Convex Analysis of Non-Convex Neural Networks

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Supervised by Mert Pilanci

Stanford University

1. Motivation: Convexity and Deep Learning

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Today, models get 99.02% top-5 accuracy [Yua+21]!

(Using all sorts of tricks like pre-training, transformers, etc.)

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Generated by DALL·E 2

A bowl of soup that is a portal to another dimension as digital art.

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DALL·E 2 has 5.5 billion parameters and took billions of Adam iterations to fit [Ram+22].

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- At \$1 per A100 hour, GPT-4 cost $\approx \$63$ million dollars.

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Takeaway: Modern deep learning models are huge and extremely expensive to train, but they have tremendous impact.

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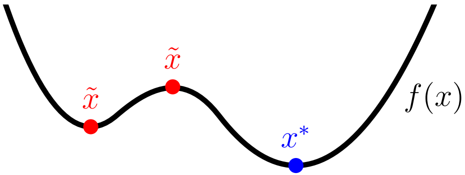

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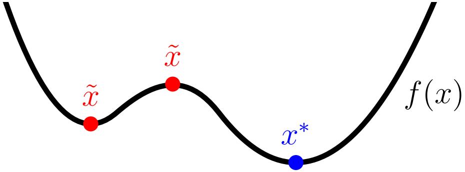

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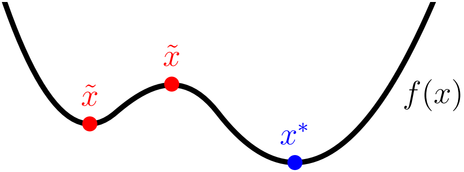

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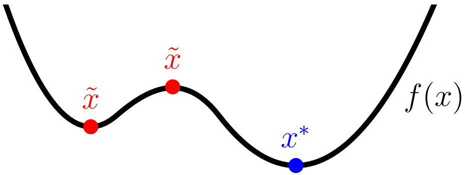

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- Optimization: saddle-points, local minima, slow convergence.
- Optimality Conditions: stationarity $\not\Rightarrow$ optimality.
- Mathematical Tools: No subgradients, no separating hyperplanes, (usually) non-zero duality gap.

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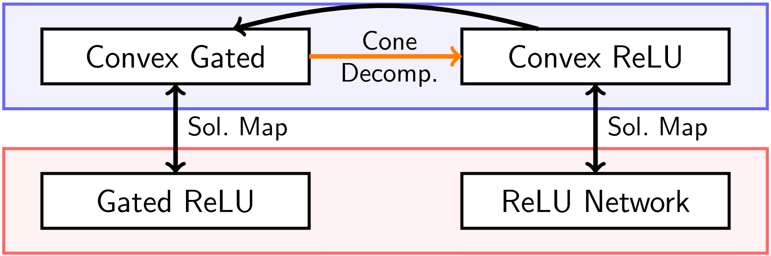

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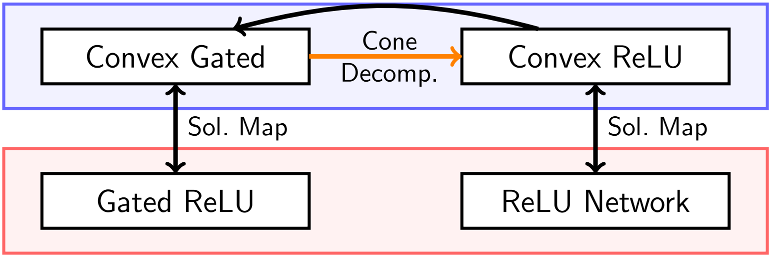

$\hookrightarrow$ This approach yields new algorithms and new insights!

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More Importantly: What can we achieve with two-layers?

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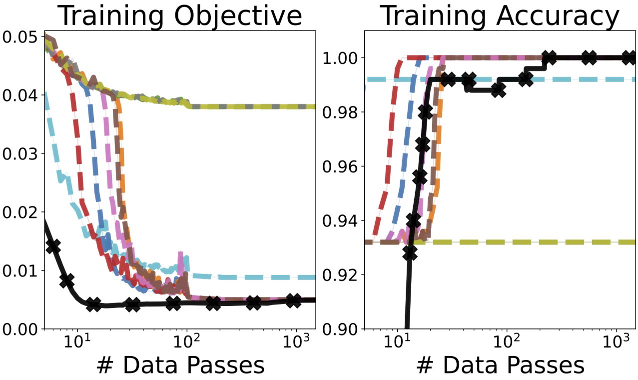

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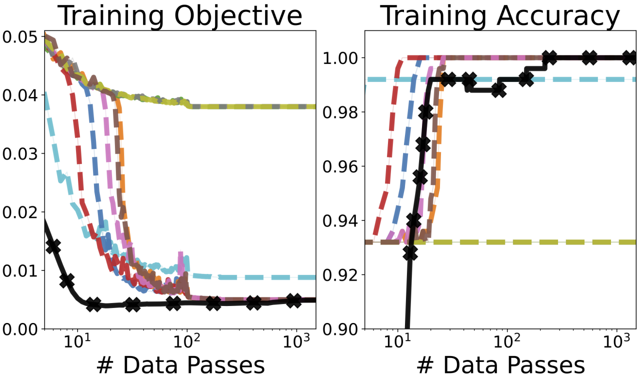

Convex reformulations enable parameter-free global optimization!

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Let's look at some examples...

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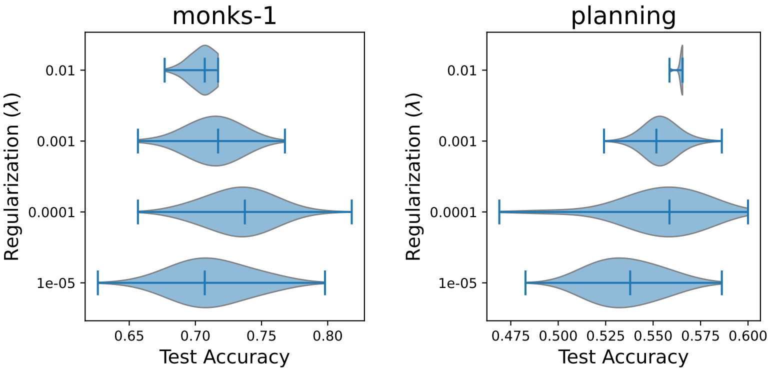

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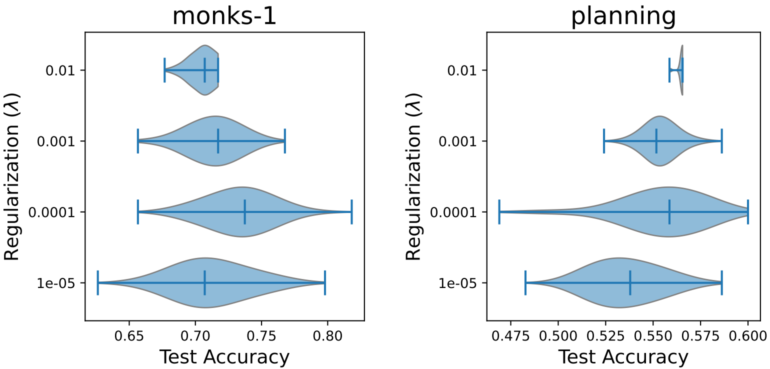

Conclusion: We need to distinguish between global optima!

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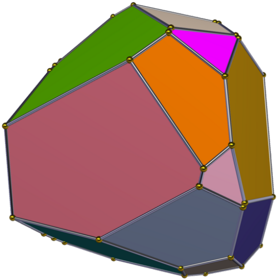

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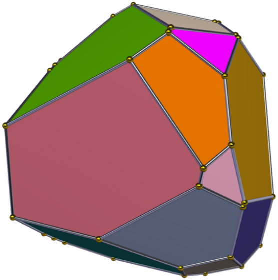

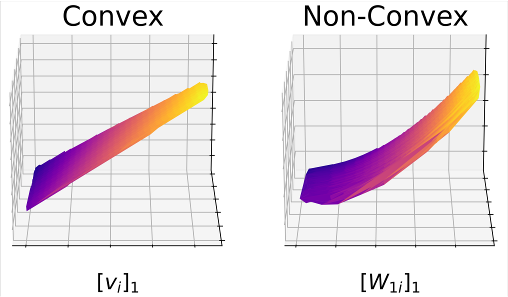

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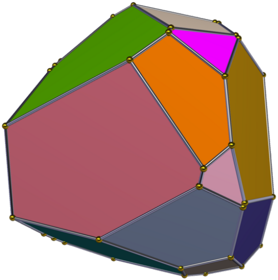

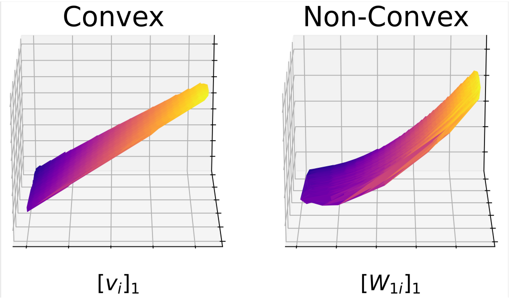

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↪ A connectivity hierarchy emerges with network width!

