Neighborhood Semantics for Modal Logic

Lecture 2

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✓ Introduction, Motivation and Background Information

Lecture 2: Basic Concepts, Non-normal Modal Logics, Completeness, Incompleteness, Relation with Relational Semantics

Lecture 3: Decidability/Complexity, Advanced Topics — Topological Semantics for Modal Logic, some Model Theory

Lecture 4: Advanced Topics — Topological Semantics for Modal Logic, some Model Theory

Lecture 5: Neighborhood Semantics in Action: Game Logic, Coalgebra, Common Knowledge, First-Order Modal Logic

Modal Logic: an Introduction, Chapters 7 - 9, by Brian Chellas

Outline of Part I

- Preliminaries
- Neighborhood Frames and Models
- Reasoning about Neighborhood Structures
- Alternative Semantics

Outline of Part II

- The Normal Situation
- Non-normal modal logics
- Completeness

Some Terminology

Let W be a set and $\mathcal{F} \subseteq \wp(W)$.

- $\mathcal{F}$ is closed under intersections if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in \mathcal{F}$, then $\cap_{i \in I} X_i \in \mathcal{F}$.
- $\mathcal{F}$ is closed under unions if for any collections of sets $\{X_i\}_{i \in I}$ such that for each $i \in I$, $X_i \in \mathcal{F}$, then $\cup_{i \in I} X_i \in \mathcal{F}$.
- $\mathcal{F}$ is closed under complements if for each $X \subseteq W$, if $X \in \mathcal{F}$, then $X^C \in \mathcal{F}$.
- $\mathcal{F}$ is supplemented, or closed under supersets or monotonic provided for each $X \subseteq W$, if $X \in \mathcal{F}$ and $X \subseteq Y \subseteq W$, then $Y \in \mathcal{F}$.

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Let W be a set and $\mathcal{F} \subseteq \wp(W)$.

- $\mathcal{F}$ contains the unit provided $W \in \mathcal{F}$
- the set $\cap_{X \in \mathcal{F}} X$ the core of $\mathcal{F}$. $\mathcal{F}$ contains its core provided $\cap_{X \in \mathcal{F}} X \in \mathcal{F}$.
- $\mathcal{F}$ is proper if $X \in \mathcal{F}$ implies $X^C \notin \mathcal{F}$.
- $\mathcal{F}$ is consistent if $\emptyset \notin \mathcal{F}$
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Lemma

$\mathcal{F}$ is supplemented iff if $X \cap Y \in \mathcal{F}$ then $X \in \mathcal{F}$ and $Y \in \mathcal{F}$.

A few more definitions

- $\mathcal{F}$ is a filter if $\mathcal{F}$ contains the unit, closed under binary intersections and supplemented. $\mathcal{F}$ is a proper filter if in addition $\mathcal{F}$ does not contain the emptyset.
- $\mathcal{F}$ is an ultrafilter if $\mathcal{F}$ is proper filter and for each $X \subseteq W$, either $X \in \mathcal{F}$ or $X^C \in \mathcal{F}$.
- $\mathcal{F}$ is a topology if $\mathcal{F}$ contains the unit, the emptyset, is closed under finite intersections and arbitrary unions.
- $\mathcal{F}$ is augmented if $\mathcal{F}$ contains its core and is supplemented.

Some Facts

Lemma
If $\mathcal{F}$ is augmented, then $\mathcal{F}$ is closed under arbitrary intersections. In fact, if $\mathcal{F}$ is augmented then $\mathcal{F}$ is a filter.

Fact
There are consistent filters that are not augmented.

Lemma
If $\mathcal{F}$ is closed under binary intersections (i.e., if $X, Y \in \mathcal{F}$ then $X \cap Y \in \mathcal{F}$), then $\mathcal{F}$ is closed under finite intersections.

Corollary
If W is finite and $\mathcal{F}$ is a filter over W, then $\mathcal{F}$ is augmented.

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- Preliminaries
- Neighborhood Frames and Models
- Reasoning about Neighborhood Structures
- Alternative Semantics

Neighborhood Frames

Let W be a non-empty set of states.

