{F}$.

7. $\mathcal{F}$ is **proper** if $X \in \mathcal{F}$ implies $X^C \notin \mathcal{F}$.

8. $\mathcal{F}$ is **consistent** if $\emptyset \notin \mathcal{F}$ and $\mathcal{F} \neq \emptyset$.

We gather some useful observations about the above properties. Our first observation provides an alternative characterization of supplemented collections.

Lemma 2.1 *$\mathcal{F}$ is supplemented iff if $X \cap Y \in \mathcal{F}$ then $X \in \mathcal{F}$ and $Y \in \mathcal{F}$.*

Proof. The left to right direction is trivial as $X \cap Y \subseteq X$ and $X \cap Y \subseteq Y$. For the right to left direction, suppose that for any $X, Y \subseteq W$, if $X \cap Y \in \mathcal{F}$ then $X \in \mathcal{F}$ and $Y \in \mathcal{F}$. Let $Z \subseteq Z' \subseteq W$ and $Z \in \mathcal{F}$. We must show $Z' \in \mathcal{F}$. Since $Z \subseteq Z'$, $Z \cap Z' = Z$ and so we have $Z \cap Z' = Z \in \mathcal{F}$. By our assumption, $Z \in \mathcal{F}$ and $Z' \in \mathcal{F}$, as desired. QED

Property 6 above deserves some comments. If $\mathcal{F}$ contains the unit, then intuitively $\mathcal{F}$ contains a maximal element (under the subset relation). Similarly, if $\mathcal{F}$ contains its core, then $\mathcal{F}$ contains a minimal element (under the subset relation). The following definition lists some well-known collections of sets.

Definition 2.2 Let W be a non-empty set and $\mathcal{F} \subseteq \wp W$, then

1. $\mathcal{F}$ is a **filter** if $\mathcal{F}$ contains the unit, closed under binary intersections and supplemented. $\mathcal{F}$ is a proper filter if in addition $\mathcal{F}$ does not contain the emptyset.

2. $\mathcal{F}$ is an **ultrafilter** if $\mathcal{F}$ is proper filter and for each $X \subseteq W$, either $X \in \mathcal{F}$ or $X^C \in \mathcal{F}$.

3. $\mathcal{F}$ is a **topology** if $\mathcal{F}$ contains the unit, the emptyset, is closed under finite intersections and arbitrary unions.

4. $\mathcal{F}$ is **augmented** if $\mathcal{F}$ contains its core and is supplemented. $\triangleleft$

Let us consider augmented collections in a bit more detail. The following fact is straightforward and left as an exercise.

Lemma 2.3 *If $\mathcal{F}$ is augmented, then $\mathcal{F}$ is closed under arbitrary intersections. In fact, if $\mathcal{F}$ is augmented then $\mathcal{F}$ is a filter.*

Proof. Left as an exercise for the reader. QED

Of course, the converse is false. But that is not very interesting since it is easy to construct collections of sets closed under intersections that are not supplemented. What is more interesting is that there are consistent filters that are not augmented.

Fact 2.4 *There are consistent filters that are not augmented.*

Proof. Suppose $W = (-1, 1) = \{x \mid x \in \mathbb{R} \text{ and } -1 \leq x \leq 1\}$ and $\mathcal{F} = \{X \subseteq (-1, 1) \mid (-\frac{1}{n}, \frac{1}{n}) \subseteq X \text{for some natural number } n \geq 1\}$. Clearly $\emptyset \notin \mathcal{F}$ (and $\mathcal{F}$ is non-empty), so $\mathcal{F}$ is consistent. By construction $\mathcal{F}$ is closed under supersets. Finally, $\mathcal{F}$ is easily seen to be closed under finite intersections: let $X_1, X_2 \in \mathcal{F}$. Then there is a $n \geq 1$ and $m \geq 1$ such that $(-\frac{1}{n}, \frac{1}{n}) \subseteq X_1$ and $(-\frac{1}{m}, \frac{1}{m}) \subseteq X_2$. Either $n \leq m$ or $m > n$. If $n = m$ we are done since in this case $(-\frac{1}{n}, \frac{1}{n}) = (-\frac{1}{m}, \frac{1}{m}) \subseteq X_1 \cap X_2$ and so $X_1 \cap X_2 \in \mathcal{F}$. This leaves the cases $n > m$ and $m > n$. Suppose $n > m$. Then $(-\frac{1}{n}, \frac{1}{n}) \subseteq (-\frac{1}{m}, \frac{1}{m})$. Hence, $(-\frac{1}{n}, \frac{1}{n}) \subseteq X_1 \cap X_2$ and so $X_1 \cap X_2 \in \mathcal{F}$. The case when $m > n$ is similar. So $\mathcal{F}$ is a consistent filter.

Now, $\cap\mathcal{F} = \emptyset$ and as noted above $\emptyset \notin \mathcal{F}$; therefore, $\mathcal{F}$ is not augmented. To see $\cap\mathcal{F} = \emptyset$, note that for each $x \in (-1, 1)$ there is a large enough n such that $x \notin (-\frac{1}{n}, \frac{1}{n})$ (this is a standard fact about real numbers). This shows that $\cap_{n\geq 1}(-\frac{1}{n}, \frac{1}{n}) = \emptyset$. Thus, since $\cap\mathcal{F} \subseteq \cap_{n\geq 1}(-\frac{1}{n}, \frac{1}{n})$, we have $\cap\mathcal{F} = \emptyset$. QED

However, as is well-known, the situation is much better if W is finite. To see this, we first note the following well-known fact.

Lemma 2.5 *If $\mathcal{F}$ is closed under binary intersections (i.e., if $X, Y \in \mathcal{F}$ then $X \cap Y \in \mathcal{F}$), then $\mathcal{F}$ is closed under finite intersections.*

Proof. The proof is standard, but we repeat it here in the interest of exposition. Suppose that $\mathcal{F}$ is closed under binary intersections. We must show for every collections of sets $\{X_i\}_{i\in I}$ with I finite and $X_i \in \mathcal{F}$ for each $i \in I$, we have $\cap_{i\in I}X_i \in \mathcal{F}$. The proof is by induction on the size of I. The base case ($|I| = 2$) is true by assumption. Suppose the statement holds for index sets of size n and consider the collection of sets $\{X_i\}_{i\in I}$ with $|I| = n+1$ and for each $i \in I$, $X_i \in \mathcal{F}$. Choose any $j \in I$ (such an index exists as $|I| > 2$). Then $\cap_{i\in I}X_i = \cap_{i\in I-\{j\}}X_i \cap X_j$. Since $|I - \{j\}| = n$, by the induction hypothesis, $\cap_{i\in I-\{j\}}X_i \in \mathcal{F}$. Thus, since $\mathcal{F}$ is closed under binary intersections and $X_j \in \mathcal{F}$, $\cap_{i\in I}X_i = \cap_{i\in I-\{j\}}X_i \cap X_j \in \mathcal{F}$, as desired. QED

Corollary 2.6 *If W is finite and $\mathcal{F}$ is a filter over W, then $\mathcal{F}$ is augmented.*

Proof. Since W is finite and $\mathcal{F} \subseteq \wp W$, $\mathcal{F}$ is finite. Since $\mathcal{F}$ is a filter, by definition $\mathcal{F}$ is closed under binary intersection. By Lemma 2.5, $\mathcal{F}$ is closed under finite intersections. In particular, since $\mathcal{F}$ is itself a finite collection of sets from $\mathcal{F}$, $\cap\mathcal{F} \in \mathcal{F}$. QED

We end this section with a number of operations on collections of sets that will be useful in the subsequent chapters. Let W be a set and $\mathcal{F} \subseteq \wp W$ any collection of sets.

- Let $sup(\mathcal{F})$ be the smallest collections of subsets of W that contains $\mathcal{F}$ and is closed under supersets.
- Let $aug(\mathcal{F})$ be the smallest augmented collection of sets containing $\mathcal{F}$. That is $aug(\mathcal{F}) = sup(\mathcal{F} \cup \{\cap\mathcal{F}\})$.

2.2 Neighborhood Frames and Models

We are now ready to present our basic object of study. Recall from our discussion of relational structures in Chapter 1 that we are after a mathematical structure that can tell us, for each state, the set of necessary *propositions*. We proceed in the simplest way possible: at each state list all the sets that are considered "necessary" (or "known" or "believed" or ...). To that end, a function $N : W \to \wp\wp W$ is called a **neighborhood function**[5].

Definition 2.7 (Neighborhood Frames) A pair $\langle W, N\rangle$ is a called a **neighborhood frame**, or a **neighborhood system**, if W a non-empty set and N is a neighborhood function. ◁

We say that $\langle W, N\rangle$ is a filter provided for each $w \in W$, $N(w)$ is a filter. Similarly for the other properties discussed above. Admittedly, this is an abuse of notation since a filter is a property of collections of sets not of neighborhood systems. However, we trust that this will not cause any confusion.

One of the main goals of the text is to highlight the similarities and differences between neighborhood frames and relational frames as a semantics for the basic modal language. With this goal in mind, we clarify which neighborhood frames *correspond* to relational frames (in a technical sense defined below). Given a relation R on a set W (i.e., $R \subseteq W \times W$), define the following functions:

1. $R^{\rightarrow} : W \to \wp W$: for each $w \in W$, let $R^{\rightarrow}(w) = \{v \mid wRv\}$.
2. $R^{\leftarrow} : \wp W \to \wp W$: for each $X \subseteq W$, $R^{\leftarrow}(X) = \{w \mid \exists v \in X \text{ such that } wRv\}$.

Definition 2.8 (R-Necessity) Given a relation R on a set W and a state $w \in W$. A set $X \subseteq W$ is **R-necessary at** w if $R^{\rightarrow}(w) \subseteq X$. Let $\mathcal{N}_w^R$ be the set of sets that are R-necessary at w (we simple write $\mathcal{N}_w$ if R is clear from context). That is,

$$\mathcal{N}_w^R = \{X \mid R^{\rightarrow}(w) \subseteq X\}$$

◁

The following lemma shows that the collection of necessary sets have very nice properties.

