Interpolation, Growth Conditions, and Stochastic Gradient Descent

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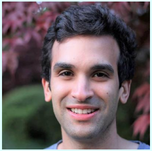

"Stochastic gradient descent (SGD) is today one of the main workhorses for solving large-scale supervised learning and optimization problems."
—Drori and Shamir [2019]

. . . and also Agarwal et al. [2017], Assran and Rabbat [2020], Assran et al. [2018], Bernstein et al. [2018], Damaskinos et al. [2019], Geffner and Domke [2019], Gower et al. [2019], Grosse and Salakhudinov [2015], Hofmann et al. [2015], Kawaguchi and Lu [2020], Li et al. [2019], Patterson and Gibson [2017], Pillaud-Vivien et al. [2018], Xu et al. [2017], Zhang et al. [2016]

Challenges in Optimization for ML

Stochastic gradient methods are the most popular algorithms for fitting ML models,

$$\textbf{SGD:} \quad w_{k+1} = w_k - \eta_k \nabla f_i\left(w_k\right).$$

But practitioners face major challenges with

- **Speed**: step-size/averaging controls convergence rate.
- **Stability**: hyper-parameters must be tuned carefully.
- **Generalization**: optimizers encode statistical tradeoffs.

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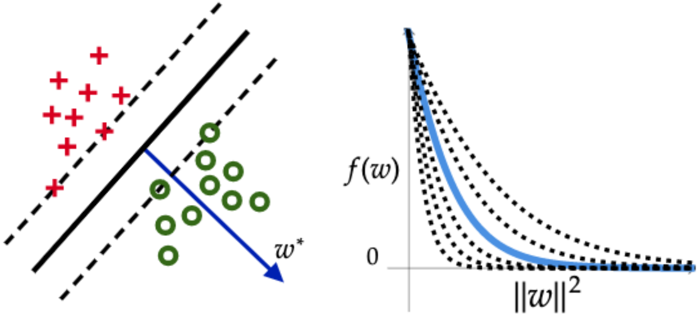
Better Optimization via Better Models



Idea: exploit "over-parameterization" for better optimization.

- Intuitively, gradient noise goes to 0 if all data are fit exactly.
- No need for decreasing step-sizes or averaging.

Assumptions for Optimization

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- (Sometimes) f is μ-**strongly-convex**: $\exists\, \mu > 0$ such that,

$$f(u) \ge f(w) + \langle \nabla f(w), u - w\rangle + \frac{\mu}{2}\|u - w\|_2^2 \quad \forall w, u \in \mathbb{R}^d.$$

Interpolation and Growth Conditions

Stochastic First-Order Oracles

Stochastic Oracles:

1. At each iteration k, query oracle $\mathcal{O}$ for stochastic estimates

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3. $\mathcal{O}$ is **unbiased**, meaning

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4. $\mathcal{O}$ is **individually-smooth**, meaning $f(\cdot, z_k)$ is L_{max}-smooth,

$$\|\nabla f(w, z_k) - \nabla f(u, z_k)\|_2 \leq L_{\text{max}} \|w - u\|_2 \quad \forall w, u \in \mathbb{R}^d,$$

almost surely.

Interpolation as a Property of Oracles

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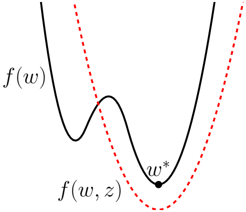

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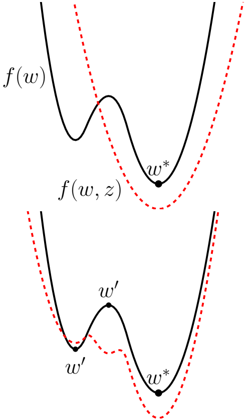

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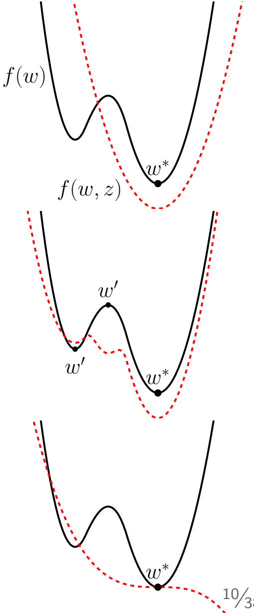

Growth Conditions: Strong and Weak Growth

Growth conditions control **global behavior** of oracles.

