Cake Cutting Algorithms

Eric Pacuit

January 7, 2007

ILLC, University of Amsterdam

staff.science.uva.nl/∼epacuit

epacuit@science.uva.nl

Plan for Today

Discuss some fair division algorithms

- What does it mean to “fairly” divide goods?
- Indivisible Goods
- Divisible Goods (Cutting a Cake)
  - Divide and Choose
  - Surplus Procedure
  - Banach-Knaster Last Diminisher
  - Dubins-Spanier Moving Knife Procedure

Main Question

How do we cut a cake fairly?

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- any desirable set of *goods* (or chores or mixtures)
- each may be *divisible* or *indivisible*
- there may be *restrictions* (such as the number of goods a player may receive)

Main Question

How do we cut a cake fairly?

- discrete procedure
- continuous moving knife procedures
- compensation procedures (using money as a divisible medium for indivisible objects)

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How do we cut a cake fairly?

- Interested not only in the *existence* of a (fair) division but also a *constructive procedure* (an algorithm) for finding it

Main Question

How do we cut a cake fairly?

- Different results known for 2,3,4,... cutters!

Main Question

How do we cut a cake fairly?

- Many ways to make this precise!

Fairness Conditions

- **Proportional:** (for two players) each player receives at least 50% of their valuation.

- **Envy-Free:** no party is willing to give up its allocation in exchange for the other player's allocation, so no players envies anyone else.

- **Equitable:** each player values its allocation the same *according to its own valuation function.*

- **Efficiency:** there is no other division better for everybody, or better for some players and not worse for the others

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Truthfulness

Some procedures ask players to represent their preferences.

This representation need not be "truthful"

Typically, it is assumed that agents will follow a maximin strategy (maximize the set of items that are guaranteed)

Main References

S. Brams and A. Taylor. *Fair Division: From Cake-Cutting to Dispute Resolution.* 1996.

J. Robertson and W. Webb. *Cake-Cutting Algorithms: Be Fair If You Can.* 1998.

J. Barbanel. *The Geometry of Efficient Fair Division.* 2005.

Indivisible Goods

S. Brams, P. Edelman and P. Fishburn. *Paradoxes of Fair Division.* *Journal of Philosophy*, **98:6** (2001).

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- Players cannot compensate each other with side payments
- All players have positive values for every item
- **Lift Preferences to Sets:** A player prefers a set S to a set T if
  - S has as many elements as T
  - for every item in $t \in T - S$ there is a distinct item $s \in S - T$ that the player prefers to t.

Indivisible Goods: Envy-Freeness and Efficiency

A unique envy-free division may be inefficient

$$\begin{array}{llllll l} A: & 1 & 2 & 3 & 4 & 5 & 6 \\ B: & 4 & 3 & 2 & 1 & 5 & 6 \\ C: & 5 & 1 & 2 & 6 & 3 & 4 \end{array}$$

$$A : \{1, 3\}$$

$$B : \{2, 4\}$$

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This is the unique *envy-free* outcome.

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However, $(12, 34, 56)$ is **not** (necessarily) envy-free

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There is no other division, including an efficient one, that guarantees envy-freeness.

Indivisible Goods: Envy-Freeness and Efficiency

There may be no envy-free division, even when all players have different preference rankings

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Trivial if all players have the same preference.

Indivisible Goods: Envy-Freeness and Efficiency

There may be no envy-free division, even when all players have different preference rankings

$$\begin{array}{llll} A: & 1 & 2 & 3 \\ B: & 1 & 3 & 2 \\ C: & 2 & 1 & 3 \end{array}$$

Three divisions are efficient: $(1,3,2)$, $(2,1,3)$ and $(3,1,2)$.
However, none of them are envy-free.

Indivisible Goods: Envy-Freeness and Efficiency

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However, none of them are envy-free.

In fact, there is **no** envy-free division.

2 Players, 1 Cake

Two players A and B

The cake is the unit interval $[0, 1]$

Only parallel, vertical cuts, perpendicular to the horizontal x-axis are made

2 Players, 1 Cake

Each player has a continuous value measure $v_A(x)$ and $v_B(x)$ on $[0,1]$ such that

- $v_A(x) \geq 0$ and $v_B(x) \geq 0$ for $x \in [0,1]$
- v_A and v_B are finitely additive, non-atomic, absolutely continuous measures
- the areas under v_A and v_B on $[0,1]$ is 1 (probability density function)

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value of finite number of disjoint pieces equals the value of their union (hence, no subpieces have greater value than the larger piece containing them).

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a single cut (which defines the border of a piece) has no area and so has no value.

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no portion of cake is of positive measure for one player and zero measure for another player.

Cutting a Cake: Divide and Choose

Procedure: one player cuts the cake into two portions and the other player chooses one.