Lemma 2.9 *Let R be a relation on W. Then for each $w \in W$, $\mathcal{N}_w$ is augmented.*

Proof. Left as an exercise. QED

[5] The motivation for calling these functions *neighborhood* functions is discussed in the historical notes at the end of this chapter (see also Chapter **??**)

Furthermore, properties of R are reflected in these collection of sets.

Observation 2.10 *Let W be a set and $R \subseteq W \times W$.*

1. *If R is reflexive, then for each $w \in W$, $w \in \cap\mathcal{N}_w$*
2. *If R is transitive then for each $w \in W$, if $X \in \mathcal{N}_w$, then $\{v \mid X \in \mathcal{N}_v\} \in \mathcal{N}_w$.*

Proof. Suppose that R is reflexive. Let $w \in W$ be an arbitrary state. Suppose that $X \in \mathcal{N}_w$. Then since R is reflexive, wRw and hence $w \in R^{\rightarrow}(w)$. Therefore by the definition of $\mathcal{N}_w$, $w \in X$. Since X was an arbitrary element of $\mathcal{N}_w$, $w \in X$ for each $X \in \mathcal{N}_w$. Hence $w \in \cap\mathcal{N}_w$.

Suppose that R is transitive. Let $w \in W$ be an arbitrary state. Suppose that $X \in \mathcal{N}_w$. We must show $\{v \mid X \in \mathcal{N}_v\} \in \mathcal{N}_w$. That is, we must show $R^{\rightarrow}(w) \subseteq \{v \mid X \in \mathcal{N}_v\}$. Let $x \in R^{\rightarrow}(w)$. Then wRx. To complete the proof we need only show $X \in \mathcal{N}_x$. That is, we must show $R^{\rightarrow}(x) \subseteq X$. Since R is transitive, $R^{\rightarrow}(x) \subseteq R^{\rightarrow}(w)$ (why?). Hence since $R^{\rightarrow}(w) \subseteq X$, $R^{\rightarrow}(x) \subseteq X$. QED

Exercise 2.11 *State and prove analogous results for the situations when R is serial, Euclidean and symmetric respectively.*

From this point of view, we can think of relational frames and neighborhood frames as two different ways of presenting the same information. It should be clear that with neighborhood frames, there is more freedom in which collection of sets can be necessary at a particular state. On the other hand, in relational frames this information is presented in a simple and elegant fashion. A natural question to ask is under what circumstances do a neighborhood frame and a relational frame represent the *same* information.

Definition 2.12 (Equivalence of Neighborhood and Relational Frames) Let $\langle W, N\rangle$ be a neighborhood frame and $\langle W, R\rangle$ be a relational frame. We say that $\langle W, N\rangle$ and $\langle V, R\rangle$ are **equivalent** if $X \subseteq W$, $X \in N(w)$ iff $X \in \mathcal{N}_w^R$. ◁

The following theorems give the situations when relational frames and neighborhood frame are equivalent.

Theorem 2.13 *Let $\langle W, R\rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*

Proof. The proof is trivial given Lemma 2.9: for each $w \in W$, let $N(w) = \mathcal{N}_w$. QED

Theorem 2.14 *Let $\langle W, N\rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

Proof. Let $\langle W, N\rangle$ be a neighborhood frame. We must define a relation R_N on W. Since $\langle W, N\rangle$ is augmented, for each $w \in W$, $\cap N(w) \in N(w)$. For each $w, v \in W$, we say wR_Nv iff $v \in \cap N(w)$. To show $\langle W, R_N\rangle$ and $\langle W, N\rangle$ are equivalent, we must show for each $w \in W$, $\mathcal{N}_w = N(w)$. Let $w \in W$ and $X \subseteq W$. If $X \in \mathcal{N}_w$. Then $R_N^{\rightarrow}(w) \subseteq X$. Since $R_N^{\rightarrow}(w) = \cap N(w)$ and N contains its core, $R_N^{\rightarrow}(w) \in N(w)$. Furthermore, since N is supplemented and $R_N^{\rightarrow}(w) = \cap N(w) \subseteq X$, $X \in N(w)$. Suppose that $X \in N(w)$. Then clearly $\cap N(w) \subseteq X$. Hence $X \in \mathcal{N}_w$. QED

2.3 Truth in Neighborhood Models

Recall the definition of the basic modal language, denoted by $\mathcal{L}(\mathsf{At})$, where At is a set of atomic sentences:

$$p \mid \neg\phi \mid \phi \wedge \phi \mid \Box\phi \mid \Diamond\phi$$

where $p \in \mathsf{At}$. The intended interpretation of $\Box\phi$ is "ϕ is *necessary*" and of $\Diamond\phi$ is "ϕ is *possible*". Of course, both *necessary* and *possible* can be interpreted in many different ways.

Definition 2.15 (Neighborhood Model) Let $\mathfrak{F} = \langle W, N\rangle$ be a neighborhood frame. A **model** based on $\mathfrak{F}$ is a tuple $\langle W, N, V\rangle$ where $V : \mathsf{At} \to 2^W$ is a valuation function. ◁

Truth of modal formulas in neighborhood models is defined in much the same way as in Definition 1.2. Of course, the only difference can be seen with the definition of truth of the modal operators.

Definition 2.16 (Truth in a Neighborhood Model) Let $\mathfrak{M} = \langle W, N, V\rangle$ be a model and $w \in W$. Truth of a formula $\phi \in \mathcal{L}(\mathsf{At})$ is defined inductively as follows:

1. $\mathfrak{M}, w \models p$ iff $w \in V(p)$
2. $\mathfrak{M}, w \models \neg\phi$ iff $\mathfrak{M}, w \not\models \phi$
3. $\mathfrak{M}, w \models \phi \wedge \psi$ iff $\mathfrak{M}, w \models \phi$ and $\mathfrak{M}, w \models \psi$
4. $\mathfrak{M}, w \models \Box\phi$ iff $(\phi)^{\mathfrak{M}} \in N(w)$
5. $\mathfrak{M}, w \models \Diamond\phi$ iff $W - (\phi)^{\mathfrak{M}} \notin N(w)$

where $(\phi)^{\mathfrak{M}}$ denotes the **truth set** of ϕ. That is $(\phi)^{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models \phi\}$. ◁

The definition of truth of the modal operators (items 4 and 5 above) where chosen to ensure that $\Box$ and $\Diamond$ would be *dual* operators. The intuition is that the neighborhood function N is simply a list of the necessary propositions. Hence, $\Box\phi$ is true at a state just in case the truth set of ϕ is a member of that list. Furthermore, ϕ is a possibility if the proposition expressed by $\neg\phi$ is not a member of that list. Other options for the definition of truth of the modal operators have been considered in the literature. Some of these will be considered below. We first spend some time getting a feel for the above definition of truth.

The following properties of the truth set will be used throughout this text. We first need some notation. Let $N : W \rightarrow \wp\wp W$ be a neighborhood function. Define $m_N : \wp W \rightarrow \wp W$ as follows: for $X \subseteq W$,

$$m_N(X) = \{w \mid X \in N(w)\}$$

Intuitively, $m_N(X)$ is the set of states in which X is necessary. Let $\mathfrak{M} = \langle W, N, V \rangle$ be a neighborhood model.

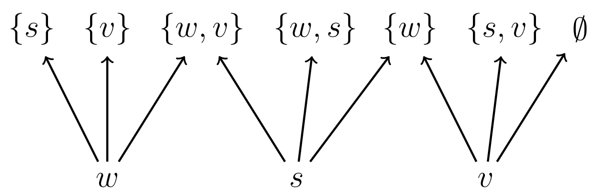
1. $(p)^{\mathfrak{M}} = V(p)$ for $p \in \mathsf{At}$
2. $(\neg\phi)^{\mathfrak{M}} = W - (\phi)^{\mathfrak{M}}$
3. $(\phi \wedge \psi)^{\mathfrak{M}} = (\phi)^{\mathfrak{M}} \cap (\psi)^{\mathfrak{M}}$
4. $(\Box\phi)^{\mathfrak{M}} = m_N((\phi)^{\mathfrak{M}})$
5. $(\Diamond\phi)^{\mathfrak{M}} = W - m_N(W - (\phi)^{\mathfrak{M}})$

The proof of the above statements is an easy application of the definition of truth and is left to the reader. We say ϕ is **valid** in $\mathfrak{M}$, denoted $\mathfrak{M} \models \phi$, if for each $w \in W$, $\mathfrak{M}, w \models \phi$. Also, ϕ is **valid** in a frame $\mathfrak{F}$ if for each model $\mathfrak{M}$ based on $\mathfrak{F}$, $\mathfrak{M} \models \phi$.

Detailed Example: Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V \rangle$ as follows:

- $N(w) = \{\{s\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

Suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$. We can picture this model as follows:



We now run through the calculation of truth of various formulas in order to develop a feeling for this semantics.

1. Since $(p)^{\mathfrak{M}} = V(p) = \{w, s\} \in N(w)$, we have $\mathfrak{M}, s \models \Box p$
2. Since $(\neg p)^{\mathfrak{M}} = \{v\} \notin N(s)$, we have $\mathfrak{M}, s \models \Diamond p$.
3. Since $(\Diamond p)^{\mathfrak{M}} = \{s, v\} \in N(v)$, we have $\mathfrak{M}, v \models \Box\Diamond p$
4. Since $(\Box p)^{\mathfrak{M}} = \{s\} \in N(w)$, we have $\mathfrak{M}, w \models \Box\Box p$.
5. Since $(\Box\Box p)^{\mathfrak{M}} = \{w\} \in N(s) \cap N(v)$, we have $\mathfrak{M}, s \models \Box\Box\Box p$ and $\mathfrak{M}, v \models \Box\Box\Box p$.
6. Since $(\Box\Box\Box p)^{\mathfrak{M}} = \{w, v\} \in N(w) \cap N(s)$, we have $\mathfrak{M}, w \models \Box\Box\Box\Box p$ and $\mathfrak{M}, s \models \Box\Box\Box\Box p$.
7. Finally, since $(\bot)^{\mathfrak{M}} = \emptyset \in N(v)$, $\mathfrak{M}, v \models \Box\bot$

Note that $\mathfrak{M}, w \models \Box(p \wedge q)$ but $\mathfrak{M}, w \not\models \Box p$. Note that if we fix the valuations of p and q, it is not possible to define a relational structure such that $\Box(p \wedge q)$ is true at w but $\Box p$ is false at w. Let us see why. The condition that $\Box p$ is false at w forces w to have an accessible world in which p is false. There is only one such world (v) where p is false. However, if v is accessible from w, then $\Box(p \wedge q)$ will no longer be true at w (since if p is false at v then so is $p \wedge q$). $\triangleleft$

The above example shows that the axiom $\Box(\phi \wedge \psi) \rightarrow \Box\phi \wedge \Box\psi$ is not valid in the class of all neighborhood frames. In addition, each of the following formulas discussed in Chapter 1 can be falsified in a neighborhood model.