Any map $N : W \to \wp\wp W$ is called a neighborhood function

Definition

A pair $\langle W, N \rangle$ is a called a neighborhood frame if W a non-empty set and N is a neighborhood function.

From Kripke Frames to Neighborhood Frames

Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

$$\text{for each } w \in W, \text{ let } R^{\rightarrow}(w) = \{v \mid wRv\}$$

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Definition

Given a relation R on a set W and a state $w \in W$. A set $X \subseteq W$ is R-necessary at w if $R^{\rightarrow}(w) \subseteq X$.

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Let $R \subseteq W \times W$, define a map $R^{\rightarrow} : W \rightarrow \wp W$:

$$\text{for each } w \in W, \text{ let } R^{\rightarrow}(w) = \{v \mid wRv\}$$

Let $\mathcal{N}_w^R$ be the set of sets that are R-necessary at w:

$$\mathcal{N}_w^R = \{X \mid R^{\rightarrow}(w) \subseteq X\}$$

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Let $\mathcal{N}_w^R$ be the set of sets that are R-necessary at w:

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Lemma

Let R be a relation on W. Then for each $w \in W$, $\mathcal{N}_w^R$ is augmented.

From Kripke Frames to Neighborhood Frames

Properties of R are reflected in $\mathcal{N}_w^R$:

- If R is reflexive, then for each $w \in W$, $w \in \cap\mathcal{N}_w$
- If R is transitive then for each $w \in W$, if $X \in \mathcal{N}_w$, then $\{v \mid X \in \mathcal{N}_v\} \in \mathcal{N}_w$.

From Neighborhood Frames to Kripke Frames

Theorem

- *Let* $\langle W, R \rangle$ *be a relational frame. Then there is an equivalent augmented neighborhood frame.*
- *Let* $\langle W, N \rangle$ *be an augmented neighborhood frame. Then there is an equivalent relational frame.*

From Neighborhood Frames to Kripke Frames

$$\boxed{\text{for all } X \subseteq W,\ X \in N(w) \text{ iff } X \in \mathcal{N}^R_w.}$$

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Theorem

- ✓ *Let $\langle W, R\rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*
- *Let $\langle W, N\rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

Proof.

For each $w \in W$, let $N(w) = \mathcal{N}_w^R$. $\square$

From Neighborhood Frames to Kripke Frames

Theorem

- *Let $\langle W, R\rangle$ be a relational frame. Then there is an equivalent augmented neighborhood frame.*

✓ *Let $\langle W, N\rangle$ be an augmented neighborhood frame. Then there is an equivalent relational frame.*

Proof.

For each $w, v \in W$, $wR_N v$ iff $v \in \cap N(w)$. □

Neighborhood Model

Let $\mathfrak{F} = \langle W, N \rangle$ be a neighborhood frame. A neighborhood model based on $\mathfrak{F}$ is a tuple $\langle W, N, V \rangle$ where $V : \mathsf{At} \rightarrow 2^W$ is a valuation function.

Truth in a Model

- $\mathfrak{M}, w \models p$ iff $w \in V(p)$
- $\mathfrak{M}, w \models \neg\varphi$ iff $\mathfrak{M}, w \not\models \varphi$
- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$

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- $\mathfrak{M}, w \models \varphi \wedge \psi$ iff $\mathfrak{M}, w \models \varphi$ and $\mathfrak{M}, w \models \psi$
- $\mathfrak{M}, w \models \Box\varphi$ iff $(\varphi)^{\mathfrak{M}} \in N(w)$
- $\mathfrak{M}, w \models \Diamond\varphi$ iff $W - (\varphi)^{\mathfrak{M}} \notin N(w)$

where $(\varphi)^{\mathfrak{M}} = \{w \mid \mathfrak{M}, w \models \varphi\}$.