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- a variational theory for the optimal set, solution path, and stability of shallow neural network optimization;
- extensions to deep, fully-connected ReLU networks.

2. Background on Convex Reformulations

Convex Reformulations: Flavor of Results

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Equivalent means:

- The global minima have the same values: $p^* = q^*$
- We can map every global minimum u^* for one problem into a global minimum v^* of the other.

Convex Reformulations: Two-Layer ReLU Networks

Non-Convex Problem (NC-ReLU)

$$\min_{W_1, w_2} \underbrace{\frac{1}{2}\| \sum_{j=1}^{m} (XW_{1j})_+ w_{2j} - y\|_2^2}_{\text{Squared Error}} + \underbrace{\lambda \sum_{j=1}^{m} \|W_{1j}\|_2^2 + |w_{2j}|^2}_{\text{Weight Decay}},$$

where $(z)_+ = \max\{z, 0\}$, $X \in \mathbb{R}^{n \times d}$, and $y \in \mathbb{R}^n$.

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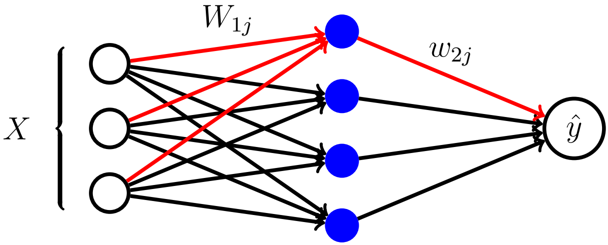

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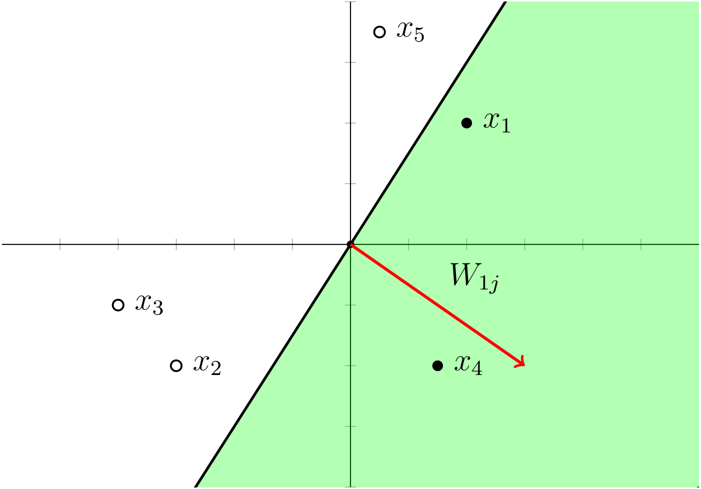

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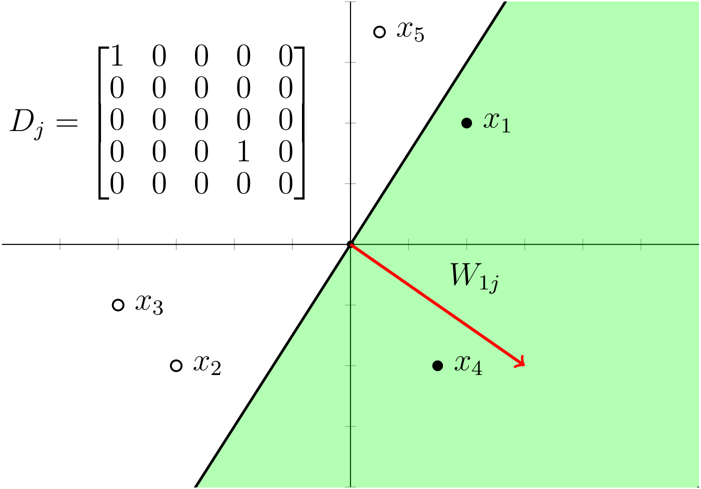

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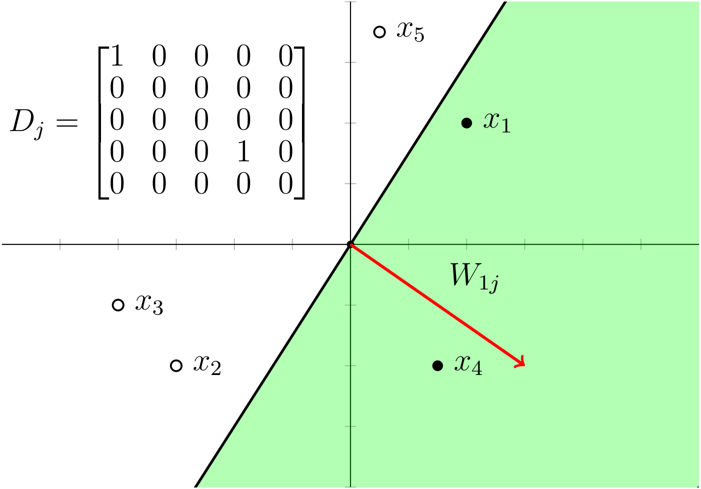

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Activation patterns linearize the ReLU: $(XW_{1j})_+ = D_j X W_{1j}$.

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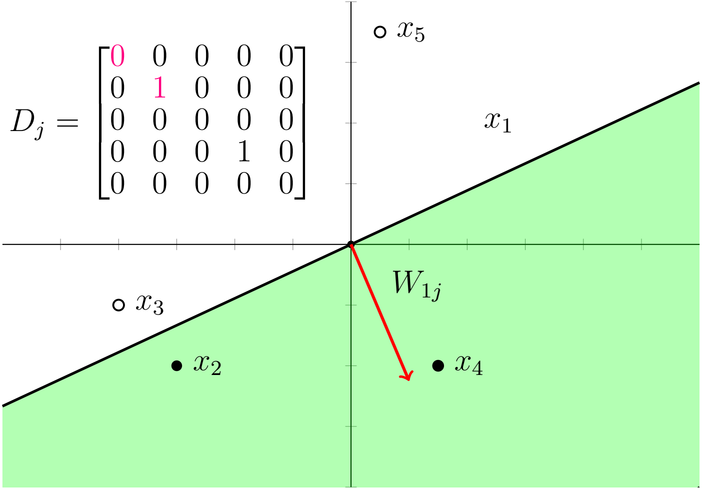
$$D_j = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$



Activation patterns linearize the ReLU: $(XW_{1j})_+ = D_j X W_{1j}$.

Convex Reformulations: Convex Problem

Convex Reformulation (C-ReLU) [PE20]

$$\min_{v,w} \frac{1}{2} \| \sum_{j=1}^{p} D_j X (v_j - w_j) - y \|_2^2 + \lambda \sum_{j=1}^{p} \|v_j\|_2 + \|w_j\|_2$$

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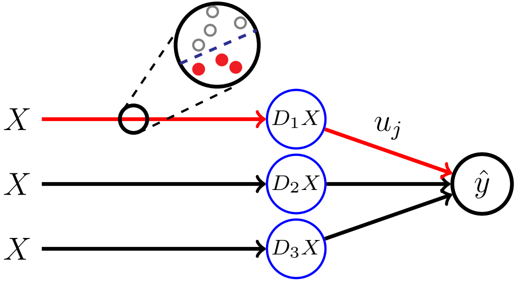

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Key Result: if network width m satisfies $m \geq m^*$ for some $m^* \leq n$, then C-ReLU and NC-ReLU are equivalent [PE20].

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Takeaway: We exchange one kind of hardness for another.

3. Better Optimization via Convex Reformulations

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[MSP22] Fast Convex Optimization for Two-Layer ReLU Networks.
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- We could solve C-ReLU using interior point methods, but computing the Hessian is infeasible for large n and d.
- We could use projected GD, but projecting onto $\mathcal{K}_i$ is an expensive quadratic program.
- Instead, we develop fast solvers based on the augmented Lagrangian method and on a "gated ReLU" relaxation.

Fast Training: Gated ReLU Networks

Recall the convex reformulation for a two-layer ReLU Network:

$$\textbf{C-ReLU}: \min_{u} \| \sum_{j=1}^{p} D_j X(v_j - w_j) - y\|_2^2 + \lambda \sum_{j=1}^{p} \|v_j\|_2 + \|w_j\|_2$$

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Theorem 3.7 [MSP22]: Let $\lambda \geq 0$ and let p^* be the optimal value of the ReLU problem. There exists a C-GReLU problem with minimizer u^* and optimal value d^* satisfying,

$$d^* \leq p^* \leq d^* + 2\lambda\kappa(\tilde{X}_{\mathcal{J}}) \sum_{D_i \in \tilde{\mathcal{D}}} \|u_i^*\|_2.$$

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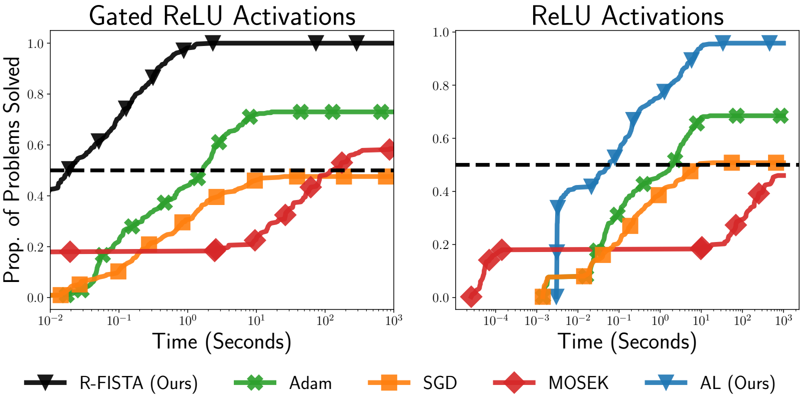