1. $\Box\phi \wedge \Box\psi \rightarrow \Box(\phi \wedge \psi)$
2. $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$
3. $\Box\top$
4. $\Box\phi \rightarrow \phi$
5. $\Box\phi \rightarrow \Box\Box\phi$

Let us consider the first two formulas and leave the last three for exercises.

Observation 2.17 *The following formulas are not valid on the class of all neighborhood frames.*

1. $\Box\phi \wedge \Box\psi \to \Box(\phi \wedge \psi)$
2. $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$

Proof. For the first formula, consider the neighborhood model $\mathfrak{M} = \langle W, N, V\rangle$ with $W = \{w, v\}$, $N(w) = \{\{w\}, \{v\}\}$, $N(v) = \{\emptyset\}$ and $V(p) = \{w\}$ and $V(q) = \{v\}$. Thus, $\mathfrak{M}, w \models \Box p \wedge \Box q$, but since $(p \wedge q)^{\mathfrak{M}} = \emptyset \notin N(w)$, $\mathfrak{M}, w \not\models \Box(p \wedge q)$.

For the second formula, we construct the following neighborhood model $\mathfrak{M} = \langle W, N, V\rangle$ with $W = \{w, v, s\}$, $N(w) = \{\{w\}, \{w, v, s\}\}$, $V(p) = \{w\}$ and $V(q) = \{w, v\}$. Then $(p)^{\mathfrak{M}} = \{w\}$, $(q)^{\mathfrak{M}} = \{w, v\}$ and $(p \to q)^{\mathfrak{M}} = (\neg p \vee q)^{\mathfrak{M}} = \{w, v, s\}$. Thus we have, $\mathfrak{M}, w \models \Box(p \to q) \wedge \Box p$ but $\mathfrak{M}, w \not\models \Box q$. QED

Exercise 2.18
1. *Show that the following formulas are not valid on the class of all neighborhood frames*
   - $\Box\top$
   - $\Box\phi \to \phi$
   - $\Box\phi \to \Box\Box\phi$
   - $\Box\Diamond\phi \to \Diamond\Box\phi$
2. *Can you find a neighborhood model with a state in which all $\Box$-formulas are false, but all $\Diamond$-formulas are true? Is this possible with relational semantics?*

So, which formulas are valid on all neighborhood frames? As noted above, Definition **??** was chosen to ensure that $\Box$ and $\Diamond$ are duals. We now make this formal.

Lemma 2.19 *The scheme $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$ is valid in all neighborhood models.*

Proof. Let $\mathfrak{M} = \langle W, N, V\rangle$ be a neighborhood model.

$$\begin{array}{rll}
\mathfrak{M}, w \models \Box\phi & \text{iff} & (\phi)^{\mathfrak{M}} \in N(w) \\
\text{(naive set theory)} & \text{iff} & W - (W - (\phi)^{\mathfrak{M}}) \in N(w) \\
\text{(Observation ??)} & \text{iff} & W - ((\neg\phi)^{\mathfrak{M}}) \in N(w) \\
\text{(Definition ??)} & \text{iff} & \mathfrak{M}, w \not\models \Diamond\neg\phi \\
\text{(Definition ??)} & \text{iff} & \mathfrak{M}, w \models \neg\Diamond\neg\phi
\end{array}$$

QED

A complete discussion of the logics is left for Chapter **??**. For now, we note that the following rule of inference is valid. The simple and instructive proof is left for the reader.

Lemma 2.20 *On the class of all neighborhood frames, If $\phi \leftrightarrow \psi$ is valid then $\Box\phi \leftrightarrow \Box\psi$ is valid.*

Proof. Left as an exercise for the reader. QED

Of course, depending on the intended interpretation of the modal operators, one may even want to work with a semantics where $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$ is not valid. Recall the discussion in Chapter **??** regarding the game-theoretic forcing operator. In light of Lemma 2.19, this can only be achieved by changing the definition of the modal operators. Let us explore some options that have been considered in the literature. Extend the basic modal language $\mathcal{L}$ with the following modalities: $[\ \rangle$, $\langle\ ]$, $\langle\ \rangle$, $[\ ]$. Truth for these modalities is defined below.

- $\mathfrak{M}, w \models \langle\ ]\phi$ iff $\exists X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \phi$
- $\mathfrak{M}, w \models [\ \rangle\phi$ iff $\forall X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \phi$
- $\mathfrak{M}, w \models \langle\ \rangle\phi$ iff $\exists X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \phi$
- $\mathfrak{M}, w \models [\ ]\phi$ iff $\forall X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \phi$

Our first easy observation is that we really only have two modalities.

Observation 2.21 *The following formulas are valid on all neighborhood models.*

- $\langle\ ]\phi \leftrightarrow \neg[\ \rangle\neg\phi$
- $[\ ]\phi \leftrightarrow \neg\langle\ \rangle\neg\phi$

Proof. Left as an exercise for the reader. QED

The first two modalities above will play an important role later in Chapter **??**. Therefore, let us look at these modalities in more detail.

Lemma 2.22 *Let $\mathfrak{M} = \langle W, N, V\rangle$ be a neighborhood model. The for each $w \in W$,*

1. *if $\mathfrak{M}, w \models \Box\phi$ then $\mathfrak{M}, w \models \langle\ ]\phi$*
2. *if $\mathfrak{M}, w \models [\ \rangle\phi$ then $\mathfrak{M}, w \models \Diamond\phi$*

However, the converses of the above statements are false.

Proof. Let $\mathfrak{M} = \langle W, N, V\rangle$ be a neighborhood model and $w \in W$. Suppose that $\mathfrak{M}, w \models \Box\phi$. Then $(\phi)^{\mathfrak{M}} \in N(w)$. Then, clearly there is an $X \in N(w)$ such that for each $v \in X$, $\mathfrak{M}, v \models \phi$ (let $X = (\phi)^{\mathfrak{M}}$). The reasoning is analogous for the second statement above and so will be left to the reader.

As for the converse of the first statement. Consider the model $\mathfrak{M} = \langle W, N, V\rangle$ where $W = \{w, v\}$, $N(w) = \{\{w\}, \{v\}\}$, and $V(p) = W$. Then $\mathfrak{M}, w \models \langle\ ]p$; however, $\mathfrak{M}, w \not\models \Box\phi$. Again the reasoning is analogous for the second statement as so is left for the reader. QED

Exercise 2.23 *Prove that*

1. *If* $\phi \to \psi$ *is valid in* $\mathfrak{M}$*, then so is* $\langle\ ]\phi \to \langle\ ]\psi$.
2. $\langle\ ](\phi \wedge \psi) \to (\langle\ ]\phi \wedge \langle\ ]\psi)$ *is valid in* $\mathfrak{M}$

Investigate analogous results for the other modal operators defined above.

2.4 Defining Properties of Neighborhood Frames

One of the main slogans from the main text on modal logic [4] describes the modal language as *a language for talking about graphs, or relational structures* ([4], pg. ??). This is made precise as follows. The key idea is that some modal formulas can be shown to **define** interesting properties of a relation. For example, it is well-known that a relational frame $\mathfrak{F} = \langle W, R\rangle$ validates $\Box\phi \to \phi$ iff R is reflexive. Similarly, in the current setting, modal formulas can be understood as expressing properties of *neighborhood frames.*

Definition 2.24 A modal formula ϕ defines a property P of neighborhood functions if any neighborhood frame $\mathfrak{F}$ has property P iff $\mathfrak{F}$ validates ϕ. $\lhd$

Consider the formulas $\Diamond\top$ and $\Box\phi \to \Diamond\phi$. On relational frames, these formulas both define the same property: that the relation is serial. This is not surprising since these formulas are semantically equivalent on the class of relational frames (i.e., they are true at exactly the same points in all relational models). However, on the class of neighborhood frames, these formulas express different properties. The first formula $\Diamond\top$ is easily seen to express the fact that the empty set is not an element of the neighborhoods. That is, $\Diamond\top$ is valid on a neighborhood frame $\mathfrak{F}$ iff the empty set is not an element of any neighborhood (the proof follows immediately from the definition of truth of modal formulas). The second formula expresses a more interesting property about neighborhood frames.

Lemma 2.25 *Let* $\mathfrak{F} = \langle W, N\rangle$ *be a neighborhood frame. Then* $\mathfrak{F} \models \Box\phi \to \Diamond\phi$ *iff* $\mathfrak{F}$ *is proper (i.e., if* $X \in \mathcal{N}(w)$ *then* $X^C \notin N(w)$*).*

Proof. The right-to-left direction is straightforward. For the other direction, suppose that $\mathfrak{F} = \langle W, N\rangle$ is not proper. Then there is a state $w \in W$ and set $X \subseteq W$ such that $X \in N(w)$ and $X^C \in N(w)$. Define a model $\mathfrak{M} = \langle W, N, V\rangle$ with $V(p) = X$. Then, by definition, $\mathfrak{M}, w \models \Box p$ and since $(\neg p)^{\mathfrak{M}} = X^C \in N(w)$ we have $\mathfrak{M}, w \models \neg\Diamond p$. QED

Lemma 2.26 *Let $\mathfrak{F} = \langle W, N\rangle$ be a neighborhood frame. Then $\mathfrak{F} \models \Box\phi \wedge \Box\psi \rightarrow \Box(\phi \wedge \psi)$ iff $\mathfrak{F}$ is closed under finite intersections.*

Proof. Suppose that $\mathfrak{F} = \langle W, N\rangle$ is a neighborhood frame that is closed under finite intersections. We must show $\mathfrak{F} \models \Box\phi \wedge \Box\psi \rightarrow \Box(\phi \wedge \psi)$. Let $\mathfrak{M} = \langle W, N, V\rangle$ be any model based on $\mathfrak{F}$ and $w \in W$. Suppose that $\mathfrak{M}, w \models \Box\phi \wedge \Box\psi$. Then $(\phi)^{\mathfrak{M}} \in N(w)$ and $(\psi)^{\mathfrak{M}} \in N(w)$. Since $N(w)$ is closed under finite intersections, $(\phi)^{\mathfrak{M}} \cap (\psi)^{\mathfrak{M}} \in N(w)$. Hence $(\phi \wedge \psi)^{\mathfrak{M}} \in N(w)$ and therefore $\mathfrak{M}, w \models \Box(\phi \wedge \psi)$.