Let $N : W \to \wp\wp W$ be a neighborhood function and define $m_N : \wp W \to \wp W$:

$$\text{for } X \subseteq W,\ m_N(X) = \{w \mid X \in N(w)\}$$

1. $(p)^{\mathfrak{M}} = V(p)$ for $p \in \mathsf{At}$
2. $(\neg\varphi)^{\mathfrak{M}} = W - (\varphi)^{\mathfrak{M}}$
3. $(\varphi \wedge \psi)^{\mathfrak{M}} = (\varphi)^{\mathfrak{M}} \cap (\psi)^{\mathfrak{M}}$
4. $(\Box\varphi)^{\mathfrak{M}} = m_N((\varphi)^{\mathfrak{M}})$
5. $(\Diamond\varphi)^{\mathfrak{M}} = W - m_N(W - (\varphi)^{\mathfrak{M}})$

Detailed Example

Suppose $W = \{w, s, v\}$ is the set of states and define a neighborhood model $\mathfrak{M} = \langle W, N, V\rangle$ as follows:

- $N(w) = \{\{s\}, \{v\}, \{w, v\}\}$
- $N(s) = \{\{w, v\}, \{w\}, \{w, s\}\}$
- $N(v) = \{\{s, v\}, \{w\}, \emptyset\}$

Further suppose that $V(p) = \{w, s\}$ and $V(q) = \{s, v\}$.

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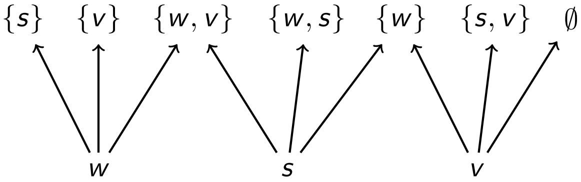

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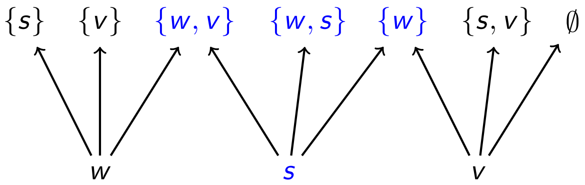

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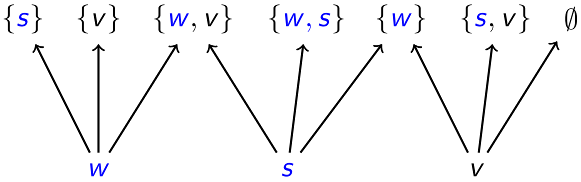

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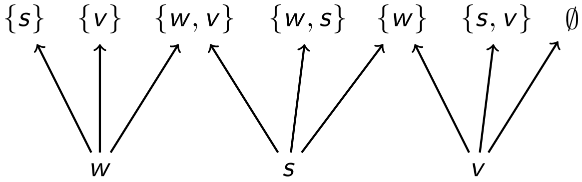

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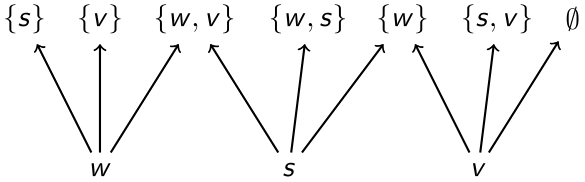

$$\mathfrak{M}, s \models \Box p$$

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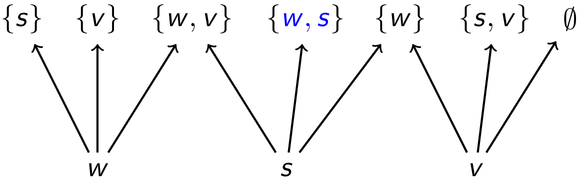

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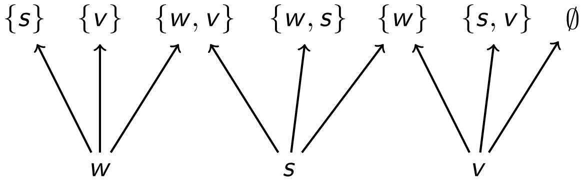

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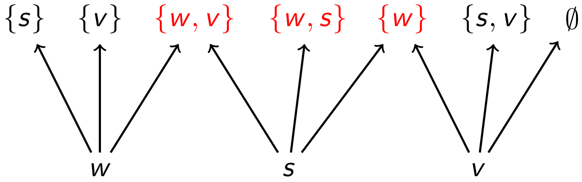

$$\mathfrak{M}, s \models \Diamond p$$
$$(\neg p)^{\mathfrak{M}} = \{v\}$$

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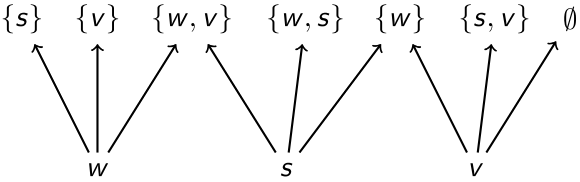