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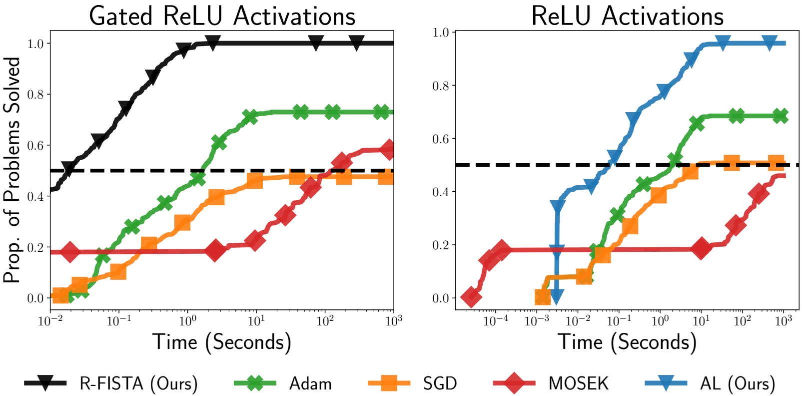

- R-FISTA/AL solve more, faster, than SGD and Adam.

4. Convex Reformulations for Theory of Neural Networks

Optimal Sets: Summary

[MP23] Optimal Sets and Solution Paths of ReLU Networks. A. Mishkin, M. Pilanci. ICML 2023

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Big Idea: use convex reformulations as an analytical tool to understand the set of all minimizers for two-layer ReLU networks,

Non-Convex Solution Set (NC-ReLU):

$$\mathcal{O}^*(\lambda) := \operatorname*{arg\,min}_{W_1, w_2} \; \frac{1}{2}\| \sum_{j=1}^{m} (XW_{1j})_+ w_{2j} - y\|_2^2 + \lambda \sum_{j=1}^{m} \|W_{1j}\|_2^2 + |w_{2j}|^2,$$

Optimal Set: Strong Duality

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   - A little care is required to handle model symmetries.

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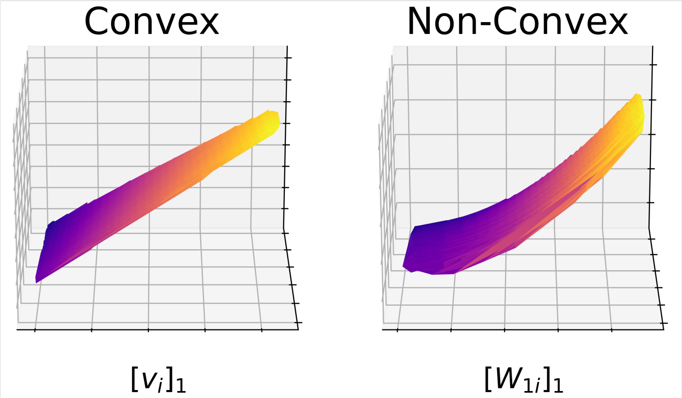
- The neuron directions are unique: $\frac{W_{1i}^*}{\|W_{1i}^*\|_2} = q_i/\lambda_i$.

Optimal Set: Appearance of Solutions



The non-convex parameterization maps a convex polytope of solutions into a curved manifold.

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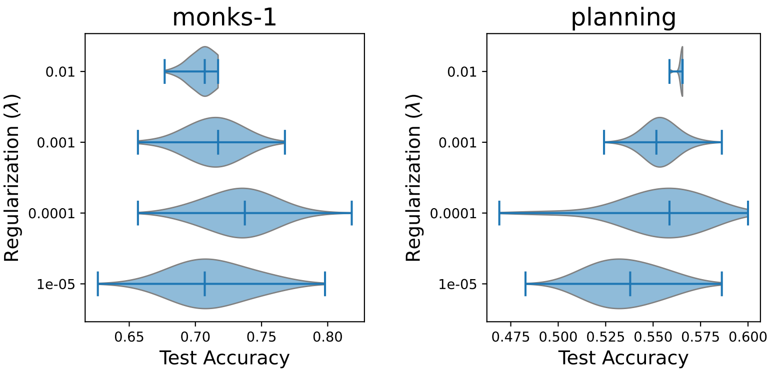

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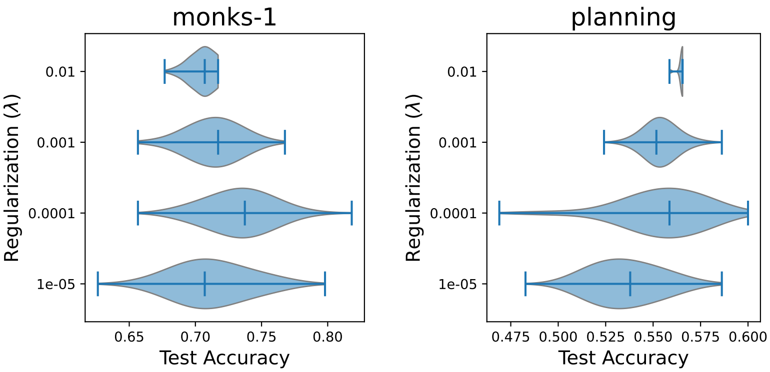

The solution you pick (implicit regularization) is crucial to good test performance!

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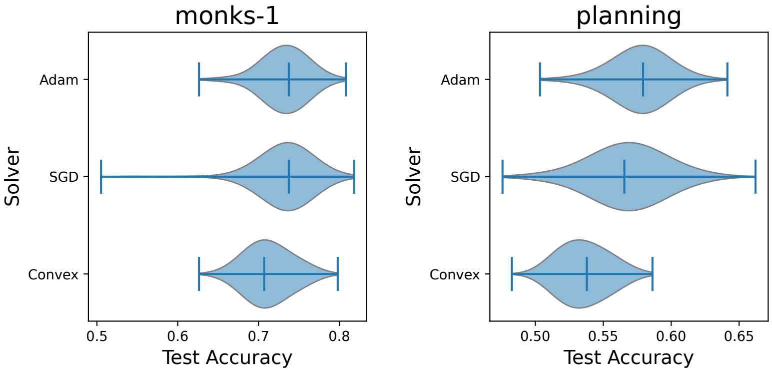

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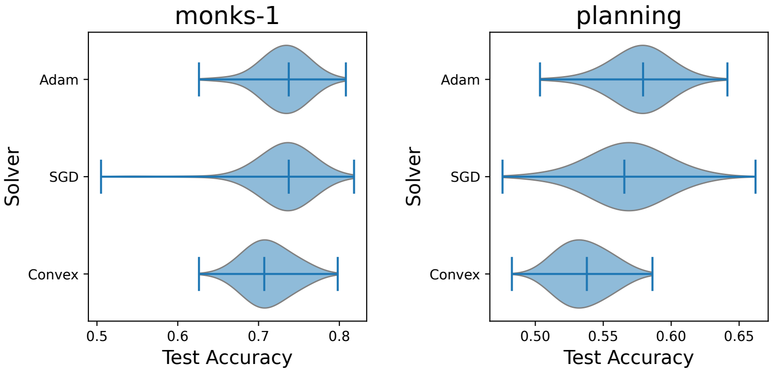

Note: SGD/Adam can converge to local minima which may perform better in this low-regularization setting.

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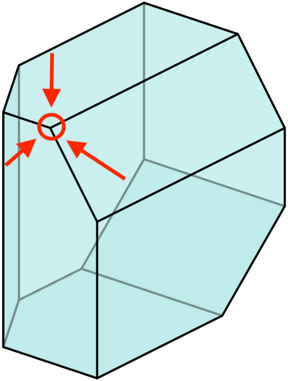

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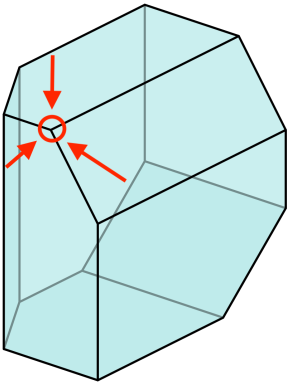

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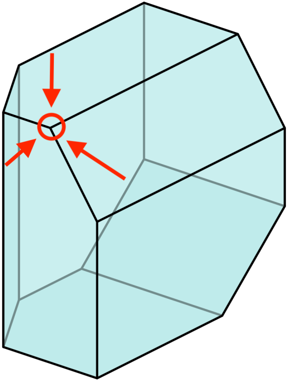

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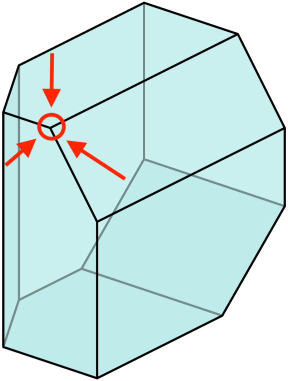

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- The smallest minimal model has exactly m^* neurons.
  - $m^* \iff$ minimum width for convex reformulation.

