Neighborhood Semantics for Modal Logic

Lecture 1

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Lecture 1: Introduction, Motivation and Background Information

Lecture 2: Basic Concepts, Non-normal Modal Logics, Completeness, Decidability, Complexity, Incompleteness, Relation with Relational Semantics

Lecture 3: Advanced Topics — Topological Semantics for Modal Logic, some Model Theory

Lecture 4: Advanced Topics — Topological Semantics for Modal Logic, some Model Theory

Lecture 5: Neighborhood Semantics in Action: Game Logic, Coalgebra, Common Knowledge, First-Order Modal Logic

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Course Website

`staff.science.uva.nl/~epacuit/nbhd_esslli.html`

Reading Material

- ✓ *Modal Logic: an Introduction*, Chapters 7 - 9, by Brian Chellas
- ✓ *Monotonic Modal Logics* by Helle Hvid Hansen, available at `www.few.vu.nl/~hhansen/papers/scriptie_pic.pdf`
- ✓ The course reader (updated version available on the website)

Concerning Modal Logic

Modal Logic by P. Blackburn, M. de Rijke and Y. Venema.

- Background
- Introduction
- Motivating Examples
- A Primer on Modal Logic

The Basic Modal Language: $\mathcal{L}$

$$p \mid \neg\varphi \mid \varphi \wedge \psi \mid \Box\varphi \mid \Diamond\varphi$$

where p is an atomic proposition (At)

Kripke (Relational) Models

$$\mathbb{M} = \langle W, R, V \rangle$$

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$$\mathbb{M} = \langle W, R, V \rangle$$

- $W \neq \emptyset$
- $R \subseteq W \times W$
- $V : \mathsf{At} \rightarrow \wp(W)$

Truth in a Kripke Model

1. $\mathbb{M}, w \models p$ iff $w \in V(p)$

2. $\mathbb{M}, w \models \neg\varphi$ iff $\mathbb{M}, w \not\models \varphi$

3. $\mathbb{M}, w \models \varphi \wedge \psi$ iff $\mathbb{M}, w \models \varphi$ and $\mathbb{M}, w \models \psi$

4. $\mathbb{M}, w \models \Box\varphi$ iff for each $v \in W$, if wRv then $\mathbb{M}, v \models \varphi$

5. $\mathbb{M}, w \models \Diamond\varphi$ iff there is a $v \in W$ such that wRv and $\mathbb{M}, v \models \varphi$

Some Validities

(M) $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$

(C) $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$

(N) $\Box\top$

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

(Dual) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

(Nec) from $\vdash \varphi$ infer $\vdash \Box\varphi$

(Re) from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

Some Validities

(M) $\Box(\varphi \wedge \psi) \rightarrow \Box\varphi \wedge \Box\psi$

(Mon) $\dfrac{\vdash \varphi \rightarrow \psi}{\vdash \Box\varphi \rightarrow \Box\psi}$

(C) $\Box\varphi \wedge \Box\psi \rightarrow \Box(\varphi \wedge \psi)$

(N) $\Box\top$

(K) $\Box(\varphi \rightarrow \psi) \rightarrow (\Box\varphi \rightarrow \Box\psi)$

(Dual) $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$

(Nec) from $\vdash \varphi$ infer $\vdash \Box\varphi$

(Re) from $\vdash \varphi \leftrightarrow \psi$ infer $\vdash \Box\varphi \leftrightarrow \Box\psi$

Neighborhoods in Topology

In a topology, a *neighborhood* of a point x is any set A containing x such that you can "wiggle" x without leaving A.

A *neighborhood system* of a point x is the collection of neighborhoods of x.