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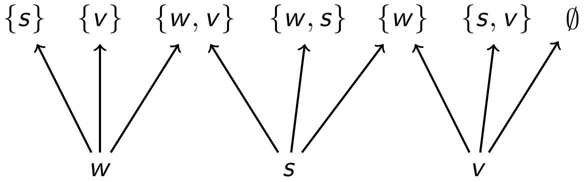

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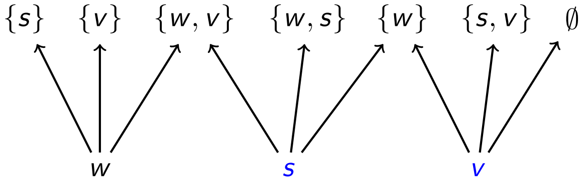

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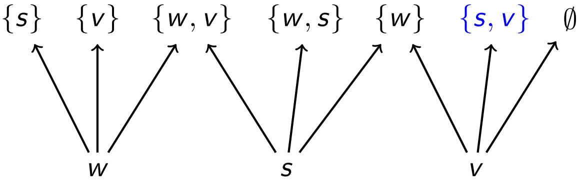

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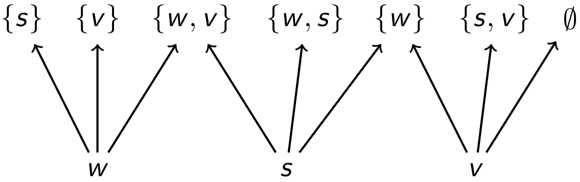

$\mathfrak{M}, w \not\models \Diamond\Box p$ $\qquad$ $\mathfrak{M}, v \models \Box\Diamond p$

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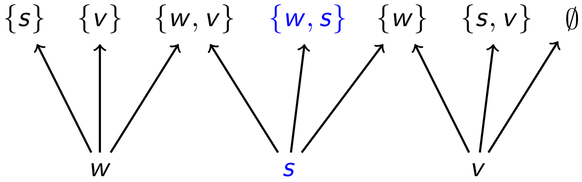

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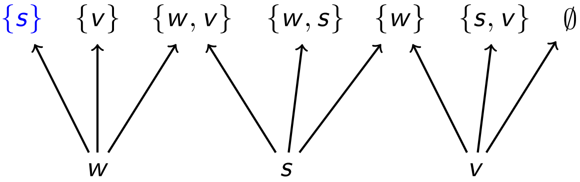
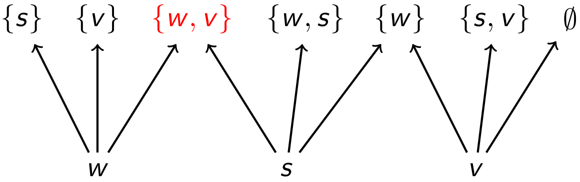


$\mathfrak{M}, w \not\models \Diamond\Box p$ $\qquad\qquad$ $\mathfrak{M}, v \models \Box\Diamond p$

$\mathfrak{M}, w \models \Box\Box p$ $\qquad\qquad$ $\mathfrak{M}, v \models \Diamond\Box p$

Detailed Example

$$V(p) = \{w, s\} \text{ and } V(q) = \{s, v\}$$



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New slogan: The basic modal language is a simple language for talking about *neighborhood structures*.

What can we say?

Definition

A modal formula φ defines a property P of neighborhood functions if any neighborhood frame $\mathfrak{F}$ has property P iff $\mathfrak{F}$ validates φ.

What can we say?

Lemma

Let $\mathfrak{F} = \langle W, N \rangle$ *be a neighborhood frame. Then* $\mathfrak{F} \models \Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$ *iff* $\mathfrak{F}$ *is closed under supersets.*

What can we say?