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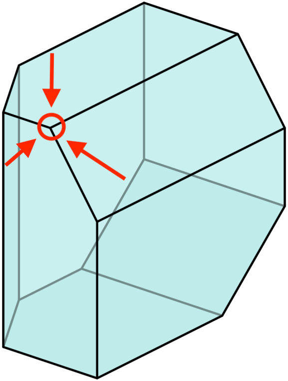

We prove:

- Vertices of the C-ReLU optimal set correspond exactly to minimal models.
- There are at most n neurons in a minimal model.
- The smallest minimal model has exactly m^* neurons.
  - $m^* \iff$ minimum width for convex reformulation.

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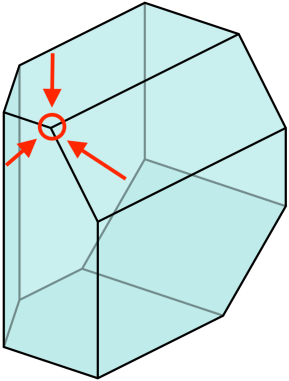

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$\hookrightarrow$ This is the first optimal pruning algorithm for neural nets!

Neuron Pruning: Performance on UCI Datasets

We also show how **optimal pruning** can be adapted to prune past m^* using a simple correction step (details in bonus!).

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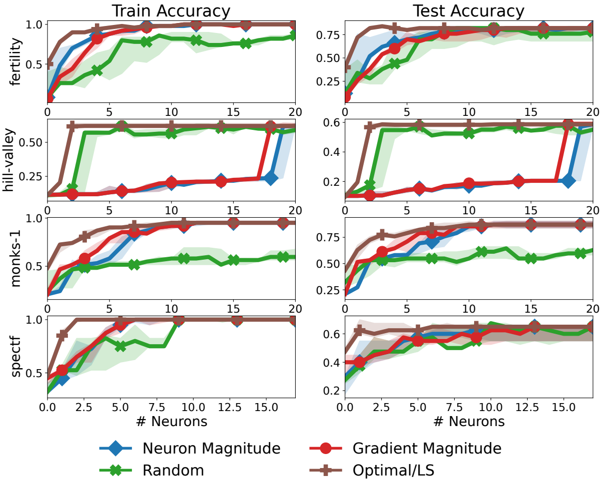

5. Extensions

Extensions: Scalar Inputs

[Zeg+24] A Library of Mirrors: Deep Neural Nets in Low Dimensions are Convex Lasso Models with Reflection Features. E. Zeger, Y. Wang, A. Mishkin, T. Ergen, E. Candès, M. Pilanci. SIMODS (In Review).

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Big Idea: ReLU networks with one-dimensional inputs admit a simpler convex reformulation as Lasso models.

- The feature matrix for the Lasso model is determined by the model architecture.
- We extend our characterization of the set of optimal ReLU neural networks to this setting.

Extensions: Mode Connectivity

[KMP25] Exploring The Loss Landscape Of Regularized Neural Networks Via Convex Duality. S. Kim, A. Mishkin, M. Pilanci. ICLR 2025 (Oral).

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Mode Connectivity: how and when are optimal ReLU networks connected to each other in weight space?

- Our previous work assumed $m \geq p$: the width of the ReLU network is at least the number of activation patterns.
- Now we study the optimal set as m ranges from m^* to p, creating a set of transitions in connectivity.

Mode Connectivity: Staircase of Connectivity

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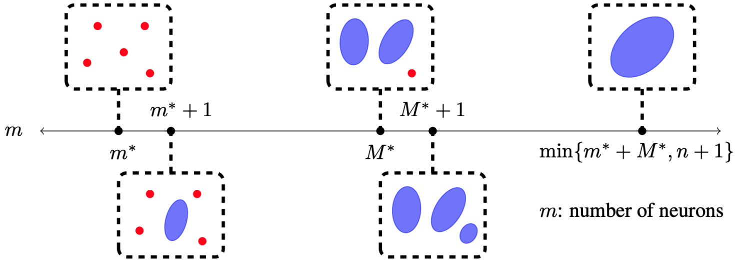
- **Credit**: this theorem is due to Sungyoon Kim, building off of my optimal set work with Mert.

Staircase of Connectivity Visualized



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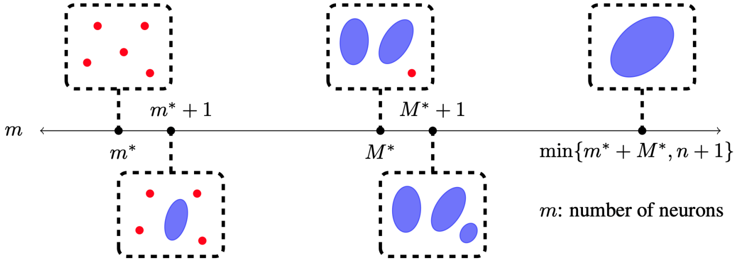

Takeaway: Connectivity increases in phases with network width.

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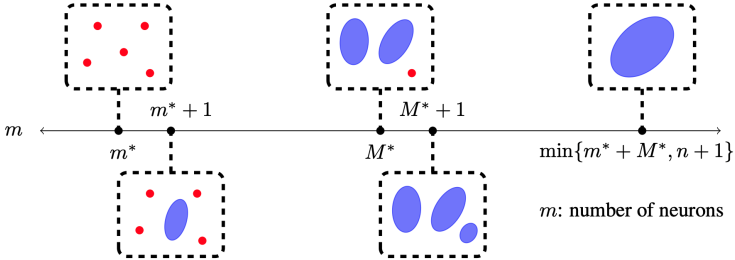

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$\hookrightarrow$ This definitively answers a long standing question in the theory of neural networks!

Extensions: Feature-Sparse Neural Networks

Convex LassoNet: Feature Sparse Convex Reformulations. A. Mishkin, T. Ergen, F. Ruan, M. Pilanci, R. Tibshirani. (Ongoing)

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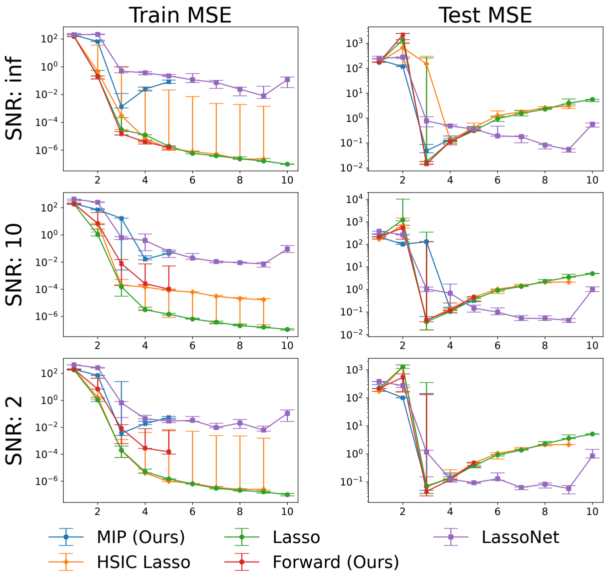
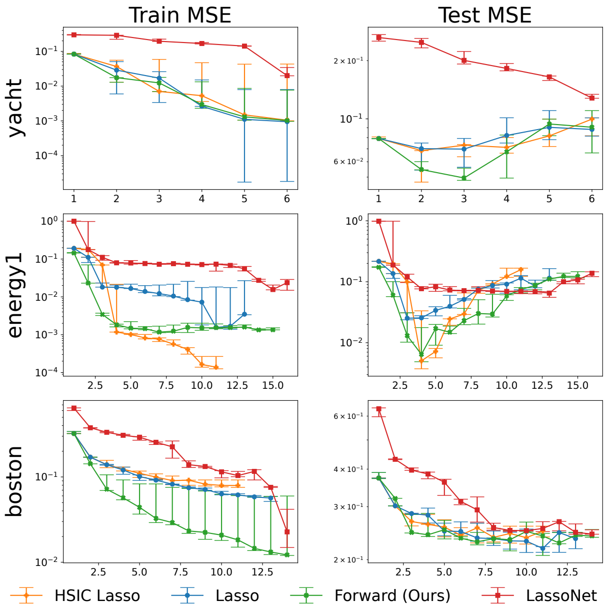
- We derive sparsity-inducing convex reformulations with global optimization guarantees.

Feature-Sparsity: Planted Neural Networks



Feature-Sparsity: Real Data



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- But, once we understand three layer networks, we understand k-layer networks for any $k \geq 1$.
- We prove that ReLU MLPs of arbitrary depth are convex functions with non-convex tensor decomposition constraints.