Suppose A is the cutter.

If A has no information about the other player's preferences, then A should cut the cake at some point x so that the value of the portion to the left of x is equal to the value of the portion to the right.

This strategy creates an **envy-free** and **efficient** allocation, but it is not necessarily **equitable**.

Cutting a Cake: Divide and Choose

Suppose A values the vanilla half twice as much as the chocolate half. Hence,

$$v_A(x) = \begin{cases} 4/3 & x \in [0, 1/2] \\ 2/3 & x \in (1/2, 1] \end{cases}$$

$$v_B(x) = \begin{cases} 1/2 & x \in [0, 1/2] \\ 1/2 & x \in (1/2, 1] \end{cases}$$

A should cut the cake at $x = 3/8$:

$$(4/3)(x - 0) = 4/3(1/2 - x) + 2/3(1 - 1/2)$$

Note that the portions are not equitable (B receive $5/8$ according to his valuation)

Surplus Procedure

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1. Independently, A and B report their value functions f_A and f_B over $[0,1]$ to a referee. These need not be the same as v_A and v_B.

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3. If a and b coincide, the cake is cut at $a = b$. One player is randomly assigned the piece to the left and the other to the right. The procedure ends.

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4. Suppose a is to the left of b (Then A receives $[0, a]$ and B receives $[b, 1]$). Cut the cake a point c in $[a, b]$ at which the players receive the *same proportion* p of the cake in this interval.

Surplus Procedure

A procedure is **strategy-proof** if maximin players always have an incentive to let $f_A = v_A$ and $f_B = v_B$.

Let c be the cut-point that guarantees proportional equitability and e the cut-point that guarantees equitability of the surplus.

Theorem The Surplus Procedure is strategy-proof, whereas any procedure that makes e the cut-point is strategy-vulnerable.

3 Players, 2 Cuts

Fact It is not always possible to divide a cake among three players into **envy-free and equitable** portions using 2 cuts.

More than 2 Players

A division is **super-envy free** if every player feels all other players received strictly less that $1/n$ of the total value of the cake.

Theorem (Barbenel) A super envy-free division *exists* if and only if the player measures are linearly independent. (in fact, there are infinitely many such divisions)

J. Barbanel. *Super envy-free cake division and independence of measures.* J. Math. Anal. Appl. (1996).

Banach-Knaster Last Diminisher Procedure

Suppose there are n different agents: $p_1, \ldots, p_n$.

Procedure:

- The first person (p_1) cuts out a piece which he claims is his fair share.

- Then, the piece goes around being inspected, in turn, by persons $p_2, p_3, \ldots, p_n$.

  – Anyone who thinks the piece is not too large just passes it. Anyone who thinks it is too big, may reduce it, putting some back into the main part.

Banach-Knaster Last Diminisher Procedure

- After the piece has been inspected by p_n, the last person who reduced the piece, takes it. If there is no such person, i.e., no one challenged p_1 , then the piece is taken by p_1.

- The algorithm continues with $n - 1$ participants.

This procedure is equitable but not envy-free

Dubins-Spanier Moving-Knife Procedure

Procedure: A referee holds a knife at the left edge of the cake and slowly moves it across the cake so that it remains parallel to its starting position.

At any time, any one of the three players (A, B or C) can call “cut”.

When this occurs, the player who called cut receives the piece to the left of the knife and exits the game.

Dubins-Spanier Moving-Knife Procedure

The game now continues moving until a second player calls cut.

The second player receives the second piece and the third player gets the remainder.

If either two or three players call cut at the same time, the cut piece is given to one of the callers at random.

This procedure is equitable but not envy-free

Open Questions

- 3-person, 2-cut envy-free procedures have been found (Stromquist, 1980; Barbanel and Brams, 2004)

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  - (Barbanel and Brams, 2004): no more than 5 cuts are needed to ensure 4-person envy-freeness.

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- 4-person, 3-cut envy free procedure? (Unknown)
  - (Barbanel and Brams, 2004): no more than 5 cuts are needed to ensure 4-person envy-freeness.

- Beyond 4 players, *no procedure is known* that yields an envy-free division of a cake unless an *unbounded* number of cuts is allowed (Brams and Taylor, 1995)

How about some pie?

A cake is a line segment and becomes a pie when its endpoints are connected to form a circle.

The cuts divide the pie into sectors each one of which is given to a player

Gale (1993): Is there an allocation of the pie that is envy-free and undominated?

Barabanel and Brams: for 2 players yes, for 3 players envy-free but not necessarily undominated, for 4 players no.

J. Barbanel and S. Brams. *Cutting a Pie Is Not a Piece of Cake.* 2005.

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S. Brams, M. Jones and C. Klamler. *Better Ways to Cut a Cake.* Notices of the AMS (2006).

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