Suppose that $\mathfrak{F} = \langle W, N\rangle$ is not closed under finite intersections. Then there are sets $X_1, \dots X_k$ such that $X_i \in N(w)$ for $i = 1, \dots, k$ but $\cap_{1\le i\le k} X_i \notin N(w)$. We claim that there are two sets Y and Y' such that $Y, Y' \in N(w)$ but $Y \cap Y' \notin N(w)$. If $\cap_{i=1}^{k-1} X_i \in N(w)$, then let $Y = \cap_{i=1}^{k-1} X_i \in N(w)$ and $Y' = X_k$. If not, then $\cap_{i=1}^{k-1} \notin N(w)$. If $\cap_{i=1}^{k-2} X_i \in N(w)$, then let $Y = \cap_{i=1}^{k-2} X_i \in N(w)$ and $Y' = X_{k-1}$. If not, then $\cap_{i=1}^{k-2} X_i \notin N(w)$. Continue in this manner until we find and $l \le k$ such that $\cap_{i=1}^{l} X_i, X_{l+1} \in N(w)$ but $\cap_{i=1}^{l+1} X_i \notin N(w)$. Notice that such an l must exists since $X_i \in N(w)$ for each $i = 1, \dots, k$. Then define a valuation function so that $V(p) = Y$ and $V(q) = Y'$. Hence, $\mathfrak{M}, w \models \Box p \wedge \Box q$. However, since $Y \cap Y' \notin N(w)$, $\mathfrak{M}, w \not\models \Box(p \wedge q)$. QED

Lemma 2.27 *Let $\mathfrak{F} = \langle W, N\rangle$ be a neighborhood frame. Then $\mathfrak{F} \models \Box(\phi \wedge \psi) \rightarrow \Box\phi \wedge \Box\psi$ iff $\mathfrak{F}$ is closed under supersets.*

Proof. The right to left direction is left as an exercise for the reader. Suppose that $\mathfrak{F} = \langle W, N\rangle$ is not closed under supersets. Then there are sets X and Y such that $X \subseteq Y$, $X \in N(w)$ but $Y \notin N(w)$. Define a valuation V such that $V(p) = X$ and $V(q) = Y$. Then since $X \subseteq Y$, $(p \wedge q)^{\mathfrak{M}} = X \in N(w)$. Hence $\mathfrak{M}, w \models \Box(p \wedge q)$. However since, $(q)^{\mathfrak{M}} = Y \notin N(w)$, $\mathfrak{M}, w \not\models \Box q$. Hence $\mathfrak{M}, w \not\models \Box p \wedge \Box q$. QED

Lemma 2.28 *Let $\mathfrak{F} = \langle W, N\rangle$ be a neighborhood frame. Then $\mathfrak{F} \models \Box\top$ iff $\mathfrak{F}$ contains the unit.*

Proof. Left as an exercise for the reader. QED

Lemma 2.29 *Let $\mathfrak{F} = \langle W, N\rangle$ be a neighborhood frame such that for each $w \in W$, $N(w) \neq \emptyset$.*

1. *$\mathfrak{F} \models \Box\phi \rightarrow \phi$ iff for each $w \in W$, $w \in \cap N(w)$*

2. *$\mathfrak{F} \models \Box\phi \rightarrow \Box\Box\phi$ iff for each $w \in W$, if $X \in N(w)$, then $\{v \mid X \in N(v)\} \in N(w)$*

Proof. Suppose that $\mathfrak{F} = \langle W, N\rangle$ is a neighborhood frame. Suppose that for each $w \in W$, $w \in \cap N(w)$. Let $\mathfrak{M}$ be any model based on $\mathfrak{F}$ and $w \in W$. Suppose that $\mathfrak{M}, w \models \Box\phi$. Then $(\phi)^{\mathfrak{M}} \in N(w)$. Since $w \in \cap N(w) \subseteq (\phi)^{\mathfrak{M}}$, $w \in (\phi)^{\mathfrak{M}}$. Hence $\mathfrak{M}, w \models \phi$. As for the converse, suppose that $w \notin \cap N(w)$. Since $N(w) \neq \emptyset$, there is an $X \in N(w)$ (note that X may be empty) such that $w \notin X$, otherwise $w \in \cap N(w)$. Define a valuation V such that $V(p) = X$. Then it is easy to see that $\mathfrak{M}, w \models \Box p$ but $\mathfrak{M}, w \not\models X$.

Suppose that for each $w \in W$, if $X \in N(w)$, then $\{v \mid X \in N(v)\} \in N(w)$. Suppose that $\mathfrak{M}$ is any model based on $\mathfrak{F}$ and $\mathfrak{M}, w \models \Box\phi$. Then $(\phi)^{\mathfrak{M}} \in N(w)$. Therefore, by assumption $\{v \mid (\phi)^{\mathfrak{M}} \in N(v)\} \in N(w)$. Since $(\Box\phi)^{\mathfrak{M}} = \{v \mid (\phi)^{\mathfrak{M}} \in N(w)\}$, $\mathfrak{M}, w \models \Box\Box\phi$. For the other direction, suppose that there is some state $w \in W$ and set X such that $X \in N(w)$ but $\{v \mid X \in N(v)\} \notin N(w)$. The define a valuation V such that $V(p) = X$. It is easy to verify that $\mathfrak{M}, w \models \Box p$ but $\mathfrak{M}, w \not\models \Box\Box p$. QED

Exercise 2.30 *Find properties on frames that are defined by the following formulas:*

1. $\Diamond\top$

2. $\neg\Box\phi \rightarrow \Box\neg\Box\phi$

3. $\Box\phi \rightarrow \Diamond\phi$

3 Classical Modal Logics: The Basics

In this section we will be interested in the following axiom schemas and rules.

PC Any axiomatization of propositional calculus

E $\Box\phi \leftrightarrow \neg\Diamond\neg\phi$

M $\Box(\phi \wedge \psi) \rightarrow (\Box\phi \wedge \Box\psi)$

C $(\Box\phi \wedge \Box\psi) \rightarrow \Box(\phi \wedge \psi)$

N $\Box\top$

K $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$

$$RE \ \frac{\phi \leftrightarrow \psi}{\Box\phi \leftrightarrow \Box\psi}$$

$$Nec \ \frac{\phi}{\Box\phi}$$

$$MP \ \frac{\phi \quad \phi \rightarrow \psi}{\psi}$$

Let **E** be the smallest set of formulas closed under instances of PC, E and the rules RE and MP. **E** is the smallest classical modal logic. The logic **EC** extends **E** by adding the axiom scheme C. Similarly for **EM**, **EN**, **ECM**, and **EMCN**. The logic **K** is the smallest set of formulas closed under instances of PC, K, E and the rules Nec and MP. Let **S** be any of the above logics, we write $\vdash_{\mathbf{S}} \phi$ if $\phi \in \mathbf{S}$.

Our first observation about these axiom systems, is that we can prove a uniform substitution theorem in **E**. Given a formulas $\phi, \psi, \psi' \in \mathcal{L}$, let $\phi[\psi/\psi']$ be the formula ϕ but replace *some* occurrences of ψ with ψ'. For example, suppose that ϕ is the formula $\Box(\Diamond p \wedge \Box\Box q) \wedge \Box p$, ψ is the formula p and ψ' is the formula $\Box p$. Then $\phi[\psi/\psi']$ can be any of the following

- $\Box(\Diamond p \wedge \Box\Box q) \wedge \Box p$
- $\Box(\Diamond\Box p \wedge \Box\Box q) \wedge \Box p$
- $\Box(\Diamond p \wedge \Box\Box q) \wedge \Box\Box p$
- $\Box(\Diamond\Box p \wedge \Box\Box q) \wedge \Box\Box p$

The uniform substitution theorem states that we can always replace logically equivalent formulas.

Theorem 3.1 (Uniform Substitution) *The following rule can be derived in* **E**

$$\frac{\psi \leftrightarrow \psi'}{\phi \leftrightarrow \phi[\psi/\psi']}$$

Proof. Suppose that $\vdash_{\mathbf{E}} \psi \leftrightarrow \psi'$. We must show $\vdash_{\mathbf{E}} \phi \leftrightarrow \phi[\psi/\psi']$. First of all, note that if ϕ and ψ are the same formula. Then either $\phi[\psi/\psi']$ is ϕ (when ψ is not replaced) or $\phi[\psi/\psi']$ is ψ' (when ψ is replaced). In the first case, $\phi \leftrightarrow \phi[\psi/\psi']$ is the formula $\phi \leftrightarrow \phi$ and so trivially, $\vdash_{\mathbf{E}} \phi \leftrightarrow \phi[\psi/\psi']$. In the second case, $\phi \leftrightarrow \phi[\psi/\psi']$ is the formula $\psi \leftrightarrow \psi'$, which can be deduced in **E** by assumption. Thus we may assume that ϕ and ψ are distinct formulas.

The proof is by induction on ϕ. The base case and boolean connectives are left as an exercise for the reader. We demonstrate the modal operator. Suppose that ϕ is $\Box\gamma$ and $\vdash_{\mathbf{E}} \psi \leftrightarrow \phi'$. The induction hypothesis is $\vdash_{\mathbf{E}} \gamma \leftrightarrow \gamma[\psi/\psi']$. Using the RE rule, $\vdash_{\mathbf{E}} \Box\gamma \leftrightarrow \Box(\gamma[\psi/\psi'])$. Note that $\Box(\gamma[\psi/\psi'])$ is the same formula as $\Box\gamma[\psi/\psi']$. Hence $\vdash_{\mathbf{E}} \Box\gamma \leftrightarrow \Box\gamma[\psi/\psi']$. QED

The above theorem will often be used (without reference). We will now proceed to show a number of basic facts about the above axiom systems. Much more information can be found in [6], Chapter 8. The first is an alternative characterization of **EM**. The alternative axiomatization is in terms of a rule. The rule, called **right monotonicity** (RM) is

$$\frac{\phi \to \psi}{\Box\phi \to \Box\psi}$$

Lemma 3.2 *The logic* **EM** *equals the logic* **E** *plus the rule* RM*.*

Proof. The proof follows easily if we can show that RM is a derived rule in **EM** and M can be derived in the logic **E** plus the rule RM. We first show that RM can be derived in **EM**.