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Lemma

Let $\mathfrak{F} = \langle W, N \rangle$ *be a neighborhood frame. Then* $\mathfrak{F} \models \Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$ *iff* $\mathfrak{F}$ *is closed under finite intersections.*

What can we say?

Consider the formulas $\Diamond\top$ and $\Box\varphi \rightarrow \Diamond\varphi$.

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- $\Diamond\top$ corresponds to the property $\emptyset \notin N(w)$

What can we say?

Consider the formulas $\Diamond\top$ and $\Box\varphi \rightarrow \Diamond\varphi$.

On relational frames, these formulas both define the same property: seriality.

On neighborhood frames:

- $\Diamond\top$ corresponds to the property $\emptyset \notin N(w)$
- $\Box\varphi \rightarrow \Diamond\varphi$ is valid on $\mathfrak{F}$ iff $\mathfrak{F}$ is proper.

What can we say?

Lemma

Let $\mathfrak{F} = \langle W, N \rangle$ be a neighborhood frame such that for each $w \in W$, $N(w) \neq \emptyset$.

1. $\mathfrak{F} \models \Box\varphi \rightarrow \varphi$ *iff for each* $w \in W$, $w \in \cap N(w)$
2. $\mathfrak{F} \models \Box\varphi \rightarrow \Box\Box\varphi$ *iff for each* $w \in W$, *if* $X \in N(w)$, *then* $\{v \mid X \in N(v)\} \in N(w)$

Find properties on frames that are defined by the following formulas:

1. $\Box\bot$
2. $\neg\Box\varphi \rightarrow \Box\neg\Box\varphi$
3. $\Diamond\varphi \rightarrow \Box\varphi$
4. $\Diamond\Box\varphi \rightarrow \Box\Diamond\varphi$
5. $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$

Neighborhood structures provide a semantics for the logics discussed yesterday.

Some Non-validities

1. $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$
2. $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$
3. $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$
4. $\Box\top$
5. $\Box\varphi \rightarrow \varphi$
6. $\Box\varphi \rightarrow \Box\Box\varphi$
7. Many more...

Validities

(Dual) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$ is valid in all neighborhood models.

(Re) If $\varphi \leftrightarrow \psi$ is valid then $\Box\varphi \leftrightarrow \Box\psi$ is valid.

Other modal operators

- $\mathfrak{M}, w \models \langle\ \rangle\varphi$ iff $\exists X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\ ]\varphi$ iff $\forall X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$

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Lemma

Let $\mathfrak{M} = \langle W, N, V\rangle$ be a neighborhood model. The for each $w \in W$,

1. *if $\mathfrak{M}, w \models \Box\varphi$ then $\mathfrak{M}, w \models \langle\ ]\varphi$*
2. *if $\mathfrak{M}, w \models [\ \rangle\varphi$ then $\mathfrak{M}, w \models \Diamond\varphi$*

However, the converses of the above statements are false.

Other modal operators

- $\mathfrak{M}, w \models \langle\ ]\varphi$ iff $\exists X \in N(w)$ such that $\forall v \in X$, $\mathfrak{M}, v \models \varphi$
- $\mathfrak{M}, w \models [\ \rangle\varphi$ iff $\forall X \in N(w)$ such that $\exists v \in X$, $\mathfrak{M}, v \models \varphi$

Lemma

1. *If $\varphi \rightarrow \psi$ is valid in $\mathfrak{M}$, then so is $\langle\ ]\varphi \rightarrow \langle\ ]\psi$.*
2. $\langle\ ](\varphi \wedge \psi) \rightarrow (\langle\ ]\varphi \wedge \langle\ ]\psi)$ *is valid in $\mathfrak{M}$*

Investigate analogous results for the other modal operators defined above.

Different Semantics

A multi-relational Kripke model is a triple $\mathbb{M} = \langle W, \mathcal{R}, V\rangle$ where $\mathcal{R} \subseteq \wp(W \times W)$.

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Are multi-relational semantics *equivalent* to neighborhood semantics?

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Are multi-relational semantics *equivalent* to neighborhood semantics? Almost

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A world is called queer if nothing is necessary and everything is possible.

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A multi-relational model with queer worlds is a quadruple $\mathbb{M} = \langle W, Q, \mathcal{R}, V \rangle$.