J. Dugundji. *Topology*. 1966.

Neighborhoods in Modal Logic

Neighborhood Structure: $\langle W, N, V\rangle$

- $W \neq \emptyset$
- $N : W \rightarrow \wp(\wp(W))$
- $V : \mathsf{At} \rightarrow \wp(W)$

Some Notation

Given $\varphi \in \mathcal{L}$ and a model $\mathbb{M}$, the

- *proposition* expressed by φ
- *extension* of φ
- *truth set* of φ

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$$(\varphi)^{\mathbb{M}} = \{w \in W \mid \mathbb{M}, w \models \varphi\}$$

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

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What does it mean to be a neighborhood?

$w \models \Box\varphi$ if the truth set of φ is a neighborhood of w

neighborhood in some topology.

J. McKinsey and A. Tarski. *The Algebra of Topology*. 1944.

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contains all the immediate neighbors in some graph

S. Kripke. *A Semantic Analysis of Modal Logic*. 1963.

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contains all the immediate neighbors in some graph

S. Kripke. *A Semantic Analysis of Modal Logic*. 1963.

an element of some distinguished collection of sets

D. Scott. *Advice on Modal Logic*. 1970.

R. Montague. *Pragmatics*. 1968.

To see the necessity of the more general approach, we could consider probability operators, conditional necessity, or, to invoke an especially perspicuous example of Dana Scott, the present progressive tense....Thus N might receive the awkward reading 'it is being the case that', in the sense in which 'it is being the case that Jones leaves' is synonymous with 'Jones is leaving'.

(Montague, pg. 73)

R. Montague. *Pragmatics and Intentional Logic*. 1970.

Segerberg's Essay

K. Segerberg. *An Essay on Classical Modal Logic*. Uppsala Technical Report, 1970.

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K. Segerberg. *An Essay on Classical Modal Logic*. Uppsula Technical Report, 1970.

> *This essay purports to deal with classical modal logic. The qualification "classical" has not yet been given an established meaning in connection with modal logic....Clearly one would like to reserve the label "classical" for a category of modal logics which—if possible—is large enough to contain all or most of the systems which for historical or theoretical reasons have come to be regarded as important, and which also posses a high degree of naturalness and homogeneity.*
>
> (pg. 1)

R. Goldblatt. *Mathematical Modal Logic: A View of its Evolution*. Handbook of the History of Logic, 2005.

Logics of High Probability

$\Box\varphi$ means "*φ is assigned 'high' probability*", where *high* means above some threshold $r \in [0, 1]$.

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H. Kyburg and C.M. Teng. *The Logic of Risky Knowledge*. Proceedings of WoLLIC (2002).

A. Herzig. *Modal Probability, Belief, and Actions*. Fundamenta Informaticae (2003).

Games

$\Box_i\varphi$ means "*player i has a strategy that guarantees φ is true.*"

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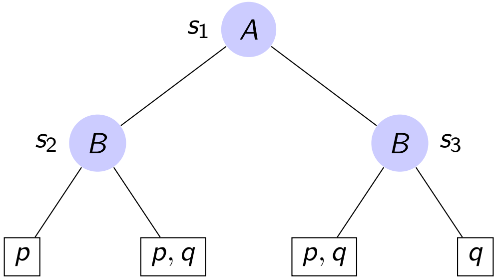

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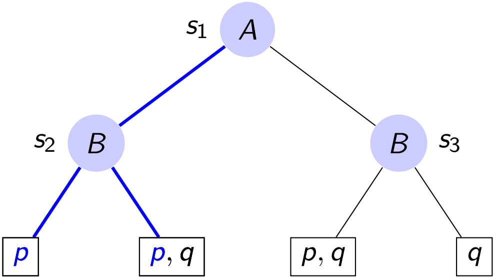

$$s_1 \models \Box_A p$$

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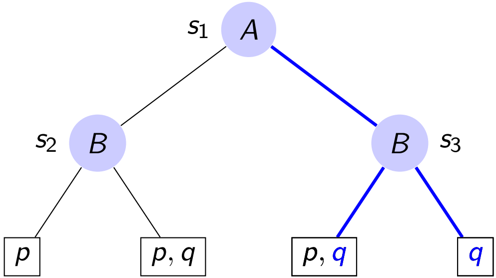