Deep Networks: Layer Elimination

Let $W^{(l)} \in \mathbb{R}^{d_{l-1} \times d_l}$ and consider the k-layer ReLU network

$$f_\theta(X) = \left(\left(\left(X W^{(1)} \right)_+ W^{(2)} \right)_+ W^{(3)} \dots \right)_+ W^{(k)}.$$

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2. This creates another two-layer block, which has a convex reformulation in terms of the activation patterns $D_j^{(2)} \ldots$

Deep Networks: Tensor Programs

Let $T^{(l)} \in \mathbb{R}^{d_0 \times \dots d_l}$ be a tensor with low-rank decomposition,

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Deep Networks: Tensor Parameterization

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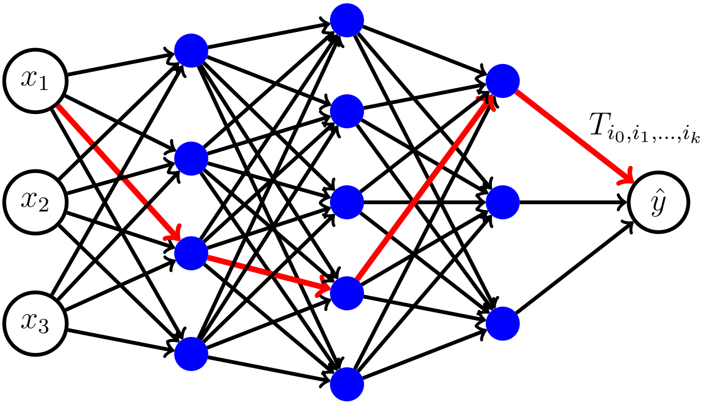

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- **Additional Results**: We provide many, many more results on continuity, stability, optimization, and other areas.

Overview: Publications and Ongoing Projects

Publications:

1. Fast Convex Optimization for Two-Layer ReLU Networks. A. Mishkin, A. Sahiner, M. Pilanci. ICML 2022.
2. Optimal Sets and Solution Paths of ReLU Networks. A. Mishkin, M. Pilanci. ICML 2023.
3. A Library of Mirrors: Deep Neural Nets in Low Dimensions are Convex Lasso Models with Reflection Features. E. Zeger, Y. Wang, A. Mishkin, T. Ergen, E. Candès, M. Pilanci. SIMODS (In Review).
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Ongoing Projects:

1. Deep Convex Reformulations: Equivalences and Optimal Sets
2. Convex LassoNet: Feature-Sparse Convex Reformulations

Overview: Additional Projects (Details in Bonus)

Additional Publications:

1. Directional Smoothness and Gradient Methods: Convergence and Adaptivity. A. Mishkin*, A. Khaled*, Y. Wang, A. Defazio, R. M. Gower. NeurIPS 2024.
2. Level Set Teleportation: An Optimization Perspective. A. Mishkin, A. Bietti, R. M. Gower. AISTATS 2025.
3. Glocal Smoothness: Line Search can really help! C. Fox, A. Mishkin, S. Vaswani, M. Schmidt. SIOPT (In Review).

Further Ongoing Projects:

1. Greedy 2-Coordinate Updates for Equality-Constrained Optimization via Steepest Descent in the 1-Norm. A. V. Ramesh*, A. Mishkin*, M. Schmidt, Y. Zhou, J. Lavington, J. She. (To Be Submitted)
2. Global Convergence of Gradient Flow on 1D Data with Sign Noise. A. Mishkin, F. Bach.

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- And, of course, my family, whose constant support made all of this possible.

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Additional Projects

Directional Smoothness: Summary

[Mis+24] Directional Smoothness and Gradient Methods: Convergence and Adaptivity. A. Mishkin*, A. Khaled*, Y. Wang, A. Defazio, R. M. Gower. NeurIPS 2024.

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Remaining Extensions:

1. Generalize the target noise to more interesting/realistic regimes (e.g. fractional Brownian motion).

Background on Convex Neural Networks

Bonus: Convex Neural Networks

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The standard ERM training problem is,

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- This is distinct from input-convex neural networks, where $x \mapsto f_\theta(x)$ is convex [AXK17].

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There are multiple flavours of convex neural networks:

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These approaches differ primarily in how they discretize the underlying infinite-width neural network.

Bonus: Function Space Viewpoint

Bengio et al. [Ben+05] and Bach [Bac17] take a function space approach:

- Let σ be an activation function and define

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- If R is sparsity inducing, then the final network may have finite width.

Bonus: Related Work Cont.

Bengio et al. [Ben+05]: algorithm-focused approach.

- Take $R(w) = \|w\|_1$ and $L(\hat{y}, y) = \max\{0, 1 - \hat{y}y\}$.

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- Take $R(w) = \|w\|_1$ and $L(\hat{y}, y) = \max\{0, 1 - \hat{y}y\}$.
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- Show that $\mathrm{nnz}(w^*) \leq n + 1$, meaning the final model is finite.
- Propose a boosting-type algorithm for iteratively adding neurons.

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- Derive an incremental algorithm based on Frank-Wolfe, but incremental steps are NP-Hard for ReLU activations.

Bonus: Key Representer Theorem

Theorem (Rogosinski [Rog58])

If $(\Omega, \mathcal{B})$ *is a Borel space,* μ *is a measure,* g_i*,* $i \in \{1, \ldots n\}$ *are measurable and* μ*-integrable, then there exists measure* $\hat{\mu}$ *with finite support at most* n *such that*

$$\int_\Omega g_i(\omega) d\mu(\omega) = \int_\Omega g_i(\omega) d\hat{\mu}(\omega)$$

for all $i \in \{1, \ldots, n\}$*.*

Prediction for dataset with n dimensions:

$$f(x_i) = \int_{\mathcal{H}} h(x_i) dw(h) = \sum_{h=1}^{m} h_j(x_i) w(h_j),$$

where $m \leq n$ and $h_j(x) = (\langle x, w_j \rangle)_+$.

Convex Reformulations

Convex Reformulations: Breaking it Down

$$\min_{u} \| \sum_{j=1}^{p} D_j X(v_j - w_j) - y\|_2^2 + \lambda \sum_{j=1}^{p} \|v_j\|_2 + \|w_j\|_2$$
$$\text{s.t. } v_j, w_j \in \mathcal{K}_j := \{w : (2D_j - I)Xw \geq 0\},$$

where $D_j = \text{diag}[\mathbb{1}(Xg_j \geq 0)]$.

- D_j is a ReLU activation pattern induced by "gate" g_j.

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$$\min_{u} \| \sum_{j=1}^{p} D_j X (v_j - w_j) - y \|_2^2 + \lambda \sum_{j=1}^{p} \|v_j\|_2 + \|w_j\|_2$$

$$\text{s.t. } v_j, w_j \in \mathcal{K}_j := \{ w : (2D_j - I) X w \geq 0 \},$$

where $D_j = \text{diag}[\mathbb{1}(X g_j \geq 0)]$.

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- Weight-decay regularization turns into "group ℓ_1" penalty.
- The constraint $v_j \in \mathcal{K}_j$ implies

$$(Xv_j)_+ = D_j X v_j.$$

That is, v_j has the activation encoded by D_j.

Bonus: Explicit Solution Mapping

Given (v^*, w^*), an optimal non-convex ReLU network is given by

C to NC:
$$W_{1i} = v_i^* / \sqrt{\|v_i^*\|}, \quad w_{2i} = \sqrt{\|v_i^*\|}$$
$$W_{1j} = w_i^* / \sqrt{\|w_i^*\|}, \quad w_{2j} = -\sqrt{\|w_i^*\|}.$$

- Optimal solution balances weight between layers.

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NC to C:
$$v_i = W_{1i}^* \left| w_{2i} \right|^* \qquad \text{if } w_{2i}^* \geq 0$$
$$w_i = W_{1i}^* \left| w_{2i} \right|^* \qquad \text{Otherwise.}$$

- Optimal solution combines weight from both layers.

Gated ReLU Networks and Cone Decompositions

Bonus: Gated ReLU Networks

Theorem 2.2 (informal): C-GReLU is equivalent to solving

$$\textbf{NC-GReLU}: \min_{W_1, \alpha} \frac{1}{2} \| \sum_{j=1}^{p} \phi_{g_j}(X, w_j)\alpha - y\|_2^2 + \frac{\lambda}{2} \sum_{j=1}^{p} \|w_j\|_2^2 + |\alpha_j|^2,$$

with the "Gated ReLU" [FMS19] activation function

$$\phi_g(X, u) = \mathsf{diag}(\mathbb{1}(Xg \geq 0))Xu,$$

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Interpretation: if $u_j \notin \mathcal{K}_j$, then the activation must be decoupled from the linear mapping in the non-convex model.

Bonus: Cone Decompositions

Question: when are Gated ReLU and ReLU networks equivalent?

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Consider special case where $\lambda = 0$.

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V.S.

$$\textbf{C-ReLU} : \min_u \| \sum_{j=1}^{p} D_j X (v_j - w_j) - y \|_2^2.$$

$$\text{s.t. } v_j, w_j \in \mathcal{K}_j := \{ w : (2D_j - I) X w \geq 0 \},$$

Bonus: Equivalent Statement

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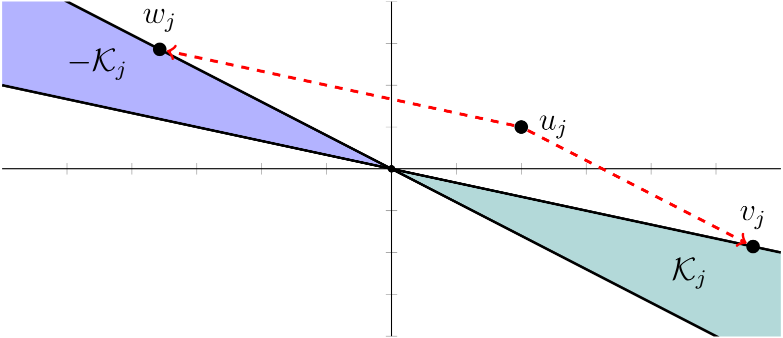

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Unfortunately, there is no extension to full-rank X.