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A multi-relational model with queer worlds is a quadruple $\mathbb{M} = \langle W, Q, \mathcal{R}, V \rangle$.

$\mathbb{M}, w \models \Box\varphi$ iff $w \notin Q$ and $\exists R \in \mathcal{R}$ such that $\forall v \in W$, if wRv then $\mathbb{M}, v \models \varphi$.

Different Semantics

M. Fitting. *Proof Methods for Modal and Intuitionistic Logics*. 1983.

L. Goble. *Multiplex semantics for Deontic Logic*. Nordic Journal of Philosophical Logic (2000).

G. Governatori and A. Rotolo. *On the axiomatization of Elgesems logic of agency and ability*. JPL (2005).

Part II

Axiomatics

- The Normal Situation
- Non-normal modal logics
- Completeness

Axiomatics

The smallest normal modal logic **K** consists of

PC Your favorite axioms of **PC**

K $\Box(\varphi \rightarrow \psi) \rightarrow \Box\varphi \rightarrow \Box\psi$

Nec $\dfrac{\vdash \varphi}{\Box\varphi}$

MP $\dfrac{\vdash \varphi \rightarrow \psi \qquad \vdash \varphi}{\psi}$

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Theorem: **K** is sound and strongly complete with respect to the class of all Kripke frames.

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Theorem: For all $\Gamma \subseteq \mathcal{L}$, $\Gamma \vdash_{\mathbf{K}} \varphi$ iff $\Gamma \models \varphi$.

Axiomatics

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Theorem: **K** $+ \Box\varphi \rightarrow \varphi + \Box\varphi \rightarrow \Box\Box\varphi$ is sound and strongly complete with respect to the class of all reflexive and transitive Kripke frames.

Incompleteness

There are (consistent) modal logics that are incomplete:

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Theorem Let **TMEQ** be the following normal modal logic:

- **K**
- $\Box\varphi \rightarrow \varphi$
- $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$
- $\Diamond(\Diamond\varphi \wedge \Box\psi) \rightarrow \Box(\Diamond\varphi \vee \Box\varphi)$
- $(\Diamond\varphi \wedge \Box(\varphi \rightarrow \Box\varphi)) \rightarrow \varphi$

There is no class of frames validating precisely the formulas in **TMEQ**.

Incompleteness

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J. van Benthem. *Two Simple Incomplete Modal Logics*. Theoria (1978).

BAO

Definition A boolean algebra with operators is a pair $\mathfrak{B} = \langle \mathfrak{A}, m \rangle$ where $\mathfrak{A}$ is a bolean algebra and m is a unary operator on $\mathfrak{A}$ such that:

- $m(x + y) = m(x) + m(y)$
- $m(0) = 0$

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Example: Given a Kripke frame $\mathbb{F} = \langle W, R \rangle$, let $\mathfrak{A} = \langle \wp(W), \cap, \cup, \cdot^C \rangle$ and $m : \wp(X) \to \wp(X)$ is defined as:

$$m(X) = \{y \in W \mid \exists x \in X \text{ such that } yRx\}$$

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Theorem *Any* normal modal logic is complete with respect to some class of boolean algebras with operators.

General Frames

Definition A general frame is a pair $\langle \mathbb{F}, \mathcal{A} \rangle$ where $\mathbb{F} = \langle W, R \rangle$ is a Kripke frame, and $\emptyset \neq \mathcal{A} \subseteq \wp(W)$ is a collection of admissible sets closed under the following operations:

- union: if $X, Y \in \mathcal{A}$ then $X \cup Y \in \mathcal{A}$
- relative complement: if $X \in \mathcal{A}$ then $W - X \in \mathcal{A}$
- modal operations: if $X \in \mathcal{A}$ then $m(X) \in \mathcal{A}$

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- modal operations: if $X \in \mathcal{A}$ then $m(X) \in \mathcal{A}$

Theorem *Any* normal modal logic **L** is sound and strongly complete with respect to some class of general frames.