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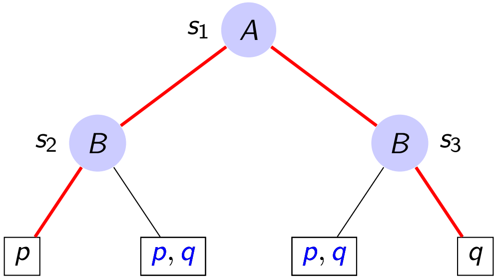

$$s_1 \models \Box_A p \wedge \Box_A q \wedge \neg\Box_A(p \wedge q)$$

Games

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R. Parikh. *The Logic of Games and its Applications*. Annals of Discrete Mathematics (1985).

M. Pauly and R. Parikh. *Game Logic — An Overview*. Studia Logica (2003).

J. van Benthem. *Logic and Games*. Course notes (2007).

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Let $\mathcal{L}_0 \subseteq \mathcal{L}$ be the set of propositional formulas.

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- $(\varphi \vee \psi)^* = (\varphi)^* \cup (\psi)^*$
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Fact: $\Box(\varphi \to \psi) \to \Box\varphi \to \Box\psi$ is not valid.

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Validities: $\varphi \to \Box\varphi$, (Mon), $\Box(\varphi \vee \Box\varphi) \to \Box\varphi$

P. Naumov. *On modal logic of deductive closure.* APAL (2005).

Deontic Logic

$\Box\varphi$ mean "*it is obliged that* φ."

1. Jones murders Smith
2. Jones ought not to murder Smith

J. Forrester. *Paradox of Gentle Murder*. 1984.

L. Goble. *Murder Most Gentle: The Paradox Deepens*. 1991.

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$\Box\varphi$ mean "*it is obliged that* φ."

1. Jones murders Smith
2. Jones ought not to murder Smith
3. If Jones murders Smith, then Jones ought to murder Smith gently

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$\Rightarrow$ If Jones murders Smith gently, then Jones murders Smith.

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7. Jones ought to murder Smith

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Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

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Note: the language is restricted so that $\Box\Box\alpha$ *is not a wff.*

Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

Consensus: α is accepted provided *everyone* accepts α.

(E) $\Box\alpha \leftrightarrow \Box\beta$ provided $\alpha \leftrightarrow \beta$ is a tautology

(M) $\Box(\alpha \wedge \beta) \rightarrow (\Box\alpha \wedge \Box\beta)$

(C) $(\Box\alpha \wedge \Box\beta) \rightarrow (\Box\alpha \wedge \Box\beta)$

(N) $\Box\top$

(D) $\neg\Box\bot$

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(D) $\neg\Box\bot$

Theorem The above axioms axiomatize consensus (provided $n \geq 2^{|\mathsf{At}|}$)).

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Majority: α is accepted if a *majority* of the agents accept α.

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(S) $\Box\alpha \rightarrow \neg\Box\neg\alpha$

(T) $([\geq]\varphi_1 \wedge \cdots \wedge [\geq]\varphi_k \wedge [\leq]\psi_1 \wedge \cdots \wedge [\leq]\psi_k) \rightarrow \bigwedge_{1\leq i\leq k}([=]\varphi_i \wedge [=]\psi_i)$ where $\forall v \in V_I$: $|\{i \mid v_(\varphi_i) = 1\}| = |\{i \mid v(\psi_i) = 1\}|$

Theorem The above axioms axiomatize majority.

Social Choice Theory

$\Box\alpha$ mean "*the group accepts* α."

M. Pauly. *Axiomatizing Collective Judgement Sets in a Minimal Logical Language*. 2006.

T. Daniëls. *Social Choice and Logic via Simple Games*. ILLC, Masters Thesis, 2007.

Other Examples

- Epistemic Logic: the logical omniscience problem.

