Some notes on limits

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1 The standard definition

The formal ("delta-epsilon") definition of a limit is as follows:

Definition 1 *We say that*

$$\lim_{x \to c} f(x) = L$$

if and only if for all $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $0 < |x - c| < \delta$ *implies* $|f(x) - L| < \epsilon$.

The good thing about this definition is that it defines the limit in terms of the ordinary ideas of subtracting numbers and comparing them with $<$. It gets rid of the vague and imprecise idea of "approaching" or "getting close to" a value. The problem with this definition is that it is very confusing. To a large extent, it's confusing because it has many of what mathematicians call "quantifiers"; there is a "for all" and a "there exists" in it, and the second quantity (δ) depends on the first (ϵ). It's not necessarily easy to wrap your head around the relationship between ϵ and δ, and how they relate to the behavior of the function f.

In fact, it gets a little worse before it gets better. The above definition has an implicit "for all" that we can make explicit:

Definition 2 *We say that*

$$\lim_{x \to c} f(x) = L$$

if and only if for all $\epsilon > 0$, *there exists* $\delta > 0$ *such that for all* $x \neq c$ *in* $(c - \delta, c + \delta)$, $|f(x) - L| < \epsilon$.

To get a clearer picture of what this is actually saying, let's negate the definition — let's write out explicitly what it means for L not to be the limit of $f(x)$ at c.

Definition 3 *We have that*

$$\lim_{x \to c} f(x) \neq L$$

if and only if there exists $\epsilon > 0$ such that for all $\delta > 0$, there exists an $x \neq c$ in $(c - \delta, c + \delta)$ with $|f(x) - L| \geq \epsilon$.

All three of these definitions are exactly equivalent. Now, compare definition 2 to definition 3. When we negated the definition, every "for all" became a "there exists", every "there exists" became a "for all", and the innermost inequality flipped around: instead of saying that the difference between $f(x)$ and L was $< \epsilon$ (i.e., small), we said that it was $\geq \epsilon$ (i.e., large).

Why does negating the definition produce this effect? Consider that the negation of a claim, "for all x, x is like so" is "there exists an x such that x isn't like so". (Such an example of an x is called a *counterexample.*) Similarly, the negation of the claim "there exists an x like so" is "for all x, x isn't like so". Thus, since the standard definition of the limit looks like "for all ϵ ... there exists δ ... for all x ...", its negation looks like "there exists ϵ ... for all δ ... there exists x ...".

In some metaphorical sense, these quantifiers are "fighting". Where the positive definition makes a claim about all values of ϵ, the negative definition tries to "show it up" by finding one ϵ that violates the claim. Where the negative definition makes a claim about all values of δ, the positive definition tries to find a δ so that that's not true. There's a natural interpretation of this "conflict" as a two-player game.

2 The game formulation

Consider a game played between two people, Player and Hater. Player has chosen a function $f(x)$, a point c on the x-axis, and a value L, and is insisting violently that $\lim_{x \to c} f(x) = L$. Hater, seized by an equally violent fit of disbelief, insists that $\lim_{x \to c} f(x) \neq L$. So they decide to play a game:

1. Hater chooses an $\epsilon > 0$.

2. Player looks at the ϵ and chooses a $\delta > 0$.

3. Hater looks at the δ and chooses a point x such that $0 < |x - c| < \delta$, i.e., an $x \neq c$ in $(c - \delta, c + \delta)$.

4. Now, the value $|f(x) - L|$ is checked. If $|f(x) - L| < \epsilon$, Player wins. If $|f(x) - L| \geq \epsilon$, Hater wins.

Claim 1 *If $\lim_{x\to c} f(x) = L$, then if Player plays cleverly, Player will win the game and Hater will lose. If $\lim_{x\to c} f(x) \neq L$, then if Hater plays cleverly, Hater will win and Player will lose.*

Why is this true? Essentially, every choice made by Player corresponds to a "there exists" statement in the positive definition of a limit, and every choice made by Hater corresponds to a "there exists" statement in the negative definition. The moves made by Player and Hater are their opportunities to "make true" their version of the statement: positive for Player, negative for Hater.