- The Normal Situation
- Non-normal modal logics
- Completeness

PC Propositional Calculus

E $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

M $\Box(\varphi \wedge \psi) \rightarrow (\Box\varphi \wedge \Box\psi)$

C $(\Box\varphi \wedge \Box\psi) \rightarrow \Box(\varphi \wedge \psi)$

N $\Box\top$

K $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

RE $\dfrac{\varphi \leftrightarrow \psi}{\Box\varphi \leftrightarrow \Box\psi}$

Nec $\dfrac{\varphi}{\Box\varphi}$

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A modal logic **L** is classical if it contains all instances of *E* and is closed under *RE*.

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E is the smallest classical modal logic.

In **E**, *M* is equivalent to

$$(Mon)\ \frac{\varphi \rightarrow \psi}{\Box\varphi \rightarrow \Box\psi}$$

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E	$\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$
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E is the smallest classical modal logic.

EM is the logic **E** $+$ *Mon*

PC 6. Propositional Calculus

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E is the smallest classical modal logic.

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PC Propositional Calculus

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EC is the logic **E** $+$ *C*

EMC is the smallest regular modal logic

A logic is normal if it contains all instances of *E*, *C* and is closed under *Mon* and *Nec*

PC Propositional Calculus

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E is the smallest classical modal logic.

EM is the logic **E** + *Mon*

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EMC is the smallest regular modal logic

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EMC is the smallest regular modal logic

K = **EMCN**

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K $= PC(+E) + K + Nec + MP$

Are there non-normal extensions of **K**?

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Let **L** be the smallest modal logic containing

- **S4** (**K** $+\ \Box\varphi \to \varphi\ +\ \Box\varphi \to \Box\Box\varphi$)
- all instances of M: $\Box\Diamond\varphi \to \Diamond\Box\varphi$

Claim: **L** is a non-normal extension of **S4**.

Useful Fact

Theorem (Uniform Substitution)

The following rule can be derived in **E**

$$\frac{\psi \leftrightarrow \psi'}{\varphi \leftrightarrow \varphi[\psi/\psi']}$$

Interesting Fact

Each of K, M and C are logically independent:

- $\mathbf{EC} \nvdash K$
- $\mathbf{EM} \nvdash K$
- $\mathbf{EK} \nvdash M$
- $\mathbf{EK} \nvdash C$

Interesting Fact

Each of K, M and C are logically independent:

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- $\mathbf{EM} \nvdash K$
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- $\mathbf{EK} \nvdash C$

"Our discussion indicates that, in a sense, C is a more fundamental schema than K; yet it is K which is most often used in axiomatizations of normal modal logics."

(pg. 45)

K. Segerberg. *An Essay on Classical Modal Logic*. 1970.

- The Normal Situation
- Non-normal modal logics
- Completeness

Some Notation

- A formula $\varphi \in \mathcal{L}$ is valid in F ($\models_{\mathsf{F}} \varphi$) if for each $\mathbb{F} \in \mathsf{F}$, $\mathbb{F} \models \varphi$.
- We say that a logic $\mathbf{L}$ is sound with respect to F, provided $\vdash_{\mathbf{L}} \varphi$ implies $\models_{\mathsf{F}} \varphi$.
- A set of formulas Γ semantically entails φ with respect to F, denoted $\Gamma \models_{\mathsf{F}} \varphi$, if for each $\mathbb{F} \in \mathsf{F}$, if $\mathbb{F} \models \Gamma$ then $\mathbb{F} \models \varphi$.
- A logic $\mathbf{L}$ is weakly complete with respect to a class of frames F, if $\models_{\mathsf{F}} \varphi$ implies $\vdash_{\mathbf{L}} \varphi$.
- A logic $\mathbf{L}$ is strongly complete with respect to a class of frames F, if for each set of formulas Γ, $\Gamma \models_{\mathsf{F}} \varphi$ implies $\Gamma \vdash_{\mathbf{L}} \varphi$.
- The $\mathbf{L}$-proof set of $\varphi \in \mathcal{L}$ is $|\varphi|_{\mathbf{L}} = \{\Gamma \mid \varphi \in \Gamma\}$.