Bonus: Not All Cones are Equal

Alternative Idea: show we don't need "singular" cones $\mathcal{K}_j$,

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Interpretation: if optimal $u_j^* \neq 0$, then set

$$u_i' = u_j^* + u_i^*.$$

It is possible to show this causes no problems.

Bonus: Cone Decomposition Proof Sketch

Proof: Works by iteratively constructing $\mathcal{K}_i$ s.t. $\mathcal{K}_j \subset \mathcal{K}_i$.

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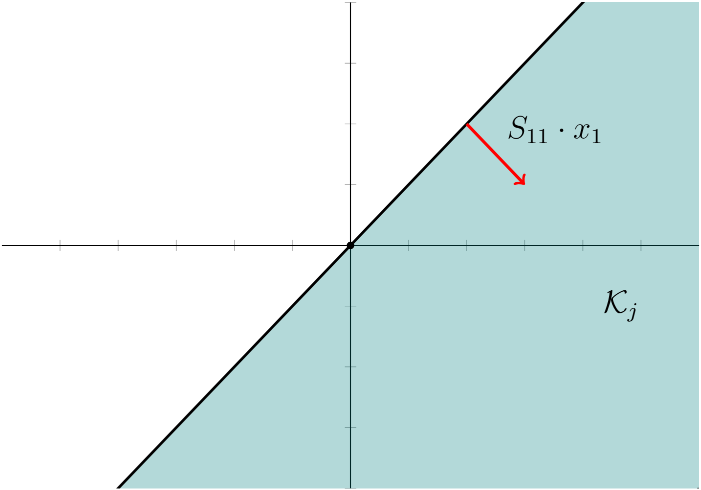
We sketch a simpler statement:

Proposition 3.2 (informal): Suppose $\mathcal{K}_j = \{0\}$. Then, there exists $\mathcal{K}_i$ for which $\mathcal{K}_i - \mathcal{K}_i = \mathbb{R}^d$ and $\mathcal{K}_j \subset \mathcal{K}_i$.

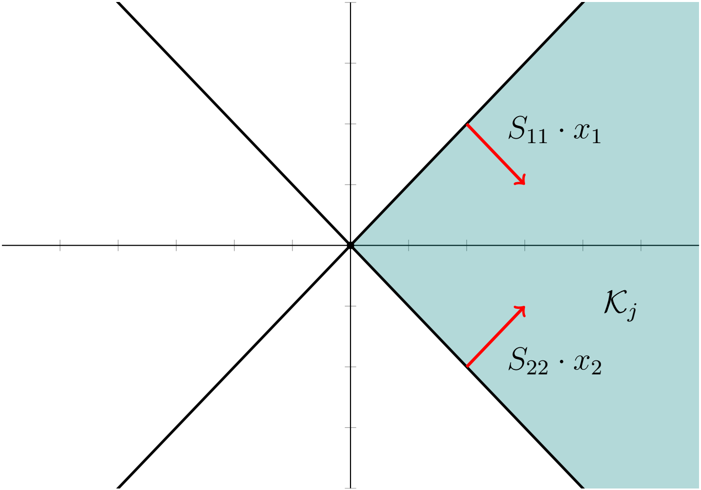
Bonus: Cone Decomposition Proof Sketch

$$\mathcal{K}'_j = \{w : [S_j]_{11} \cdot \langle x_1, w \rangle \geq 0\}$$



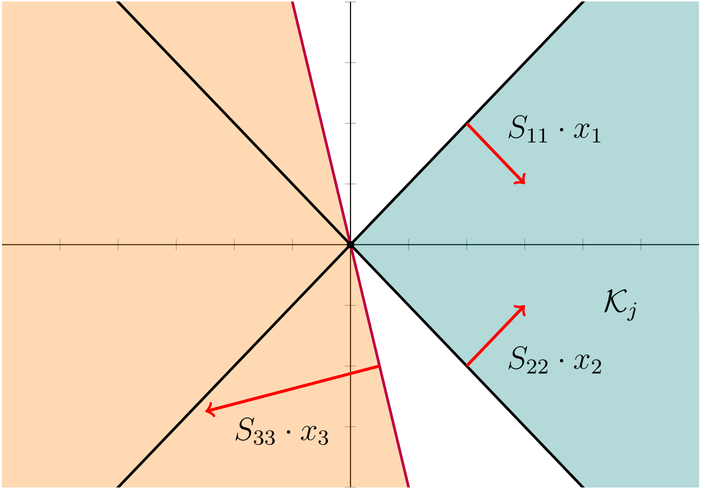
Cone Decompositions: Proof Sketch

$$\mathcal{K}_j'' = \mathcal{K}_j' \cap \{w : [S_j]_{22} \cdot \langle x_2, w \rangle \geq 0\}$$



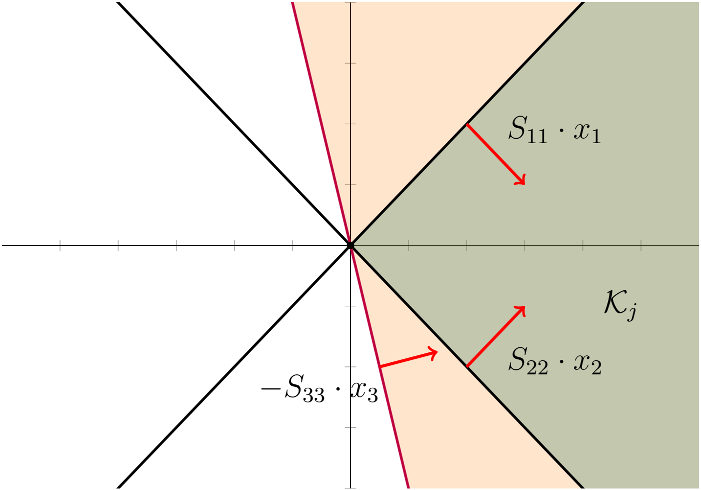
Bonus: Cone Decomposition Proof Sketch

$$\mathcal{K}_j''' = \mathcal{K}_j'' \cap \{w : [S_j]_{33} \cdot \langle x_3, w \rangle \geq 0\}$$



Bonus: Cone Decomposition Proof Sketch

$$\tilde{\mathcal{K}}_j''' = \mathcal{K}_j'' \cap \{w : -[S_j]_{33} \cdot \langle x_3, w \rangle \geq 0\}$$



Bonus: Main Cone Decomposition Result

- The real proof is more complex, but this is the core idea.
  - Build $\mathcal{K}_i$ by switching signs of $[S_j]_{ii}$.
  - Equivalent to turning on/off activations.
- Leads to our main approximation result.

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- Leads to our main approximation result.

Theorem 3.7 (informal): Let $\lambda \geq 0$ and let p^* be the optimal value of the ReLU problem. There exists a C-GReLU problem with minimizer u^* and optimal value d^* satisfying,

$$d^* \leq p^* \leq d^* + 2\lambda\kappa(\tilde{X}_{\mathcal{J}}) \sum_{D_i \in \tilde{\mathcal{D}}} \|u_i^*\|_2.$$

Details on Optimization Algorithms

Bonus: ReLU by Cone Decomposition

1. Solve the gated ReLU problem:

$$u^* \in \arg\min_{u} \| \sum_{j=1}^{p} D_j X u_j - y\|_2^2 + \lambda \sum_{j=1}^{p} \|u_j\|_2$$

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- $L(v, w) = \|v\|_2 + \|w\|_2$ gives an SOCP.
- $L(v, w) = 0$ yields a linear feasibility problem.

Bonus: R-FISTA

Consider "composite" optimization problem:

$$\min_x f(x) + g(x),$$

where f is L-smooth and g is convex. Smoothness implies

$$\begin{aligned} f(y) &\le Q_{x_k,1/L}(y) \\ &= f(x_k) + \langle \nabla f(x_k), y - x_k \rangle + \frac{L}{2}\|y - x_k\|_2^2. \end{aligned}$$

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The **FISTA** algorithm minimizes Q_{y_k,η_k} and handles g exactly:

$$\begin{aligned} x_{k+1} &= \arg\min_y Q_{y_k,\eta_k}(y) + g(y) \\ y_{k+1} &= x_{k+1} + \frac{t_k - 1}{t_{k+1}} (x_{k+1} - x_k) \end{aligned}$$

where $t_{k+1} = (1 + \sqrt{1 + 4t_k^2})/2$.

Bonus: R-FISTA Continued

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- **Line-search**: backtrack on η_k until:

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  as proposed by [BT09].

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- And lots of other **convex tricks**...