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Strong Growth+Noise : $\mathbb{E}\left[\|\nabla f(w, z_k)\|^2\right] \leq \rho \|\nabla f(w)\|^2 + \sigma^2.$

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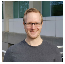


Strong Growth : $\mathbb{E}\left[\|\nabla f(w, z_k)\|^2\right] \leq \rho \|\nabla f(w)\|^2.$

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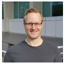


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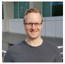

Weak Growth : $\mathbb{E}\left[\|\nabla f(w, z_k)\|^2\right] \leq \alpha \left(f(w) - f(w^*)\right).$

- Implies **mixed** interpolation.

Growth Conditions: Interpolation + Smoothness

Smoothness relates local and global behavior.

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Lemma (Interpolation and Weak Growth)

If minimizer interpolation holds, then weak growth also holds with

$$\alpha \leq \frac{L_{max}}{L}.$$

Lemma (Interpolation and Strong Growth)

Assume f is μ strongly-convex. If minimizer interpolation holds, then strong growth also holds with

$$\rho \leq \frac{L_{max}}{\mu}.$$

Stochastic Gradient Descent

Fixed Step-size SGD

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0. Choose an initial point $w_0 \in \mathbb{R}^d$.

1. For each iteration $k \geq 0$:
    1.1 Query $\mathcal{O}$ for $\nabla f(w_k, z_k)$.
    1.2 Update input as

$$w_{k+1} = w_k - \eta \nabla f(w_k, z_k).$$

Fixed Step-size SGD

Prior work for SGD under growth conditions or interpolation:

- Convergence under strong growth [Cevher and Vu, 2019, Schmidt and Le Roux, 2013, Solodov, 1998, Tseng, 1998].
- Convergence under weak growth [Vaswani et al., 2019a].
- Convergence under interpolation [Bassily et al., 2018].

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We provide many improved results:

- **Bigger** step-sizes and **faster** rates for convex and strongly-convex objectives.
- **Almost-sure** convergence under weak/strong growth.
- **Trade-offs** between growth conditions and interpolation.

Convergence for Fixed Step-size SGD

Theorem (Convex + Interpolation)

Assume f is convex and minimizer interpolation holds. Then SGD with $\eta = \frac{1}{L_{max}}$ converges as

$$\mathbb{E}\left[f(\bar{w}_K)\right] - f(w^*) \leq \frac{L_{max}}{2\,K}\|w_0 - w^*\|^2.$$

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- Improves over worst-case rate with weak growth.
- If $L_{\max} = L$, then guarantee is tight with deterministic GD!
- Otherwise, stochasticity worsens conditioning of problem.

Weakness of Fixed Step-size SGD

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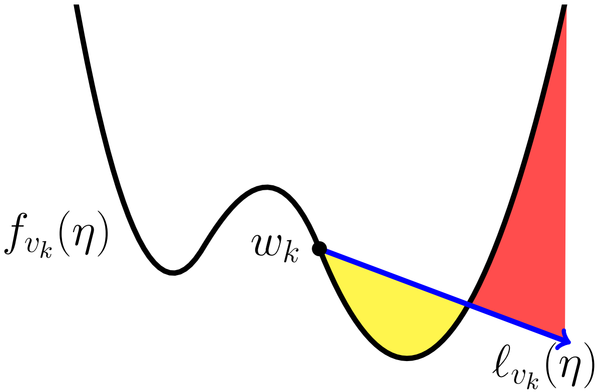
Is **grid-search** really the best way to pick η?

```python

for i, step_size in enumerate(np.logspace(-4,1,12)):
    opt_params["step_size"] = step_size
    results[i] = run_experiment(opt_params, exp_params, data_params, model_fn,
                                objective, error_fn, training_set, test_set)

```

SGD: the Armijo Line-search

The **Armijo line-search** is a classic solution to step-size selection.

$$f(\underbrace{w_k - \eta_k \nabla f(w_k)}_{w_{k+1}}) \leq f(w_k) - c \cdot \eta_k \|\nabla f(w_k)\|^2.$$



SGD with Armijo Line-search: Procedure

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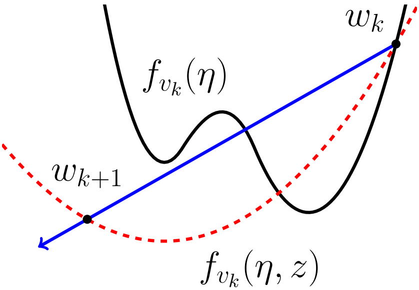
0. Choose an initial point $w_0 \in \mathbb{R}^d$.
1. For each iteration k:
   1.1 Query $\mathcal{O}$ for $f(w_k, z_k)$, $\nabla f(w_k, z_k)$.
   1.2 Set $w_{k+1} \leftarrow w_k - \eta_k \nabla f(w_k, z_k)$.
   1.3 Backtrack (decrease η_k) until

$$f(w_{k+1}, z_k) \leq f(w_k, z_k) - c \cdot \eta_k \|\nabla f(w_k, z_k)\|^2.$$

Note: Evaluates Armijo condition on $f(\cdot, z_k)$ instead of f and needs direct access to $f(\cdot, z_k)$ to backtrack.