1.	$\phi \to \psi$	Assumption
2.	$\phi \leftrightarrow (\phi \wedge \psi)$	Follows from 1 and propositional reasoning
3.	$\Box\phi \leftrightarrow \Box(\phi \wedge \psi)$	2 and RE
4.	$\Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi$	Instance of M
5.	$\Box\phi \to \Box\phi \wedge \Box\psi$	Follows from 3,4 using propositional reasoning
6.	$\Box\phi \to \Box\psi$	Follows form 5 using propositional reasoning

Thus RM is a derived rule of **EM**. We now show that we can derive M in the logic **E** plus the rule RM.

1.	$(\phi \wedge \psi) \to \phi$	Propositional Tautology
2.	$\Box(\phi \wedge \psi) \to \Box\phi$	Follows from 1 using RM
3.	$(\phi \wedge \psi) \to \psi$	Propositional Tautology
4.	$\Box(\phi \wedge \psi) \to \Box\psi$	Follows from 3 using RM
5.	$\Box(\phi \wedge \psi) \to \Box\phi \wedge \Box\psi$	Follows from 2,4 using propositional reasoning

Thus M can be derived using the rule RM. QED

The logic **K** is the smallest **normal**, or Kripkean, modal logic. The logics **E**, **EM**, etc. are strictly *weaker* than **K**.

Lemma 3.3 *The logic* **EMC** *equals the logic* **E** *plus the axiom scheme* K*.*

Proof.
The fact that $\vdash_{\mathbf{K}} M$ and $\vdash_{\mathbf{K}} C$ is left for the reader. We only show that K can be derived in **EMC**. First of all note that since **EMC** contains M, the rule RM is derivable (see Lemma 3.2).

1.	$((\phi \to \psi) \wedge \phi) \to \psi$	Propositional Tautology
2.	$\Box((\phi \to \psi) \wedge \phi) \to \Box\psi$	From 1 and RM
3.	$(\Box(\phi \to \psi) \wedge \Box\phi) \to \Box((\phi \to \psi) \wedge \phi)$	Instance of C
4.	$(\Box(\phi \to \psi) \wedge \Box\phi) \to \Box\psi$	Follows from 2,3 using[6] PR
5.	$\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$	Follows from 4 using PR

QED

Lemma 3.4 *The logic* **EN** *equals the logic* **E** *plus the rule* Nec.

Proof. It is easy to see that using Nec, we can prove $\Box\top$. Suppose that $\vdash_{\mathbf{EN}} \phi$. We must show that $\vdash_{\mathbf{EN}} \Box\phi$. Using propositional reasoning since ϕ is derivable in **EN**, $\vdash_{\mathbf{EN}} (\top \leftrightarrow \phi)$. Using RE, $\vdash_{\mathbf{EN}} \Box\top \leftrightarrow \Box\phi$. Hence $\vdash_{\mathbf{EN}} \Box\top \to \Box\phi$. Since $\Box\top$ is obviously provable in **EN**, $\vdash_{\mathbf{EN}} \Box\phi$. QED

Putting these two lemmas together, we see that **EMCN** is the smallest normal modal logic.

Corollary 3.5 *The logic* **EMCN** *equals the logic* **K**.

In this section we have define eight modal logics between **E** and **K**. Of course, there still remains a question as to whether all eight logics are in fact distinct. This will be answered in the following section.

3.1 Soundness and Completeness

In this section we will prove soundness and completeness results. It is assumed that the reader is familiar with basic soundness and completeness results in modal logic (with respect to relational frames), see [4, 6] for more information. We quickly review some some basic terminology.

3.1.1 Preliminaries

Let F be a collection of neighborhood frames. A formula $\phi \in \mathcal{L}$ is **valid in** F, or F**-valid** if for each $\mathbb{F} \in \mathsf{F}$, $\mathbb{F} \models \phi$. We say that a logic $\mathbf{L}$ is **sound** with respect to F, provided $\mathbf{L} \subseteq \mathsf{F}$. That is for each formula $\phi \in \mathcal{L}$, $\vdash_{\mathbf{L}} \phi$ implies ϕ is valid in F. Given a set of formulas Γ, a formula ϕ and a collection of frames F, we say Γ **semantically entails** ϕ with respect to F, denoted $\Gamma \models_{\mathsf{F}} \phi$, if for each $\mathbb{F} \in \mathsf{F}$, if

$\mathbb{F} \models \Gamma$ then $\mathbb{F} \models \phi$. Here $\mathbb{F} \models \Gamma$ means for each $\phi \in \Gamma$, $\mathbb{F} \models \phi$. Finally we write $\models_{\mathsf{F}} \phi$ if for each $\mathbb{F} \in \mathsf{F}$, $\mathbb{F} \models \phi$. A logic $\mathbf{L}$ is **weakly complete** with respect to a class of frames F, if $\models_{\mathsf{F}} \phi$ implies $\vdash_{\mathbf{L}} \phi$. A logic $\mathbf{L}$ is **strongly complete** with respect to a class of frames F, if for each set of formulas Γ, $\Gamma \models_{\mathsf{F}} \phi$ implies $\Gamma \vdash_{\mathbf{L}} \phi$.

Let $\mathbf{L}$ be any modal logic. A formula $\phi \in \mathcal{L}$ is said to be **inconsistent in L**, or**L-inconsistent** if $\vdash_{\mathbf{L}} \neg\phi$. A set of formulas Γ is said to be **L-inconsistent** if $\Gamma \vdash_{\mathbf{L}} \bot$. The set Γ is **L-consistent** if it is not inconsistent. A consistent set of formulas Γ is called a **maximally consistent set** if for each formula ϕ, either $\phi \in \Gamma$ or $\neg\phi \in \Gamma$.

Let $M_{\mathbf{L}}$ be the set of $\mathbf{L}$-maximally consistent sets of formulas. Recall that *Lindenbaum's Lemma* (see [6] and [4] for an extended discussion) states that given any consistent collection of formulas Γ' there is a maximally consistent set of formulas that contains Γ'. Given a formula $\phi \in \mathcal{L}$, let $|\phi|_{\mathbf{L}}$ be the **proof set** of ϕ in $\mathbf{L}$. Formally, $|\phi|_{\mathbf{L}} = \{\Delta \mid \Delta \in W_{\mathbf{L}} \text{ and } \phi \in \Delta\}$. We first note that proof sets share a number of properties in common with truth sets.

Lemma 3.6 *Let* $\mathbf{L}$ *be a logic and* $\phi, \psi \in \mathcal{L}$. *Then*

1. $|\phi \wedge \psi|_{\mathbf{L}} = |\phi|_{\mathbf{L}} \cap |\psi|_{\mathbf{L}}$
2. $|\neg\phi|_{\mathbf{L}} = M_{\mathbf{L}} - |\phi|_{\mathbf{L}}$
3. $|\phi \vee \psi|_{\mathbf{L}} = |\phi|_{\mathbf{L}} \cup |\psi|_{\mathbf{L}}$
4. *If* $|\phi|_{\mathbf{L}} \subseteq |\psi|_{\mathbf{L}}$ *then* $\vdash_{\mathbf{L}} \phi \rightarrow \psi$
5. $\vdash_{\mathbf{L}} \phi \leftrightarrow \psi$ *iff* $|\phi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$

Proof. The proofs are standard facts about maximally consistent sets and left for the reader. QED

Finally, one more fact about proof sets that will be of interest.

Lemma 3.7 *For each* $\phi \in \mathcal{L}$, $\psi \in \cap|\phi|_{\mathbf{L}}$ *iff* $\vdash_{\mathbf{L}} \phi \rightarrow \psi$.

Proof. Suppose that $\vdash_{\mathbf{L}} \phi \rightarrow \psi$. Then for each maximally consistent set Γ, $\phi \rightarrow \psi \in \Gamma$. Hence since for each $\Gamma \in |\phi|_{\mathbf{L}}$, $\phi \in \Gamma$, we have $\psi \in \Gamma$. Thus $\phi \in \cap|\phi|_{\mathbf{L}}$.

Suppose that $\psi \in \cap|\phi|_{\mathbf{L}}$ but it is not the case that $\vdash_{\mathbf{L}} \phi \rightarrow \psi$. Then $\neg(\phi \rightarrow \psi)$ is consistent in $\mathbf{L}$. Using Lindenbaum's Lemma, we can construct a maximally consistent set Γ such that $\neg(\phi \rightarrow \psi) \in \Gamma$. That is, $\phi, \neg\psi \in \Gamma$. Since $\phi \in \Gamma$, $\Gamma \in |\phi|_{\mathbf{L}}$. But then $\neg\psi \in \Gamma$ contradicts the fact that $\psi \in \cap|\phi|_{\mathbf{L}}$. QED

3.2 The Proofs

Given any model $\mathbb{M} = \langle W, N, V\rangle$ and a set $X \subseteq W$, we say that X is **definable** in $\mathbb{M}$ if there is a formula $\phi \in \mathcal{L}$ such that $(\phi)^{\mathbb{M}} = X$. Let $\mathcal{D}_{\mathbb{M}}$ be the set of all sets that are definable in $\mathbb{M}$. First of all note, that $\mathcal{D}_{\mathbb{M}} \neq 2^W$. A simple counting argument will show this fact. For there are only countably many formulas of $\mathcal{L}$ and hence only countably many definable subsets. However (if W is countable) 2^W is uncountable. A subset $X \subseteq M_{\mathbf{L}}$ is called a **proof set** provided there is some formula $\phi \in \mathcal{L}$ such that $X = |\phi|_{\mathbf{L}}$. Again notice that there are only countably many proof sets; however, if At is countable, then $M_{\mathbf{L}}$ is uncountable, and hence there are uncountably many subsets of $M_{\mathbf{L}}$.