Bonus: AL Method

Let $\tilde{X}_i = (2D_i - I)X$ so that $v_j \in \mathcal{K}_j \iff X_j v_j \geq 0$ and define

$$F(v, w) = \| \sum_{j=1}^{p} D_j X(v_j - w_j) - y\|_2^2 + \lambda \sum_{j=1}^{p} \|v_j\|_2 + \|w_j\|_2.$$

Now we can form the augmented Lagrangian:

$$\begin{aligned} \mathcal{L}_\delta(v,w,\gamma,\zeta) := (\delta/2) \sum_{D_i \in \tilde{\mathcal{D}}} \big[\|(\gamma_i/\delta - \tilde{X}_i v_i)_+\|_2^2 \\ + \|(\zeta_i/\delta - \tilde{X}_i w_i)_+\|_2^2 \big] + F(v, w). \end{aligned} \tag{1}$$

We use the multiplier method to update the dual parameters:

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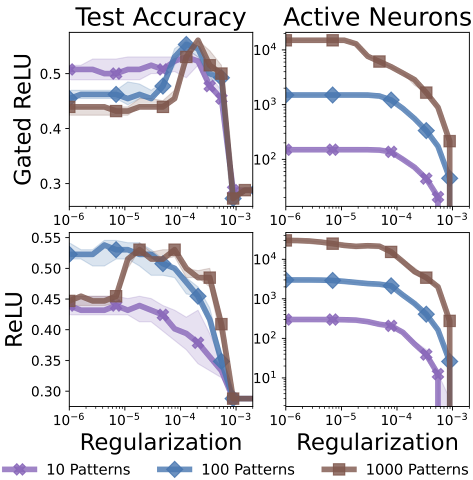
$$\gamma_{k+1} = (\gamma_k - \delta \tilde{X}_i v_i)_+, \quad \zeta_{k+1} = (\zeta_k - \delta \tilde{X}_i w_i)_+.$$

We use warm starts and propose a heuristic for δ.

Additional Optimization Experiments

Bonus: Sub-sampling Patterns



- Variance induced by resampling $\tilde{\mathcal{D}}$ is minimal.
- Standard bias-variance trade-off.

Bonus: Generalization Performance

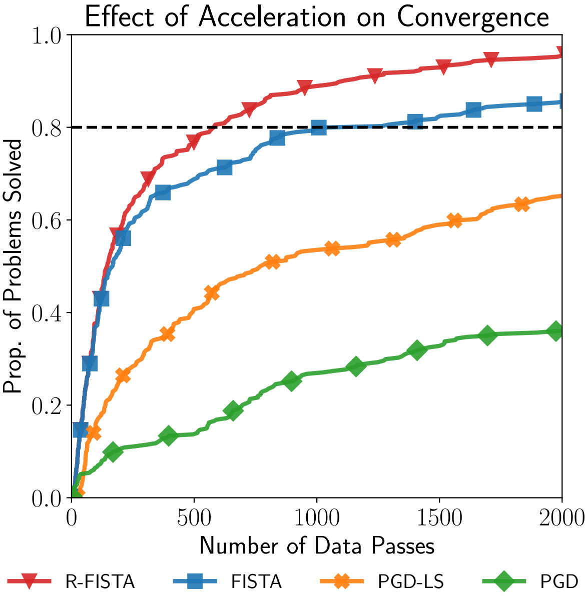
Generalization performance is equivalent to non-convex solvers.

Dataset	Convex	Adam	SGD
magic	85.9	**86.9**	86.4
statlog-heart	**83.3**	**83.3**	79.6
vertebral-col.	**90.3**	**90.3**	88.7
cardiotocogr.	**89.9**	36.5	88.9
abalone	**66.2**	65.3	66.1
annealing	90.6	**93.7**	88.7
car	87.8	**94.8**	90.1
bank	89.8	**90.8**	90.5
breast-cancer	**68.4**	64.9	**68.4**
page-blocks	94.0	**97.1**	96.9
contrac	**55.1**	54.4	53.7
congressional	63.2	62.1	**67.8**
spambase	93.3	**93.5**	93.2
synthetic	**98.3**	96.7	96.7
hill-valley	**65.3**	62.8	55.4

Bonus: Comparison to Standard Baselines

Dataset	**C-GReLU**	**C-ReLU**	**RF**	**Linear**	**RBF**
blood	79.9	**80.5**	75.8	74.5	77.9
chess-krvkp	**99.2**	98.6	98.9	97.2	98.4
conn-bench	**90.2**	85.4	73.2	68.3	85.4
cylinder-bands	76.5	**78.4**	77.5	71.6	71.6
fertility	**80.0**	**80.0**	75.0	75.0	75.0
heart-hung.	**86.2**	**86.2**	84.5	84.5	**86.2**
hill-valley	**76.0**	68.6	57.9	62.0	70.2
ilpd-liver	72.4	**74.1**	66.4	71.6	71.6
mammographic	77.6	78.6	**80.7**	**80.7**	80.2
monks-1	**100**	**100**	95.8	79.2	83.3
musk-1	94.7	**95.8**	92.6	86.3	**95.8**
ozone	**97.6**	**97.6**	97.4	97.2	97.4
pima	74.5	74.5	**76.5**	75.2	73.2
planning	**69.4**	63.9	66.7	66.7	**69.4**
spambase	93.5	93.6	**94.1**	92.2	93.6
spectf	**87.5**	75.0	68.8	68.8	68.8
statlog-german	74.0	**77.5**	73.5	75.0	75.5
tic-tac-toe	99.0	99.0	99.5	98.4	**100**

Bonus: Acceleration Ablation



Example: Discontinuous Regularization Paths

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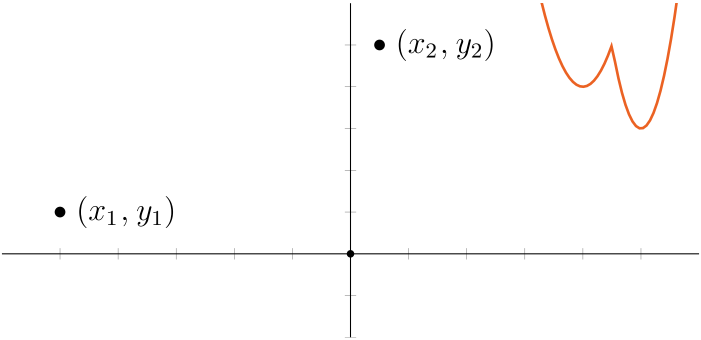

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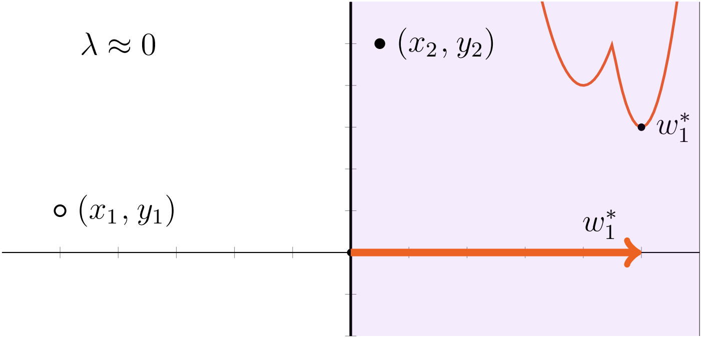

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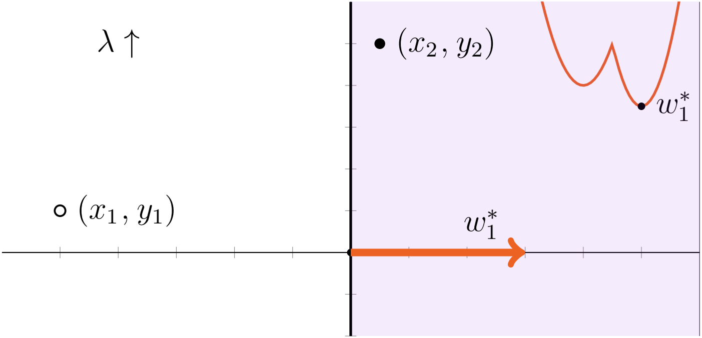

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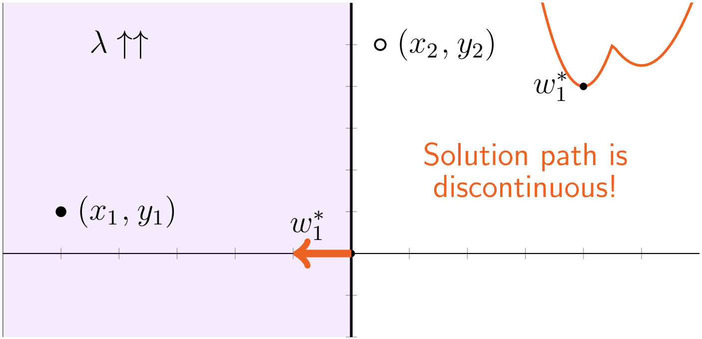

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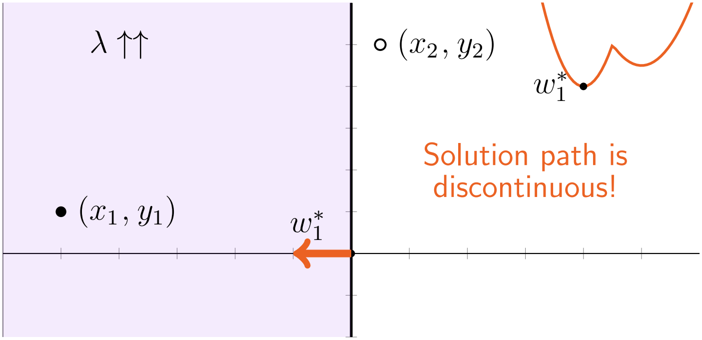

Goal: Overcome these problems via convexification.