SGD with Armijo Line-search: Visualization



No Interpolation

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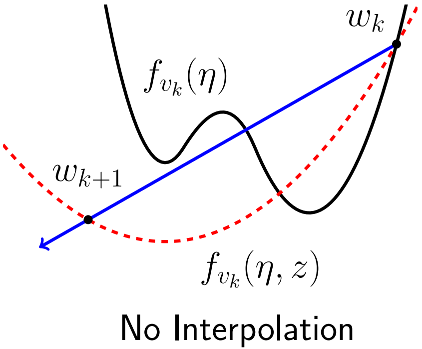


SGD with Armijo Line-search: Key Lemma

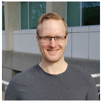
Lemma (Step-size Bound)

Assume minimizer interpolation holds.

Then the **maximal** *step-size satisfying the stochastic Armijo condition satisfies the following:*

$$\frac{2(1-c)}{L_{max}} \leq \eta_{max} \leq \frac{f(w_k, z_k) - f(w^*, z_k)}{c\|\nabla f(w_k, z_k)\|^2}.$$




Comments:

- Mirrors classic result in deterministic optimization.
- Easy to relax to a backtracking line-search.

SGD with Armijo Line-Search: Lemma Geometry

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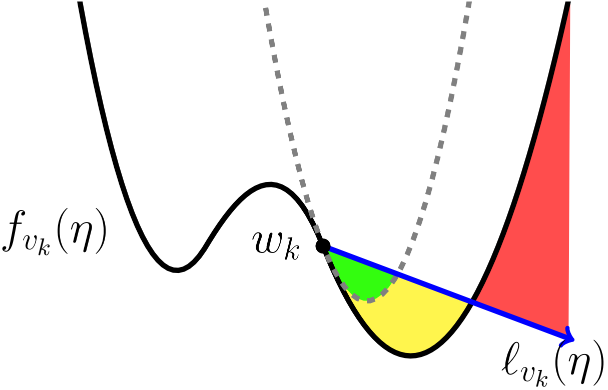

SGD with Armijo Line-search: Convergence

Theorem (Convex + Interpolation)

Assume minimizer interpolation holds and $f(\cdot, z)$ is convex. Then SGD with the Armijo line-search and $c = 1/2$ converges as

$$\mathbb{E}\left[f(\bar{w}_K)\right] - f(w^*) \leq \frac{L_{max}}{2\,K}\|w_0 - w^*\|^2.$$

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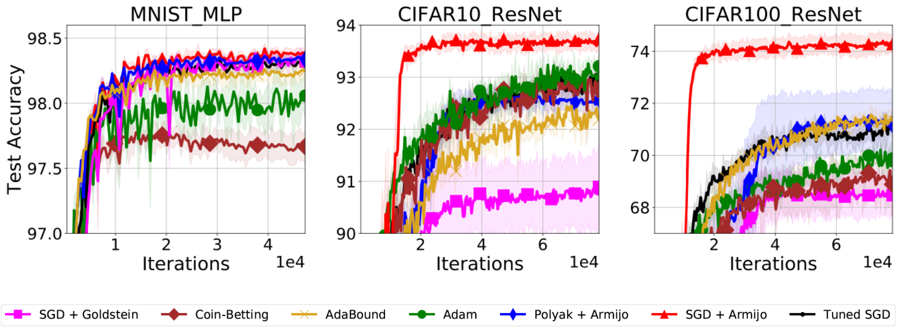
- Using the Armijo line-search is (nearly) parameter-free and recovers the deterministic rate when $L_{\text{max}} = L$.
- **Strongly-convex** f: we improve rate from $\bar{\mu}$ to μ.

