The Evolution from the Law of Gravitation to General Relativity

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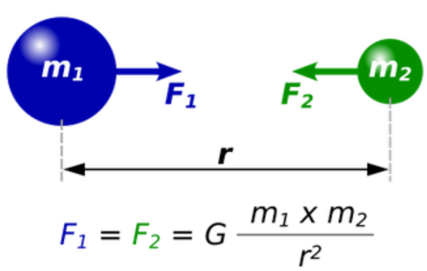
Introduction

Albert Einstein's field equation lies at the core of general relativity. However, it did not emerge solely from thought experiments; rather, it represents the culmination of scientific advancements across multiple fields, shaped by the contributions of many great thinkers. By tracing the progression from Newton's law of gravitation, we can gain valuable insight into how Einstein formulated the field equation. This historical perspective not only deepens our understanding of the underlying physics but also enhances our appreciation of the groundbreaking contributions made by both Newton and Einstein.

This article begins by exploring Newton's discovery of the law of universal gravitation. Inspired by this groundbreaking work, Coulomb and his contemporaries formulated a corresponding law describing the electrostatic force between electric charges. Both Newton's law and Coulomb's law express the force between interacting bodies as inversely proportional to the square of the distance separating them. Building on these foundational principles, mathematicians developed Gauss's law of field flux, which later became a cornerstone of Maxwell's equations in electromagnetism. Further abstraction of these force fields led to Poisson's Equation, which relates potential fields to the density of the matter that generates them. Ultimately, Einstein's field equation can be viewed as a profound generalization of Poisson's equation.

Newton's Law of Gravitation

Much of early science originated from philosophical ideas, including the concept of gravity. For example, Aristotle believed that rocks fell to the ground because it was in their nature to seek their natural place. Well before Isaac Newton, foundational ideas about matter and motion were already taking shape. Galileo Galilei conducted numerous experiments, carefully observing the behavior of falling and rolling objects. Meanwhile, by studying the motion of celestial bodies, Johannes Kepler formulated his three laws describing planetary motion and gravitational interactions.



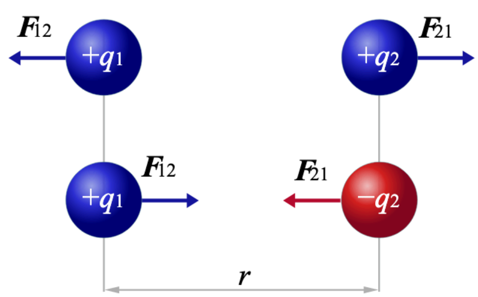
Isaac Newton extended the idea that Kepler's laws must also apply to the Moon's orbit around the Earth—and, more broadly, to all objects on Earth. By combining his own laws of motion with mathematical analysis, he developed a theoretical foundation for Kepler's empirical observations. This led to the formulation of Newton's law of universal gravitation in 1687: Any two bodies attract each other with a force proportional to the product of their masses and inversely proportional to the square of the distance between them:

(1) $$F = G\frac{m_1 m_2}{r^2}$$

In this equation, F is the gravitational force between two masses m_1 and m_2 separated by a distance r, and G is the Newtonian constant of gravitation (6.67×10^{-11} $m^3kg^{-1}s^{-2}$).

Coulomb's Law

Ancient observers noticed that certain objects, when rubbed with materials like a cat's fur, could attract lightweight items such as feathers. Building on these early observations, William Gilbert conducted systematic studies of electricity and magnetism, focusing in particular on the effects of static electricity generated by rubbing amber. Early investigators observed that electrical forces decreased with distance in a way similar to gravitational forces. Through experiments with electrically charged spheres, Franz Aepinus and Joseph Priestley were among the first to propose that electrical force obeys an inverse-square law, much like Newton's law of universal gravitation. However, unlike gravity, which is always attractive, the force between two spheres carrying the same sign of charges is repulsive, John Robison discovered.



$$\left|F_{12}\right| = \left|F_{21}\right| = k_e \frac{|q_1 \times q_2|}{r_2}$$

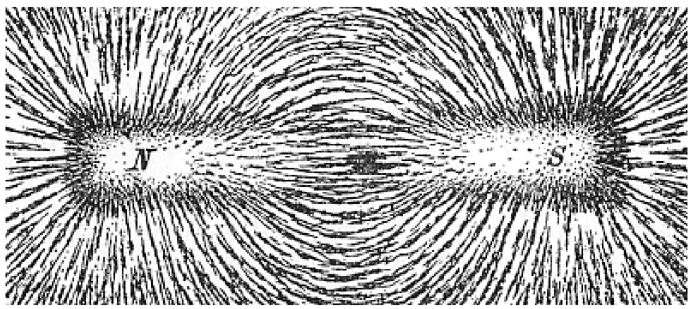
Using a torsion balance to study the attractive and repulsive forces between charged particles, Charles-Augustin de Coulomb determined that the magnitude of the electric force between two point charges is directly proportional to the product of their charges and inversely proportional to the square of the distance between them. In 1785, he published what became known as Coulomb's law:

(2) $$F = k\frac{q_1 q}{r^2}$$

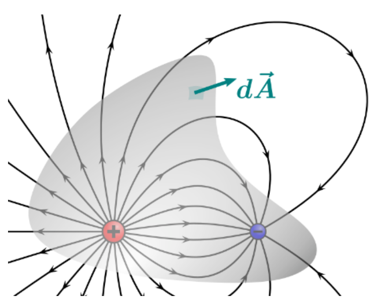
Here, the positive/negative value of F is the repulsive/attractive electrostatic force between two charges q_1 and q_2 separated by a distance r, and k is the Coulomb constant (8.99 x 10^9 Nm^2C^{-2}). Coulomb's law is a foundational principle in the development of the theory of electromagnetism.

Gauss's Law

When iron filings are sprinkled on a sheet of paper placed over a bar magnet, they reveal the direction of magnetic field lines, as illustrated in the figure below. These lines can also be constructed by measuring the strength and direction of the magnetic field at numerous points in space. At each location, a vector is drawn pointing in the direction of the local magnetic field, with its length proportional to the field's strength. Connecting these vectors forms a set of magnetic field lines. These lines are analogous to streamlines in fluid flow, representing a continuous distribution of the magnetic field.



Inspired by the observation of magnetic field lines, scientists have conceptualized that both gravitational and electrostatic forces are distributed through space as vector fields surrounding their sources. The strength of these fields, typically visualized by the length of field vectors, can also be represented by the density of field lines, as illustrated in the figure below.



Gauss's law is a mathematical abstraction that applies to both gravitational and electromagnetic forces. It states that the flux of a field through an arbitrary closed surface is proportional to the total quantity of the source enclosed by the surface, regardless of how the source is distributed. These sources may be masses in the case of gravitational fields or electric charges in the case of electrical fields. While Gauss's law alone is not sufficient to determine the exact distribution of a field, it can be used effectively in cases where symmetry ensures uniformity. In situations lacking such symmetry, Gauss's law is applied in its differential form, which states that the divergence of a field is proportional to the local density of its source:

(3) $$\nabla \bullet F = k\rho$$

Here, ∇ represents the del operator, ***F*** denotes the field distribution, ρ is the density of the field source, and k is a constant specific to the type of field. In the case of the electric field, this equation is typically written in the form:

(4) $$\nabla \bullet E = \frac{1}{\varepsilon_0}\rho$$

Here, ***E*** is the electrical field, ρ is the charge density, and ε_0 is the vacuum permittivity. Note that equation (4) is the first equation in Maxwell's equations.

Poisson's Equation

In the presence of a gravitational field ***F*** caused by a massive object with density ρ, Gauss's law, as given in equation (3), can be expressed in the following form:

(5) $$\nabla \bullet F = -4\pi G\rho$$

It can be used to obtain the corresponding Poisson equation for gravity. Since the gravitational field is conservative, it can be expressed in terms of a scalar potential φ:

(6) $$F = -\nabla\varphi$$

Substituting this into Gauss's law in the form of equation (5)

(7) $$\nabla \bullet (-\nabla\varphi) = -4\pi G\rho$$

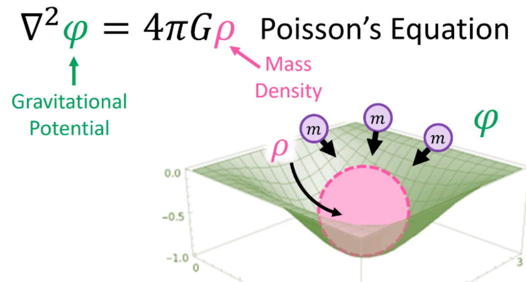
we obtain Poisson's equation for gravity:

(8) $$\nabla^2\varphi = 4\pi G\rho$$

Here, ∇^2 represents the Laplace operator. When the mass density is zero, Poisson's equation reduces to Laplace's equation. The corresponding Green's function can be employed to calculate the potential at a distance r from a central point mass m, given by:

(9) $$\varphi(r) = \frac{-Gm}{r}$$

This is equivalent to Newton's law of universal gravitation. Physically, Poisson's equation describes how the mass density ρ creates a local dip in the gravitational potential φ, pulling nearby masses toward it. This effect is illustrated in the figure below.