As usual, when constructing a canonical model the states of the world will be maximally consistent sets, i.e., elements of $M_{\mathbf{L}}$. Suppose that the logic $\mathbf{L}$ contains $\mathbf{E}$. Consider any function $N_{\mathbf{L}} : M_{\mathbf{L}} \rightarrow 2^{2^{M_{\mathbf{L}}}}$ such that for each $\Gamma \in M_{\mathbf{L}}$

$$|\phi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma) \text{ iff } \Box\phi \in \Gamma$$

Intuitively, given a set Γ, $N_{\mathbf{L}}(\Gamma)$ contains at least all the proof sets of necessary formulas from Γ. When defining any function, the first question one should ask is *is the function well-defined*? This may be a strange question to ask about $N_{\mathbf{L}}$, since we were *assuming* that it is in fact a function. So, what can go wrong? Well, $N_{\mathbf{L}}$ is any function that maps maximally consistent sets Γ to sets of subsets of $M_{\mathbf{L}}$. The only requirement on $N_{\mathbf{L}}$ is that for each $\Gamma \in M_{\mathbf{L}}$, $|\phi|_{\mathbf{L}} \in N(\Gamma)$ iff $\Box\phi \in \Gamma$. In particular, this requirement ensures that for each set $X \in N_{\mathbf{L}}(\Gamma)$ such that $X = |\phi|_{\mathbf{L}}$, $\Box\phi \in \Gamma$. Say that a proof set $|\phi|_{\mathbf{L}}$ is **necessary at** Γ provided $\Box\phi \in \Gamma$. Thus the functions we are interested in map maximally consistent sets Γ to collections of subsets of $M_{\mathbf{L}}$ that can be broken into to parts: the proof sets that are necessary at Γ and non-proof sets (i.e., sets not of the form $|\phi|_{\mathbf{L}}$ for some ϕ). However, as we see from Lemma 3.6 it is possible that a proof set X is defined by two different formulas. Thus to ensure that any functions can be defined that satisfy the above property, we must check that if $|\phi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ and $|\phi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$, then $\Box\psi \in \Gamma$.

Lemma 3.8 *For any logic* $\mathbf{L}$ *containing the rule RE, if* $N_{\mathbf{L}} : M_{\mathbf{L}} \rightarrow 2^{2^{M_{\mathbf{L}}}}$ *is a function such that for each* $\Gamma \in M_{\mathbf{L}}$*,* $|\phi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ *iff* $\Box\phi \in \Gamma$*. Then if* $|\phi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ *and* $|\phi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$*, then* $\Box\psi \in \Gamma$*.*

Proof. Let ϕ and ψ be two formulas such that $|\phi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$. Further suppose that $\Box\phi \in \Gamma$ and $|\phi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$. Since $|\phi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$, $\Box\phi \in \Gamma$. Also, by Lemma 3.6, since $|\phi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$, $\vdash_{\mathbf{L}} \phi \leftrightarrow \psi$. Using the *RE* rule, $\vdash_{\mathbf{L}} \Box\phi \leftrightarrow \Box\psi$. Hence $\Box\phi \leftrightarrow \Box\psi \in \Gamma$. Hence $\Box\psi \in \Gamma$. QED

Define the **canonical valuation**, $V_{\mathbf{L}} : \mathsf{At} \rightarrow 2^{M_{\mathbf{L}}}$ as follows. Let $p \in \mathsf{At}$, then $V_{\mathbf{L}}(p) = |p|_{\mathbf{L}} = \{\Gamma \mid \Gamma \in M_{\mathbf{L}} \text{ and } p \in \Gamma\}$.

Definition 3.9 (Canonical for L) A neighborhood model $\mathbb{M} = \langle W, N, V \rangle$ is called a **canonical for L** if

1. $W = M_{\mathbf{L}}$
2. for each $\Gamma \in W$ and each formula ϕ, $|\phi|_{\mathbf{L}} \in N(\Gamma)$ iff $\Box\phi \in \Gamma$
3. $V = V_{\mathbf{L}}$

◁

For example, let $\mathbb{M}^{min}_{\mathbf{L}} = \langle M_{\mathbf{L}}, N^{min}_{\mathbf{L}}, V_{\mathbf{L}} \rangle$, where for each $\Gamma \in M_{\mathbf{L}}$, $N^{min}_{\mathbf{L}}(\Gamma) = \{|\phi|_{\mathbf{L}} \mid \Box\phi \in \Gamma\}$. The model $\mathbb{M}^{min}_{\mathbf{L}}$ is easily seen to be a canonical for $\mathbf{L}$. Furthermore, it is the minimal canonical for $\mathbf{L}$ in the sense that for each Γ, $N_{\mathbf{L}}(\Gamma)$ is defined to be the smallest set still satisfying the requisite property. Let $P_{\mathbf{L}}$ be the set of all proof sets of $M_{\mathbf{L}}$. Alternatively the largest canonical, $\mathbb{M}^{max}_{\mathbf{L}}$ can be defined as $\langle M_{\mathbf{L}}, N^{max}_{\mathbf{L}}, V_{\mathbf{L}} \rangle$, where for each $\Gamma \in M_{\mathbf{L}}$, $N^{max}_{\mathbf{L}}(\Gamma) = N^{min}_{\mathbf{L}}(\Gamma) \cup \{X \mid X \subseteq M_{\mathbf{L}}, X \notin P_{\mathbf{L}}\}$

Lemma 3.10 (Truth Lemma) *For any consistent logic* $\mathbf{L}$ *and any consistent formula* ϕ*, if* $\mathbb{M}$ *is canonical for* $\mathbf{L}$*,*

$$(\phi)^{\mathbb{M}} = |\phi|_{\mathbf{L}}$$

Proof. The boolean connectives are as usual and left for the reader. We focus on the modal case. Let $\mathbb{M} = \langle W, N, V \rangle$ be canonical for $\mathbf{L}$. Suppose that $\Gamma \in (\Box\phi)^{\mathbb{M}}$, then by definition $(\phi)^{\mathbb{M}} \in N(\Gamma)$. By the induction hypothesis, $(\phi)^{\mathbb{M}} = |\phi|_{\mathbf{L}}$, hence $|\phi|_{\mathbf{L}} \in N(\Gamma)$. By part 2 of Definition 3.9, $\Box\phi \in \Gamma$. Hence $\Gamma \in |\Box\phi|_{\mathbf{L}}$. Conversely, suppose that $\Gamma \in |\Box\phi|_{\mathbf{L}}$. Then by definition of a truth set, $\Box\phi \in \Gamma$. Hence by part 2 of Definition 3.9, $|\phi|_{\mathbf{L}} \in N(\Gamma)$. By the induction hypothesis, $|\phi|_{\mathbf{L}} = (\phi)_{\mathbf{L}}$, hence $(\phi)^{\mathbb{M}} \in N(\Gamma)$. Hence $\Gamma \in (\Box\phi)^{\mathbb{M}}$. QED

Theorem 3.11 *The logic* $\mathbf{E}$ *is sound and strongly complete with respect to the class of all neighborhood frames.*

Proof. The proof is standard and so will only be sketched. Soundness is straightforward (and in fact already shown in earlier exercises). As for strong completeness, we will show that every consistent set of formulas can be satisfied in some model. Before proving this, we briefly explain why this implies strong completeness. The proof is by contrapositions. Suppose that it is not the case that $\Gamma \vdash_{\mathbf{L}} \phi$.

Then $\Gamma \cup \{\neg\phi\}$ is consistent. Since any such set has a model, there $\Gamma \cup \{\neg\phi\}$ is true in some model. But then Γ cannot semantically entail ϕ. Thus if $\Gamma \not\vdash_{\mathbf{L}} \phi$ then $\Gamma \not\models_{\mathsf{F}} \phi$ (where F is the class of all neighborhood frames).

Let Γ be a consistent set of formulas. By Lindenbaum's Lemma, there is a maximally consistent set Γ' such that $\Gamma \subseteq \Gamma'$. Then consider the model $\mathbb{M}^{min}_{\mathbf{E}}$. By the Truth Lemma (Lemma 3.10), $\mathbb{M}^{min}_{\mathbf{L}}, \Gamma' \models \Gamma'$. Thus Γ is true is some model, namely the minimal canonical model. QED

Notice that in the above proof, the choice to use the *minimal* canonical model for **E** was somewhat arbitrary. It is easy to see that the proof would go through if we had used $\mathbb{M}^{max}_{\mathbf{E}}$ instead of $\mathbb{M}^{min}_{\mathbf{E}}$. Indeed, *any* canonical model for **E** could have been used in the above proof. That there is such a choice of canonical models, will be of great use when proving completeness of systems above **E**. The strategy for proving strong completeness for systems above **E** is similar to the strategy for proving strong completeness of some well-known normal modal logics, such as **S4** or $\mathsf{S5}$. Given the above construction and proof lemma, all that remains is to show that a particular canonical model belongs to the class of frames under consideration. This argument, called by [4] *completeness-via-canonicity*, can be adapted to the neighborhood setting. For example, consider the system **EC**. We argued in the previous section that **C** corresponds to the neighborhoods being closed under (finite) intersections. We now show that **EC** is sound and strongly complete with respect to neighborhood frames that are closed under intersections. We first show that C is canonical for this property (see [4] Chapter 4 for an extended discussion of this notion).

Lemma 3.12 *If $C \in \mathbf{L}$, then $\langle M_{\mathbf{L}}, N^{min}_{\mathbf{L}}\rangle$ is closed under finite intersections.*

Proof. Suppose that $C \in \mathbf{L}$. Further, suppose that $X, Y \in N^{min}_{\mathbf{L}}(\Gamma)$. By definition of $N^{min}_{\mathbf{L}}$, $X = |\phi|_{\mathbf{L}}$ and $Y = |\psi|_{\mathbf{L}}$ where $\Box\phi \in \Gamma$ and $\Box\psi \in \Gamma$. Hence $\Box\phi \wedge \Box\psi \in \Gamma$ and so using C, $\Box(\phi \wedge \psi) \in \Gamma$. Hence, $|\phi \wedge \psi|_{\mathbf{L}} \in N^{min}_{\mathbf{L}}(\Gamma)$. Therefore, since $|\phi|_{\mathbf{L}} \cap |\psi|_{\mathbf{L}} = |\phi \wedge \psi|_{\mathbf{L}}$, $N^{min}_{\mathbf{L}}$ is closed under intersections. QED

Given the above proof, strong completeness is straightforward.