Optimal Sets

Bonus: C-ReLU Optimality Conditions

We form the Lagrangian for the convex reformulation:

$$\mathcal{L}(v, w, \rho^+, \rho^-) = \frac{1}{2}\| \sum_{D_i \in \tilde{\mathcal{D}}} D_i X(v_i - w_i) - y\|_2^2 + \lambda \sum_{D_i \in \tilde{\mathcal{D}}} \|v_i\|_2 + \|w_i\|_2$$
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  - It turns out each q_i^+ is **unique** WLOG!

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Facts: let (θ, ρ) be primal dual optimal.

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- Every solution is a non-negative multiple of these q_i vectors.

Bonus: Explicit Optimal Set

We gave a characterization of $\mathcal{W}^*(\lambda)$ that depends on

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More complex, but also **explicit**.

Bonus: Solution Mapping for C-ReLU

Given (v^*, w^*), an optimal non-convex ReLU network is given by

C to NC:
$$W_{1i} = v_i^* / \sqrt{\|v_i^*\|}, \quad w_{2i} = \sqrt{\|v_i^*\|}$$
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Recall structure of **non-convex optima**:

$$\begin{aligned}\mathcal{O}_\lambda = \big\{ &(W_1, w_2) : \; f_{W_1, w_2}(X) = \hat{y}, \\ &\forall\, i \in \mathcal{S}_\lambda, W_{1i} = ({}^{\alpha_i}/\lambda)^{1/2} q_i, w_{2i} = (\alpha_i \lambda)^{1/2}, \alpha_i \geq 0 \\ &\forall\, i \in [2p] \setminus \mathcal{S}_\lambda, W_{1i} = 0, \; w_{2i} = 0 \big\}.\end{aligned}$$

Optimal Pruning

Optimal Pruning: the Polytope of Solutions

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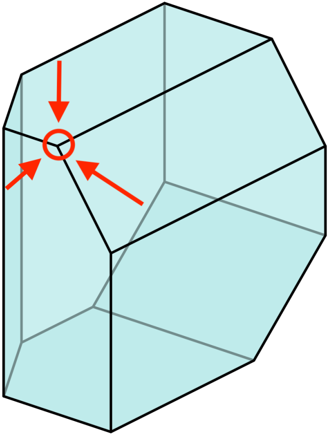

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Are these vertices special in some way?

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1. We prove vertices of $\mathcal{W}^*(\lambda)$ are minimal models.
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3. We give a poly-time algorithm for computing minimal models starting from any model θ.

Bonus: Optimal Pruning Pseudo-code

Algorithm Pruning solutions

Input: data matrix X, solution θ.
$k \leftarrow 0$.
$\theta^k \leftarrow \theta$.
while $\exists \beta \neq 0$ s.t. $\sum_{i \in \mathcal{A}_\lambda(\theta^k)} \beta_i D_i X \theta_i^k = 0$ **do**
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Let $r = \text{rank}(X)$. Complexity to compute a minimal model:

$$O\left(d^3 r^3 (\frac{n}{r})^{3r} + (n^3 + nd) r (\frac{n}{r})^r \right).$$

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Total complexity: $O(ndp + n^3 p)$.

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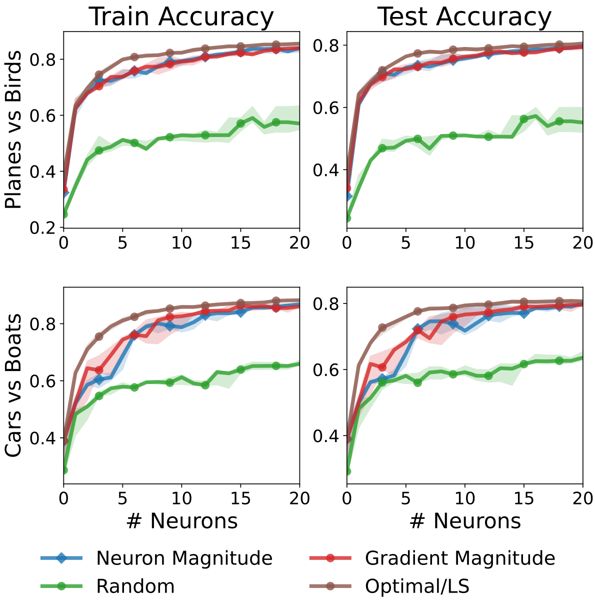
- Algorithm is just structured pruning with a correction step!
- We use existing literature for structured pruning to select j.
- Brute-force search works best:

$$\arg\min_{j} \left\{ \min_{\beta} \frac{1}{2} \| \sum_{i \in \mathcal{A}_\lambda(\theta^k) \backslash j} \beta_i D_i X \theta_i^k - D_j X \theta_j \|_2^2 \right\}$$

Neuron Pruning: Performance on CIFAR-10



Convex Reformulations of Deep ReLU Networks

Deep Reformulations: Setup

- Let $X^{(l)}$ and $T^{(l)}$ be tensors of order $l+1$ indexed by $i_0, \ldots, i_l$.
- We assume $R^{(l)}_{i_1,\ldots,i_{l-1}} \in \mathbb{R}^{n \times d_0}$ and $T^{(l)}_{i_1,\ldots,i_{l-1}} \in \mathbb{R}^{d_0 \times d_l}$.

We equip these tensors with the reduction product

$$R^{(l)} \odot T^{(l)} = \sum_{i_1,\ldots,i_{l-1}} R^{(l)}_{i_1,\ldots,i_{l-1}} T^{(l)}_{i_1,\ldots,i_{l-1}},$$

- This sums over the product of all the matrix slices $R^{(l)}_{i_1,\ldots,i_{l-1}}$ and $T^{(l)}_{i_1,\ldots,i_{l-1}}$.

Deep Reformulations: Recursive Patterns

- Let $\mathcal{D}_X^{(1)}$ be the set of achievable ReLU patterns in the first layer.
- Let $X^{(1)} = X \in \mathbb{R}^{n \times d_0}$.
- Define $X_{i_1,\ldots,i_l}^{(l+1)} = D_{i_l}^{(l)} X_{i_1,\ldots,i_{l-1}}^{(l)}$ so that we have,

$$X_{i_1}^{(2)} = D_{i_1}^{(1)} X^{(1)} = D_{i_1}^{(1)} X$$
$$X_{i_1,i_2}^{(3)} = D_{i_2}^{(2)} X_{i_1}^{(2)} = D_{i_2}^{(2)} D_{i_1}^{(1)} X.$$

- Here, $D_{i_l}^{(l)} \in \mathcal{D}_X^{(l)}$ is the set of ReLU patterns achievable by our tensor product in the l^{th} layer,

$$\mathcal{D}_X^{(l)} = \left\{ \mathbb{1}\left(X^{(l)} \odot T^{(l)} \geq 0 \right) : T^{(l)} \in \mathbb{R}^{d_0 \times \cdots \times d_l} \right\}.$$

Deep Reformulations: Tensor Decomposition

Each tensor $T_{i_l}^{(l)}$ is contained in at least one activation cone,

$$\mathcal{K}_{i_l}^{(l)} = \left\{ T_{i_l}^{(l)} \in \mathbb{R}^{d_0 \times \cdots \times d_{(l-1)}} : (2D_{i_l}^{(l)} - I) \left[X^{(l)} \odot T_{i_l}^{(l)} \right] \geq 0 \right\}.$$

Now, let $\mathcal{G}^{(1)} = \mathbb{R}^{d_0 \times d_1}$ and define

$$\begin{aligned} \mathcal{G}^{(l+1)} := \Bigg\{ & T^{(l+1)} \in \mathbb{R}^{d_0 \times p_1 \ldots \times p_l \times d_{l+1}} \\ & : \exists\, T^{(l)} \in \mathcal{G}^{(l)} \text{ where } \mathcal{I}_{j_l}^{(l)} = \left\{ i_l : T_{i_l}^{(l)} \in \mathcal{K}_{j_l}^{(l)} \right\}, \\ & \exists\, W^{(l+1)} \in \mathbb{R}^{d_l \times d_{l+1}}, \\ & \text{s.t. } T_{j_1, \ldots, j_l}^{(l+1)} = \sum_{i_l \in \mathcal{I}_{j_l}^{(l)}} T_{j_1, \ldots, j_{l-1}, i_l}^{(l)} \otimes W_{i_l}^{(l+1)}, \\ & \text{and } \sum_{j_l = 1}^{p_l} \left| \mathcal{I}_{j_l}^{(l)} \right| \leq d_l \Bigg\}. \end{aligned}$$

Deep Reformulations: Layer-Merging Lemma

Lemma (Rank-Controlled Layer Elimination)

Let $T^{(l)} \in \mathcal{G}^{(l)}$. *Then the activations at layer* $l+2$ *are given by*

$$Z^{(l+2)} = \left(\sum_{i_l=1}^{d_l} \left(X^{(l)} \odot T_{i_l}^{(l)} \right)_+ W_{i_l}^{(l+1)} \right)_+, \tag{3}$$

if and only if the activations are also equal to

$$Z^{(l+2)} = \left(X^{(l+1)} \odot T^{(l+1)} \right)_+,$$

for some $T^{(l+1)} \in \mathcal{G}^{(l+1)}$.