  M. Vardi. *On Epistemic Logic and Logical Omniscience*. TARK (1986).

- Reasoning about coalitions

  M. Pauly. *Logic for Social Software*. Ph.D. Thesis, ILLC (2001).

- Knowledge Representation

  V. Padmanabhan, G. Governatori, K. Su . *Knowledge Assesment: A Modal Logic Approach*. KRAQ (2007).

- Program logics: modeling concurrent programs

  D. Peleg. *Concurrent Dynamic Logic*. J. ACM (1987).

- ??????

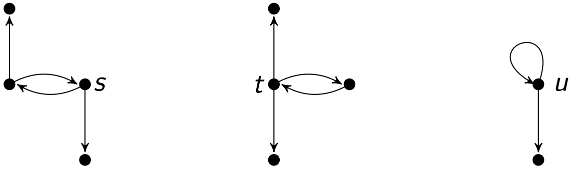
- Background
- Introduction
- Motivating Examples
- A Primer on Modal Logic

Slogan 1: Modal languages are simple yet expressive languages for talking about relational structures.

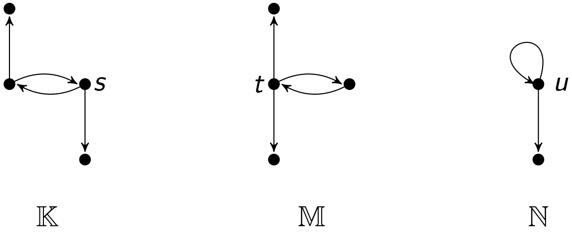
Slogan 2: Modal languages provide an internal, local perspective on relational structures.

P. Blackburn, M. de Rijke and Y. Venema. *Modal Logic*. 2001.

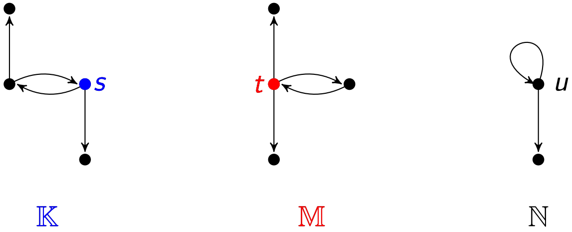
P. Blackburn and J. van Benthem. *Modal Logic: A Semantics Perspective*. Handbook of Modal Logic (2007).



$\mathbb{K}$ $\mathbb{M}$ $\mathbb{N}$



$$\Box(\Box\bot \vee \Diamond\Box\bot)$$



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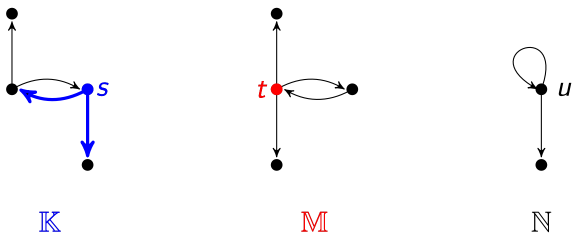

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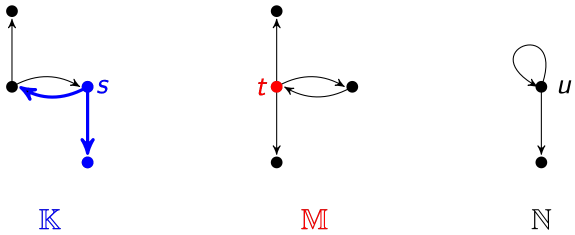

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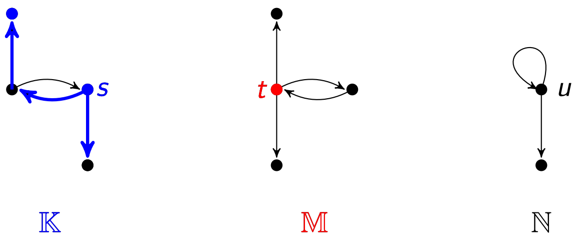