Theorem 3.13 *The logic* **EC** *is sound and strongly complete with respect to the class of neighborhood frames that are closed under intersections.*

Proof. The proof is left as an exercise for the reader. QED

Exercise 3.14 *Prove that* **EN** *is sound and strongly complete with respect to neighborhood frames that contain the unit.*

Moving on to the logic **EM**, we see that the argument is not quite so straightforward. It is here where we make use of the fact that we have a choice of canonical models. The main roadblock is that $\langle M_{\mathbf{EM}}, N^{min}_{\mathbf{EM}}\rangle$ is not closed under supersets.

Observation 3.15 *Show that* $\langle M_{\mathbf{EM}}, N^{min}_{\mathbf{EM}}\rangle$ *is not closed under supersets.*

Proof. Let p be a propositional variable and let Γ be a maximally consistent set such that $\Box p \in \Gamma$ (such a set exists by Lindenbaum's Lemma since $\Box p$ is consistent). Then $|p|_{\mathbf{EM}} \in N^{min}_{\mathbf{EM}}(\Gamma)$. Let Y be any non-proof set that extends $|p|_{\mathbf{EM}}$. To see that such a set exists, let Y' be any non-proof set (there are uncountably many subsets of $M_{\mathbf{EM}}$ but only countably many proof sets.) Then $Y = Y' \cup |p|_{\mathbf{EM}}$ is not a proof set. For if $Y = |\psi|_{\mathbf{EM}}$ for some formula ψ, then $Y' = |\psi \wedge \neg p|_{\mathbf{EM}}$ (why?), which contradicts the fact that Y' is a non-proof set. Clearly $Y \notin N^{min}_{\mathbf{EM}}(\Gamma)$ (why?). But then we have found a set X in $N^{min}_{\mathbf{EM}}(\Gamma)$ such there is a superset of X not contained in $N^{min}_{\mathbf{EM}}(\Gamma)$. QED

However, we can easily skirt this difficulty by choosing a different, better behaved, canonical models. Recall from Section 2, that if $\mathcal{F}$ is any collection of subsets of W, then $sup(\mathcal{F}) = \{X \mid \exists Y \in \mathcal{F} \text{ where } Y \subseteq X\}$. Given any frame $\mathbb{F} = \langle W, N\rangle$, let the **supplementation** of $\mathbb{F}$, denoted $sup(\mathbb{F})$, be the frame $\langle W, N^{sup}\rangle$, where for each $w \in W$, $N^{sup}(w) = sup(N(w))$. Use a similar definition for models, i.e., given $\mathbb{M} = \langle W, N, V\rangle$, the $sup(\mathbb{M}) = \langle W, N^{sup}, V\rangle$. The main argument is to show that the supplementation of the minimal canonical model is a canonical for **EM**.

Lemma 3.16 *Suppose that* $\mathbb{M} = sup(\mathbb{M}^{min}_{\mathbf{EM}})$*. Then* $\mathbb{M}$ *is canonical for* **EM**.

Proof. Suppose that $\mathbb{M} = \langle W, N, V\rangle$, where $W = M_{\mathbf{EM}}$ and for each $\Gamma \in W$, $N(\Gamma) = sup(N^{min}_{\mathbf{EM}}(\Gamma)$, and $V = V_{\mathbf{EM}}$. Let $\Gamma \in W$ be arbitrary. We must show for each formula ϕ,

$$|\phi|_{\mathbf{EM}} \in N(\Gamma) \text{ iff } \Box\phi \in \Gamma$$

The right to left direction is trivial since $\mathcal{N}^{min}_{\mathbf{EM}}(\Gamma) \subseteq N(\Gamma)$. Suppose that $|\phi|_{\mathbf{EM}} \in N(\Gamma)$. Then there is some proof set X such that $X \subseteq |\phi|_{\mathbf{EM}}$. Since X is a proof set, there is some formula ψ such that $X = |\psi|_{\mathbf{EM}}$. Since $|\psi|_{\mathbf{EM}} \subseteq |\phi|_{\mathbf{EM}}$, by Lemma **??** $\vdash_{\mathbf{EM}} \phi \rightarrow \psi$. Since **EM** satisfies the RM rule, $\vdash_{\mathbf{EM}} \Box\phi \rightarrow \Box\psi$. Thus $\Box\phi \rightarrow \Box\psi \in \Gamma$. Hence $\Box\psi \in \Gamma$. QED

Theorem 3.17 *The logic* **EM** *is sound and strongly complete with respect to the class of supplemented frames.*

Proof. Left as an exercise for the reader. QED

Putting everything together, we have a characterization of the smallest normal modal logic **K**.

Theorem 3.18 *The logic* **K** *is sound and strongly complete with respect to the class of filters.*

Exercise 3.19 *Prove that* **K** *is sound and strongly complete with respect to the class of augmented frames.*

Exercise 3.20 *Find an axiomatization and prove soundness and completeness for modal logics with the following modalities:* $[\ \rangle, \langle\ ], [\ ], \langle\ \rangle$

3.2.1 General Neighborhood Frames

General frames are an important tool for modal logicians. There is no inherent difficulty in adapting the definition to the neighborhood setting.

Definition 3.21 (General Neighborhood Frames) A **general neighborhood frame** is a tuple $\mathbb{F}^g = \langle W, N, \mathcal{A}\rangle$, where W is a non-empty set of states, N is a neighborhood function, and $\mathcal{A}$ is a collection of subsets of W closed under intersections, complements, and the m_N operator. $\triangleleft$

We say a valuation $V : \mathsf{At} \to 2^W$ is admissible for a general frame $\langle W, N, \mathcal{A}\rangle$ if for each $p \in \mathsf{At}$, $V(p) \in \mathcal{A}$.

Definition 3.22 (General Neighborhood Model) Suppose that $\mathbb{F}^g = \langle W, N, \mathcal{A}\rangle$ is a general neighborhood frame. A general modal based on $\mathbb{F}^g$ is a tuple $\mathbb{M}^g = \langle W, N, \mathcal{A}, V\rangle$ where V is an admissible valuation. $\triangleleft$

Truth in a general model is defined as in the beginning of this section.

Lemma 3.23 *Let* $\mathbb{M}^g$ *be an general neighborhood model. Then for each* $\phi \in \mathcal{L}$*,* $(\phi)^{\mathbb{M}^g} \in \mathcal{A}$*.*

Proof. The proof follows from an easy induction over the structure of ϕ. QED

Given a logic **L**, it is easy to show that the set $\mathcal{A}_{\mathbf{L}} = \{|\phi|_{\mathbf{L}} \mid \phi \in \mathcal{L}\}$ is a boolean algebra and closed under the m_N operator. A general frame is called a **L**-frame, if **L** is valid on that frame. We show that for each modal logic **L** the canonical frame is a **L**-frame.

Lemma 3.24 *Let* **L** *be any logic extending* **E***. Then the general canonical fram* $\mathbb{F}^g_{\mathbf{L}} \models \mathbf{L}$*.*

Proof. Let $\phi \in \mathbf{L}$ and V an arbitrary admissible valuation. We must show that $\mathbb{M}^g = \langle M_{\mathbf{L}}, N, V\rangle$ validates ϕ. Since V is admissible, for each propositional letter p_i occurring in ϕ, $V(p_i) \in \mathcal{A}_{\mathbf{L}}$. Hence for each (there are only finitely many), p_i, $V(p_i) = |\psi_i|_{\mathbf{L}}$ for some formula ψ_i. Let ϕ' be ϕ where each p_i is replaced with ψ_i. We prove by induction of ϕ that $(\phi)^{\mathbb{M}^g} = (\phi')^{\mathbb{M}^g_{\mathbf{L}}}$.

The base case is when $\phi = p$. Then $\phi' = \psi$ for some $\psi \in \mathcal{L}$ where $V(p) = |\psi|_{\mathbf{L}} \in \mathcal{A}_{\mathbf{L}}$. Then $\Gamma \in (p)^{\mathbb{M}^g}$ iff $\Gamma \in V(p) = |\psi|_{\mathbf{L}}$ iff $\Gamma \in (p)^{\mathbb{M}^g_{\mathbf{L}}}$. The boolean connectives are straightforward. Suppose that ϕ is of the form $\Box\gamma$ and $(\gamma)^{\mathbb{M}^g} = (\gamma')^{\mathbb{M}^g_{\mathbf{L}}}$. Note that $\phi' = \Box\gamma'$. Hence $\Gamma \in (\phi)^{\mathbb{M}^g}$ iff $(\gamma)^{\mathbb{M}^g} \in N(\Gamma)$ iff $(\gamma')^{\mathbb{M}^g_{\mathbf{L}}} \in N(\Gamma)$ iff $\Gamma \in \phi'$.

QED

4 Advanced Topics

The majority of the course will be spent discussing the following topics.

Tableaux for Classical Modal Logics: Tableaux systems have been developed for many non-normal modal logics (see, for example, [17, 19]).

Decidability and Complexity: It is not surprising that classical modal logic is decidable. What perhaps is more surprising is that the complexity of many classical systems is NP-complete (as opposed to the "usual" $PSPACE$-completeness for normal modal logics). This is discussed in detail in [40, 39].

Neighborhood Incompleteness: Martin Gerson [12] found two logics that are incomplete with respect to neighborhood semantics. This phenomena is explored in more detail in [35, 34, 5, 24].

Relation with Relational Semantics: As we have seen in Section 1, there is a 1-1 correpsondance between augmented neighborhood frames and relational frames. This fact can be used to show:

Theorem 4.1 *For every Kripke model $\langle W, R, V\rangle$, there is an pointwise equivalent classical model $\langle W, N, V\rangle$, and vice versa*

But does this result imply that metatheoretic results using relational semantics can be transfered to the neighborhood setting? The following list of results demonstrates the difficulty in answering this question.