Some Notation

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- for all $p \in \mathsf{At}$, $V(p) = |p|_{\mathbf{L}}$

Examples of Canonical Models

$\mathbb{M}_{\mathbf{L}}^{min} = \langle M_{\mathbf{L}}, N_{\mathbf{L}}^{min}, V_{\mathbf{L}} \rangle$, where for each $\Gamma \in M_{\mathbf{L}}$,
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Let $P_{\mathbf{L}} = \{|\varphi|_{\mathbf{L}} \mid \varphi \in \mathcal{L}\}$ be the set of all proof sets.

$\mathbb{M}^{max}_{\mathbf{L}} = \langle M_{\mathbf{L}}, N^{max}_{\mathbf{L}}, V_{\mathbf{L}} \rangle$, where for each $\Gamma \in M_{\mathbf{L}}$,
$N^{max}_{\mathbf{L}}(\Gamma) = N^{min}_{\mathbf{L}}(\Gamma) \cup \{X \mid X \subseteq M_{\mathbf{L}}, X \notin P_{\mathbf{L}}\}$

The canonical model works...

Lemma

For any logic $\mathbf{L}$ *containing the rule RE, if* $N_{\mathbf{L}} : M_{\mathbf{L}} \to \wp(\wp(M_{\mathbf{L}}))$ *is a function such that for each* $\Gamma \in M_{\mathbf{L}}$, $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ *iff* $\Box\varphi \in \Gamma$. *Then if* $|\varphi|_{\mathbf{L}} \in N_{\mathbf{L}}(\Gamma)$ *and* $|\varphi|_{\mathbf{L}} = |\psi|_{\mathbf{L}}$, *then* $\Box\psi \in \Gamma$.

Lemma (Truth Lemma)

For any consistent classical modal logic $\mathbf{L}$ *and any consistent formula* φ, *if* $\mathbb{M}$ *is canonical for* $\mathbf{L}$,

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The Proofs

Theorem

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Lemma

If $C \in \mathbf{L}$, *then* $\langle M_{\mathbf{L}}, N_{\mathbf{L}}^{min} \rangle$ *is closed under finite intersections.*

Theorem

The logic **EC** *is sound and strongly complete with respect to the class of neighborhood frames that are closed under intersections.*

The Proofs

Fact: $\langle M_{\mathbf{EM}}, N^{min}_{\mathbf{EM}} \rangle$ is not closed under supersets.

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Lemma

Suppose that $\mathbb{M} = sup(\mathbb{M}^{min}_{\mathbf{EM}})$*. Then* $\mathbb{M}$ *is canonical for* **EM***.*

Theorem

The logic **EM** *is sound and strongly complete with respect to the class of supplemented frames.*

The Proofs

Theorem

The logic **K** *is sound and strongly complete with respect to the class of filters.*

Theorem

The logic **K** *is sound and strongly complete with respect to the class of augmented frames.*

Incompleteness?

Are all modal logics complete with respect to some class of neighborhood frames?

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Incompleteness

Martin Gerson. *The Inadequacy of Neighbourhood Semantics for Modal Logic*. Journal of Symbolic Logic (1975).

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Presents two logics **L** and **L**$'$ that are incomplete with respect to neighborhood semantics.

(there are formulas φ and φ' that are valid in the class of frames for **L** and **L**$'$ respectively, but φ and φ' are not deducible in the respective logics).

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Presents two logics **L** and **L**′ that are incomplete with respect to neighborhood semantics.

L is between **T** and **S4**

L′ is above **S4** (adapts Fine's incomplete logic)

Comparing Relational and Neighborhood Semantics

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Fact: If a (normal) modal logic is complete with respect to some class of relational frames then it is complete with respect to some class of neighborhood frames.

What about the converse?

Are there normal modal logics that are incomplete with respect to relational semantics, but complete with respect to neighborhood semantics?

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- An extension of **S4**

  M. Gerson. *An Extension of S4 Complete for the Neighbourhood Semantics but Incomplete for the Relational Semantics*. Studia Logica (1975).

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Notable exceptions:

L. Chagrova. *On the Degree of Neighborhood Incompleteness of Normal Modal Logics*. AiML 1 (1998).

V. Shehtman. *On Strong Neighbourhood Completeness of Modal and Intermediate Propositional Logics (Part I)*. AiML 1 (1998).

T. Litak. *Modal Incompleteness Revisited*. Studia Logica (2004).

Thank You!