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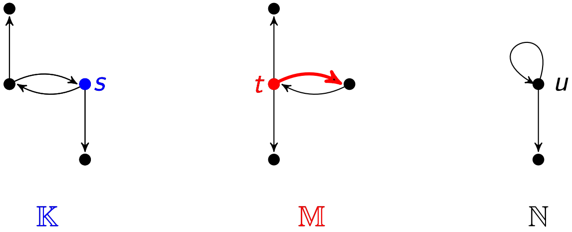

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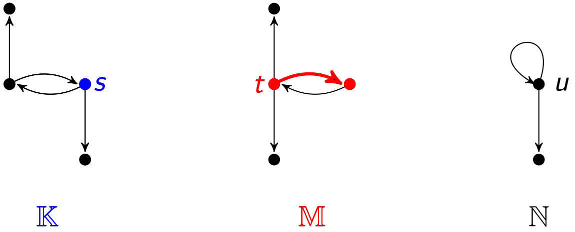

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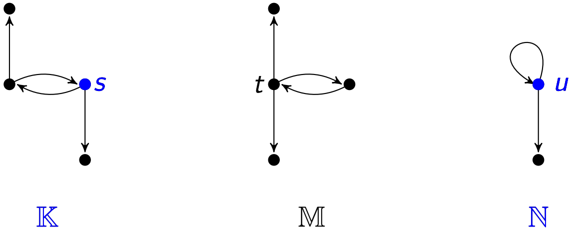
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Notation

A Kripke frame is a pair $\langle W, R\rangle$ where $R \subseteq W \times W$.

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Let $\mathbb{F} = \langle W, R \rangle$ be a Kripke frame and $\mathbb{M} = \langle W, R, V \rangle$ a model based on $\mathbb{M}$.

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φ is satisfiable in $\mathbb{M}$ if there exists $w \in W$ such that $\mathbb{M}, w \models \varphi$

Notation

A Kripke frame is a pair $\langle W, R \rangle$ where $R \subseteq W \times W$.

Let $\mathbb{F} = \langle W, R \rangle$ be a Kripke frame and $\mathbb{M} = \langle W, R, V \rangle$ a model based on $\mathbb{M}$.

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φ is valid on a frame $\mathbb{F}$ ($\mathbb{F} \models \varphi$) if for all models $\mathbb{M}$ based on $\mathbb{F}$, $\mathbb{M} \models \varphi$.

Definable Properties

A modal formula φ defines a class of frames K provided

$$\mathbb{F} \in \mathsf{K} \text{ iff } \mathbb{F} \models \varphi$$

- $\Box\varphi \to \Box\Box\varphi$ defines the class of transitive frames.
- $\varphi \leftrightarrow \Box\varphi$ defines the class of frames consisting of isolated reflexive points ($\forall x \in W,\ xRy \to x = y$).
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Some modal formulas correspond to genuine second-order properties: Löb ($\Box(\Box\varphi \rightarrow \varphi) \rightarrow \Box\varphi$), *McKinsey* ($\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$)

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The Sahlqvist Theorem gives an algorithm *for finding a first-order correspondant for certain modal formulas.*

Slogan 3: Modal logics are not isolated formal systems.

The Standard Translation

$$st_x : \mathcal{L} \to \mathcal{L}_1$$

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First-order language with an appropriate signature

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Fact: Modal logic falls in the two-variable fragment of $\mathcal{L}_1$.

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Lemma For each $w \in W$, $\mathbb{M}, w \models \varphi$ iff $\mathbb{M} \Vdash st_x(\varphi)[x/w]$.

What can we say with modal logic? What about in comparison with first-order logic?