- There are logics which are complete with respect to a class of neighborhood frames but not complete with respect to relational frames. This was first shown by Dov Gabbay [9] and later Martin Gerson [11] showed that in fact there is such a logic which is an extension of **S4**.

- Martin Gerson [13] showed that there is a neighborhood frame for **T** with no equivalent relational frame. That is, there is a neighborhood frame such that no relational frame can model exactly the same formulas.

A second line of inquiry discovered independently by Kracht and Wolter [22] and Gasquet and Herzig [10] is that non-normal modal logics can be *simulated* by a normal modal logic with three modalities. The key idea is to translate the *model* as well as the formulas.

Definition 4.2 (Normal Translation) Given a neighborhood model $\mathcal{M} = \langle W, N, V\rangle$, define a Kripke model $\mathcal{M}^\circ = \langle V, R_N, R_{\not\ni}, R_\nu, Pt, V\rangle$ as follows:

- $V = W \cup 2^W$
- $R_\ni = \{(v, w) \mid w \in W, v \in 2^W, v \in w\}$
- $R_{\not\ni} = \{(v, w) \mid w \in W, v \in 2^W, v \notin w\}$
- $R_N = \{(w, v) \mid w \in W, v \in 2^W, v \in N(w)\}$
- $Pt = W$

$\triangleleft$

We can interpret a standard modal language in such models. Let $\mathcal{L}'$ be the language

$$\phi := p \mid \neg\phi \mid \phi \wedge \psi \mid [\ni]\phi \mid [\not\ni]\phi \mid [N]\phi \mid \mathsf{Pt}$$

where $p \in \mathsf{At}$ and Pt is a unary modal operator. Truth is defined as usual: for example $[\ni]\phi$ is true at w just in case for each v, if $wR_\ni v$ then ϕ is true at v. We can now define a truth-preserving translation from sentences of $\mathcal{L}$ to sentences of $\mathcal{L}'$. Define $ST : \mathcal{L} \to \mathcal{L}'$ as follows

- $ST(p) = p$
- $ST(\neg\phi) = \neg ST(\phi)$
- $ST(\phi \wedge \psi) = ST(\phi) \wedge ST(\phi)$
- $ST(\Box\phi) = \langle\nu\rangle([\ni]ST(\phi) \wedge [\not\ni]\neg ST(\phi))$

Lemma 4.3 *For each neighborhood model $\mathcal{M} = \langle W, \nu, V\rangle$ and each formula $\phi \in \mathcal{L}$, for any $w \in W$,*

$$\mathcal{M}, w \models \phi \text{ iff } \mathcal{M}^\circ, w \models \mathsf{Pt} \rightarrow ST(\phi)$$

Duality Results: As with normal modal logics, there is a duality between neighborhood frames and modal algebras. This was first studied by Dosen [7] and later by Hansen [18] in the context of *monotonic* modal logic.

Topology and Modal Logic: Much of the original motivation for neighborhood frames as a semantics form modal logics comes from elementary point-set topology. The idea is to think of the propositions in $N(w)$ to be *close* to the point w. We begin by reviewing some very basic point-set topology. More information can be found in any point-set topology text book (Dugundji [8] is an excellent choice).

Definition 4.4 (Topological Space) A **topological space** is a neighborhood frame $\langle W, \mathcal{T}\rangle$ where W is a nonempty set and

1. $W \in \mathcal{T}$, $\emptyset \in W$
2. $\mathcal{T}$ is closed under finite intersections
3. $\mathcal{T}$ is closed under arbitrary unions.

$\triangleleft$

The collection $\mathcal{T}$ is called a **topology**. Elements $O \in \mathcal{T}$ are called **opens**. A set C such that $W - C \in \mathcal{T}$ is called **closed**. Given a topology $\langle W, \mathcal{T}\rangle$, let $\mathcal{T}_C$ be the collection of closed subsets of W, i.e., $\mathcal{T}_C = \{C \mid W - C \in \mathcal{T}\}$. The following observation is an easy consequence of the above definition.

Observation 4.5 *Let $\langle W, \mathcal{T}\rangle$ be a topological space. Then $\mathcal{T}_C$ has the following properties:*

1. $\emptyset, W \in \mathcal{T}_C$

2. $\mathcal{T}_C$ is closed under finite unions

3. $\mathcal{T}_C$ is closed under arbitrary intersections

Proof. Left as an exercise for the reader. QED

Given a topological space $\langle W, \mathcal{T}\rangle$ and a point $w \in W$, a **neighborhood of** w is any open set that contains w. Let $\mathcal{T}_w = \{O \mid O \in \mathcal{T} \text{ and } w \in O\}$ be the collection of all neighborhoods of w.

Lemma 4.6 *Let $\langle W, \mathcal{T} \rangle$ be a topological space. Then for each $w \in W$, the collection $\mathcal{T}_w$ contains W, is closed under finite intersections and closed under arbitrary unions.*

Definition 4.7 (Neighborhood System) Let $\langle W, \mathcal{T} \rangle$ be a topological space. A pair $\langle W, N \rangle$ is called a **neighborhood system** provided $N : W \to \mathcal{T}$ is defined as follows: $N(w) = \mathcal{T}_w$. $\triangleleft$

Let $\langle W, \mathcal{T} \rangle$ be a topological space and $X \subseteq W$ any set. The largest open subset of X is called the **interior** of X, denoted $Int(X)$. Formally,

$$Int(X) = \cup\{O \mid O \in \mathcal{T} \text{ and } O \subseteq X\}$$

The smallest closed set containing X is called the **closure** of X, denoted $Cl(X)$. Formally,

$$Cl(X) = \cap\{C \mid W - C \in \mathcal{T} \text{ and } X \subseteq C\}$$

It is easy to see that a set X is open if $Int(X) = X$ and closed if $Cl(X) = X$. The following Lemma will be helpful when studying the topological semantics of the next section.

Lemma 4.8 *Let $\langle W, \mathcal{T} \rangle$ be a topological space and $X \subseteq W$. Then*

1. $Int(X \cap Y) = Int(X) \cap Int(Y)$
2. $Int(\emptyset) = \emptyset$, $Int(W) = W$
3. $Int(X) \subseteq X$
4. $Int(Int(X)) = Int(X)$
5. $Int(X) = W - Cl(W - X)$

Exercise 4.9 *Use the last fact in the above lemma to derive corresponding properties for the closure operator.*

Topological semantics for modal logic has been around for nearly 60 years and is usually attributed to McKinsey and Tarski [25]. The literature on this topic is much to vast to survey here (the reader is referred to [38] for a complete discussion). A **topological model** is a tuple $\mathbb{M}^T = \langle W, \mathcal{T}, V \rangle$, where $\langle W, \mathcal{T} \rangle$ is a topological space and V is a valuation function. Formulas in $\mathcal{L}$ are interpreted at states $w \in W$. The boolean connectives and atomic propositions are interpreted as usual. We only give the definition of truth of the modal operator:

$$\mathbb{M}^T, w \models \Box\phi \text{ iff } \exists O \in \mathcal{T}, w \in O \text{ such that } \forall v \in O, \mathbb{M}^T, v \models \phi$$

Notice the similarity between this definition an the definition of truth of the modal operator $\langle\ ]$. The only difference is the extra clause $w \in O$. However, recall the function $w \mapsto \mathcal{T}_w$, where $\mathcal{T}_w$ is the set of neighborhoods of w. Then, the above clause can be written as

$$\mathbb{M}^T, w \models \Box\phi \text{ iff } \exists O \in \mathcal{T}_w \text{ such that } \forall v \in O, \mathbb{M}^T, v \models \phi$$

Although this difference is a trivial change in terminology, it demonstrates a close connection between neighborhood frames and topological frames. Finally it is easy to see that

$$(\Box\phi)^{\mathbb{M}^T} = Int((\phi)^{\mathbb{M}^T})$$

The classic result of Tarski and McKinsey shows that **S4** is sound and complete with respect to the class of all topologies

Let $\langle W, N, V\rangle$ be a neighborhood models. Suppose that N satisfies the following properties

- for each $w \in W$, $N(w)$ is a filter
- for each $w \in W$, $w \in \cap N(w)$
- for each $w \in W$ and $X \subseteq W$, if $X \in N(w)$, then $m_N(X) \in N(w)$

We can now show that there is a topological model that is point-wise equivalent to $\mathcal{M}$. Consider the set $\mathcal{B} = \{m_N(X) \mid X \subseteq W\}$. We will show that $\mathcal{B}$ is a base. That is we must show that 1. $\cup\mathcal{B} = W$ and 2. for each $X, Y \in \mathcal{B}$ and each $x \in X \cap Y$ there is a $Z \in \mathcal{B}$ such that $x \in Z \subseteq X \cap Y$. Suppose that $X = m_N(x_1)$ and $Y = m_N(X_2)$ and $x \in X \cap Y$. Since $N(w)$ is a filter, $m_N(X_1) \cap m_N(X_2) = m_N(X_1 \cap X_2)$. Thus $x \in m_N(X_1 \cap X_2) \subseteq m_N(X_1) \cap m_N(X_2)$. Thus we are done if we can show that $m_N(X_1 \cap X_2) \in \mathcal{B}$. But this follows form the third property above since $X_1 \cap X_2 \in N(w)$.

Model Theory of Modal Logic with Respect to Neighborhood Semantics: Modal logic has a rich model theory with respect to relational semantics (cf.[16]). What about the situation with respect to neighborhood semantics? Concerning *monotonic* neighborhood models, the situation is well-understood [18, 30]. We will explore the more general setting with discussions on various model constructions (eg. bounded morphisms, disjoint unions, bisimulations, generated submodels and ultrafilter extensions) [21], generalizations of some classic results in modal logic (eg. the van Benthem characterization theorem) [21] and a notion of *neighborhood canonicity* [36]

Applications and Extensions: Much of the interest in neighborhood semantics is generated by the fact that a neighborhood frame is a natural example of a *coalgebra* [41]. We will discuss this fact and other relations between non-normal modal logics and work in coalgebra (cf. [20]).

We will end the course with two "applications" of neighborhood models for modal logic. This first is the somewhat surprising result that one can prove a *strong* completeness result for certain classical modal logics with a common knowledge operator [23]. Secondly, we show how the neighborhood models can be used to give an interesting new perspective on classic results in first-order modal logic [2].

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