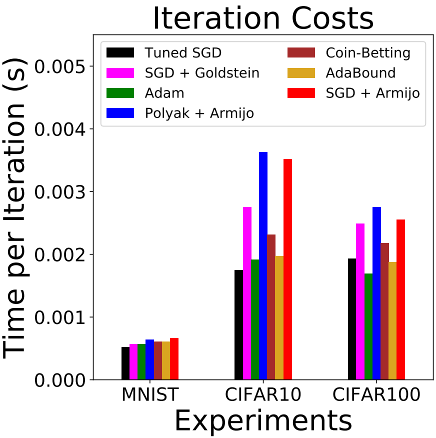
Painless SGD: Stochastic Armijo in Practice

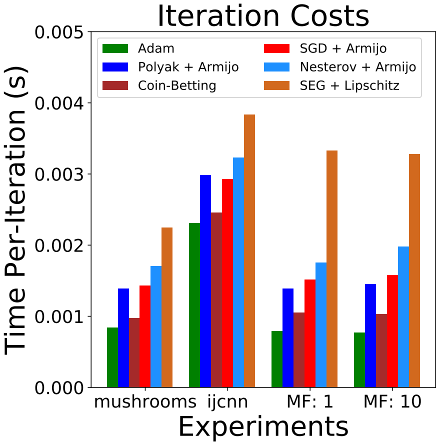
Classification accuracy for ResNet-34 models trained on MNIST, CIFAR-10, and CIFAR-100.



Painless SGD: Added Cost of Backtracking

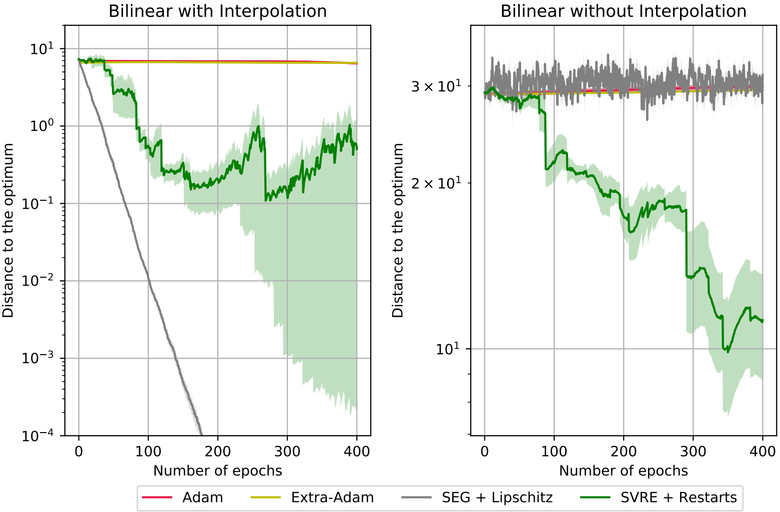
Backtracking is low-cost and averages once per-iteration.





Painless SGD: Sensitivity to Assumptions

SGD with line-search is **robust**, but can still fail catastrophically.



Acceleration

Stochastic Acceleration

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We follow Vaswani et al. [2019a] and close the convergence gap.

Stochastic Acceleration: Main Result

Strong growth implies a modified **descent lemma**,

$$\mathbb{E}_{z_k}[f(w_{k+1})] - f(w_k) \leq \eta \left(1 - \frac{\rho L \eta}{2}\right) \|\nabla f(w_k)\|_2^2.$$

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Theorem (Acceleration)

Assume f is strongly convex and strong growth holds. Then stochastic acceleration with step-size $\eta = 1/\rho L$ converges as

$$\mathbb{E}\left[f(w_K)\right] - f(w^*) \leq \left(1 - \sqrt{\frac{\mu}{\rho L}}\right)^K \left(f(w_0) - f(w^*) + \frac{\mu}{2}\delta_0\right).$$

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- Improves dependence from ρ to $\sqrt{\rho}$
  - Recall: $\sqrt{\rho} = \sqrt{\kappa_{\text{max}}} = \sqrt{L_{\text{max}}/\mu}$ in the worst case.

Takeaways

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- **Growth Conditions**: "smooth" oracles satisfying interpolation are well-behaved globally.
- **SGD**: improved rates show SGD under interpolation is tight with the deterministic setting.
- **Line-Search**: the Armijo line-search yields fast, parameter-free optimization under interpolation.

Takeaways

- **Interpolation**: the oracle model extends interpolation to general stochastic optimization problems.
- **Growth Conditions**: "smooth" oracles satisfying interpolation are well-behaved globally.
- **SGD**: improved rates show SGD under interpolation is tight with the deterministic setting.
- **Line-Search**: the Armijo line-search yields fast, parameter-free optimization under interpolation.
- **Acceleration**: stochastic acceleration is possible with a penalty of only $\sqrt{\rho}$.

Thanks for Listening!

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