Disjoint Union

Definition Let $\mathbb{M}_1 = \langle W_1, R_1, V_1 \rangle$ and $\mathbb{M}_2 = \langle W_2, R_2, V_2 \rangle$. The disjoint union is the structure $\mathbb{M}_1 \uplus \mathbb{M}_2 = \langle W, R, V \rangle$ where

- $W = W_1 \cup W_2$
- $R = R_1 \cup R_2$
- for all $p \in \mathsf{At}$, $V(p) = V_1(p) \cup V_2(p)$

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Lemma For each collection of Kripke structures $\{\mathbb{M}_i \mid i \in I\}$, for each $w \in W_i$, $\mathbb{M}_i, w \models \varphi$ iff $\biguplus_{i \in I} \mathbb{M}_i, w \models \varphi$

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Fact The universal modality is not definable in the basic modal language.

Generated Submodel

Definition $\mathbb{M}' = \langle W', R', V' \rangle$ is a generated submodel of $\mathbb{M} = \langle W, R, V \rangle$ provided

- $W' \subseteq W$ is R-closed:
  for each $w' \in W$ and $v \in W$, if wRv then $v \in W'$.
- $R' = R \cap W' \times W'$
- for all $p \in \mathrm{At}$, $V'(p) = V(p) \cap W'$

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Lemma If $\mathbb{M}'$ is a generated submodel of $\mathbb{M}$ then for each $w \in W'$, $\mathbb{M}', w \models \varphi$ iff $\mathbb{M}, w \models \varphi$

Generated Submodel

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Fact The backwards looking modality is not definable in the basic modal language.

Bounded Morphism

Definition A bounded morphism between models $\mathbb{M} = \langle W, R, V\rangle$ and $\mathbb{M}' = \langle W', R', V'\rangle$ is a function f with domain W and range W' such that:

Atomic harmony: for each $p \in \mathrm{At}$, $w \in V(p)$ iff $f(w) \in V'(p)$

Morphism: if wRv then $f(w)Rf(v)$

Zag: if $f(w)R'v'$ then $\exists v \in W$ such that $f(v) = v'$ and wRv

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Lemma If $\mathbb{M}'$ is a bounded morphic image of $\mathbb{M}$ then for each $w \in W$, $\mathbb{M}, w \models \varphi$ iff $\mathbb{M}', f(w) \models \varphi$

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Fact Counting modalities are not definable in the basic modal language (eg., $\Diamond_1\varphi$ iff φ is true in more than 1 accessible world).

Bisimulation

A bisimulation between $\mathbb{M} = \langle W, R, V\rangle$ and $\mathbb{M}' = \langle W', R', V'\rangle$ is a non-empty binary relation $Z \subseteq W \times W'$ such that whenever wZw':

Atomic harmony: for each $p \in \mathsf{At}$, $w \in V(p)$ iff $w' \in V'(p)$

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We write $\mathbb{M}, w \underline{\leftrightarrow} \mathbb{M}', w'$ if there is a Z such that wZw'.

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We write $\mathbb{M}, w \leftrightsquigarrow \mathbb{M}', w'$ *iff* $\forall \varphi \in \mathcal{L}$, $\mathbb{M}, w \models \varphi$ *iff* $\mathbb{M}', w' \models \varphi$.

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Lemma On finite frames, if $\mathbb{M}, w \leftrightsquigarrow \mathbb{M}', w'$ then $\mathbb{M}, w \,\underline{\leftrightarrow}\, \mathbb{M}', w'$.

The Van Benthem Characterization Theorem

Modal logic is the bisimulation invariant fragment of first-order logic.

The Van Benthem Characterization Theorem

For any first-order formula $\varphi(x)$, TFAE:

1. $\varphi(x)$ is invariant for bisimulation
2. $\varphi(x)$ is equivalent to the standard translation of a basic modal formula.

The Goldblatt-Thomason Theorem

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The Goldblatt-Thomason Theorem

An elementary class of frames K is modally definiable iff it is closed under disjoint unions, bounded morphic images, generated subframes, and reflects ultrafilter extensions.

End of lecture 1.