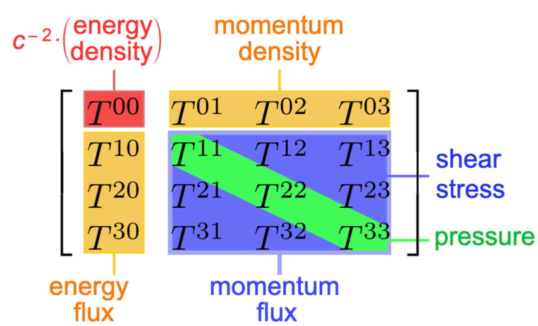


Einstein's Field Equation

Einstein's field equation is a generalization of Poisson's equation (8), where the gravitational potential term in the equation is replaced by the Einstein tensor G^{uv} and the mass density term is replaced by the energy-momentum tensor T^{uv}:

(10) $$G^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

Each tensor in this equation is a 4x4 tensor. Therefore, Einstein's field equation is a compact form of 16 equations. For instance, the energy-momentum tensor T^{uv} represents a 4x4 matrix:



At first glance, it might appear straightforward to transition from Poisson's equation to Einstein's field equations. However, in practice, a substantial conceptual and mathematical gap separates these two. The closest analogy to Poisson's equation in general relativity involves replacing the gravitational potential with the time-time component of the Ricci tensor, R^{00}, and substituting the mass density with the corresponding component of the energy-momentum tensor, T^{00}:

(11) $$R^{00} = \frac{8\pi G}{c^4} T^{00}$$

This is known as the Newton-Cartan reformulation of Poisson's equation. To convert the mass density term in Poisson's equation into the energy-momentum tensor in the Newton–Cartan framework, the density must be integrated over the volume and multiplied by the square of the speed of light. However, this equation by itself is not fully relativistic, as the remaining components of the Ricci tensor and the energy-momentum tensor do not transform consistently across all reference frames due to relativistic effects like time dilation and length contraction. To address this issue, the solution is to include all 4x4 components of both the Ricci tensor and the energy-momentum tensor:

$$(12) \qquad R^{\mu\nu} \approx \frac{8\pi G}{c^4} T^{\mu\nu}$$

Here, the indices μ and v range from 0 to 3, corresponding to the time dimension and the three spatial dimensions. As a result, each tensor in the equation is represented as a 4x4 matrix. In Einstein's notation, summation over repeated indices is implied—a convention known as the Einstein summation convention. At first glance, the equation appears nearly balanced; however, a subtle discrepancy arises due to the divergence properties of the tensors when taking their covariant derivatives and performing metric contractions. To correct for this imbalance, Einstein introduced an additional term on the left-hand side of the equation:

$$(13) \qquad R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

Here, R denotes the Ricci scalar, and g^{uv} is the metric tensor used in the additional term. The entire left-hand side of the equation forms the Einstein tensor, G^{uv}, as in Equation (10). It's important to note that indices in these equations can be lowered or raised using the inverse metric through index contraction. As a result, subscripts may be used in some equations, depending on the conventions adopted in the literature.

According to the Big Bang theory, the universe is finite in age and space. Applying Equation (13) to cosmology initially suggested that the universe should eventually contract under the influence of its own gravity. However, modern astronomical observations indicate that the universe is not only expanding but doing so at an accelerating rate. To account for this expansion, Einstein introduced the cosmological constant term on the left-hand side of the equation, representing a form of energy that drives this expansion:

$$(14) \qquad R^{\mu\nu} - \frac{1}{2} R g^{\mu\nu} + \Lambda g^{\mu\nu} = \frac{8\pi G}{c^4} T^{\mu\nu}$$

Here, Λ denotes the cosmological constant (1.4657×10^{-52} m^{-2}). A positive vacuum energy density associated with the cosmological constant implies a negative pressure, which in turn drives the accelerated expansion of the universe, as observed in cosmological data. Because this energy has not been directly detected, it is commonly referred to as **dark energy**. Since the value of the cosmological constant is extremely small, its effects only become significant on cosmic scales. When Einstein's field equations are applied to smaller or local-scale phenomena, such as black holes, the influence of the cosmological constant is typically negligible.

Revision History

- 07/21/2025: Initial Post on Stanford Site
- 11/02/2025: Published on Zenodo
- 12/17/2025: Adding Links to Summaries of Related Articles

Links to Summaries of Related Articles

- https://cs.stanford.edu/people/zjl/abstract.html, PDF
- https://sites.google.com/view/zjl/abstracts, PDF
- https://xenon.stanford.edu/~zjl/abstract.html, PDF
- https://doi.org/10.5281/zenodo.17967154, PDF

Further Literature

- Misconceptions in Thermodynamics (PDF: DOI) (中文: DOI)
- The Mechanism Driving Crookes Radiometers (PDF: DOI) (中文: DOI)
- The Cause of Brownian Motion (PDF: DOI) (中文: DOI)
- Can Temperature Represent Average Kinetic Energy? (PDF: DOI) (中文: DOI)
- The Nature of Absolute Zero Temperature (PDF: DOI) (中文: DOI)
- The Triangle of Energy Transformation (PDF: DOI) (中文: DOI)
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- Superconductor Origin of Earth's Magnetic Field (PDF: DOI) (中文: DOI)
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- The Evolution from the Law of Gravitation to General Relativity (PDF: DOI) (中文: DOI)
- The Simplest Derivation of $E = mc^2$ (PDF: DOI) (中文: DOI)
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