Note that we did not say that Player or Hater will *always* win the game if their condition is true — only that they will if they play correctly. It's like Tic-Tac-Toe; if two Tic-Tac-Toe players play intelligently, the game will end in a draw, but it's certainly possible for either player to make a mistake and lose the game. Similarly, even if $\lim_{x\to c} f(x) = L$, Player can make a clumsy choice of δ and get caught out by Hater — or vice versa.

Now, some examples.

3 Example: Glory to the Hater!

Let's concretize a little. Consider the function:

$$f(x) = \begin{cases} 1 & : x \neq 0 \\ 3 & : x = 0 \end{cases}$$

We can see that $\lim_{x\to 0} f(x) = 1$; the fact that $f(0) = 3$ is just an isolated "removable discontinuity". Let's prove that $\lim_{x\to 0} f(x) \neq 3$.

Apply the negative definition of a limit. We need an $\epsilon > 0$ such that for any $\delta > 0$, the interval $(0 - \delta, 0 + \delta)$ will contain $x \neq 0$ such that $f(x) > \epsilon$. Note that since we are using the negative definition, or under the game interpretation, since we are being the Hater, we only have to pick one particular fixed value of ϵ.

Intuitively, our hope of doing this rests on the fact that there is always a gap of size 2 between the claimed value of the limit (3) and the actual values attained by the function (1). So let's choose $\epsilon = 1$ — half of that unbridgeable gap.

Now, consider any possible value of $\delta > 0$ (under the game interpretation, any move Player might make). Given δ, we need a nonzero x in $(-\delta, \delta)$ such that $|f(x) - 3| > 1$. So let's choose $x = \frac{\delta}{2}$; since Player had to choose a positive/nonzero δ, this value is also nonzero, and $f(x) = 1$.

Now, $|f(x) - L| = |1 - 3| = 2 > \epsilon$, so the negative definition is satisfied and we've shown that $\lim_{x\to 0} f(x) \neq 3$! And under the game interpretation, we've provided a plan that Hater can follow in order to always win the game.

Note that the success of our proof hinged on choosing the right ϵ. If we had chosen $\epsilon = 4$ (i.e., if Hater had made a bad move), then Player could have chosen $\delta = 1$ and won the game. Again, the important thing is not who wins the game for a particular sequence of moves; the important thing is who has a *winning strategy* for the game.

4 Example: a "delta-epsilon proof"

The kind of problem commonly called a "delta-epsilon proof" is of the form: show, using the formal definition of a limit, that $\lim_{x\to c} f(x) = L$ for some c, f, L. Conceptually, your task in such a proof is to step into Player's shoes: given that Hater can throw any $\epsilon > 0$ at you, you need to find a scheme for turning that ϵ into a $\delta > 0$.

Typically, these proofs have three steps:

1. An informal "fishing expedition", where you assume that $|x - c| < \delta$ for some δ, then use that fact to manipulate $|f(x) - L|$ and compute a bound on it in terms of δ.

2. Use the results of the fishing expedition to write down a scheme for turning ϵ into δ.

3. Check that your scheme actually works, i.e., that if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. The reasoning in this step will closely resemble the reasoning in the fishing expedition (typically, both will include a long

chain of inequalities), but this time δ has actually been firmly chosen in terms of ϵ and the end result will be a formal proof that f, c, and L satisfy the definition of a limit.

Let's see an example (section 2.4, exercise 36). We will show that $\lim_{x\to 2} \frac{1}{x} = \frac{1}{2}$.

4.1 The fishing expedition

Consider x near 2, i.e., with $|x-2| < \delta$ for some small δ, and consider the values of $|f(x) - \frac{1}{2}|$. We can reason as follows:

$$|f(x) - \frac{1}{2}| = |\frac{1}{x} - \frac{1}{2}| = |\frac{2-x}{2x}| = \frac{|2-x|}{|2x|}$$

We want to get an upper bound on this quantity. On top, we have $|2-x| = |x-2| < \delta$, so that's fine. If we can get a lower bound on the bottom, then we can get an upper bound on the whole thing. But x is at the very least $2-\delta$. Peeking ahead in our argument, we'll be able to choose δ as small as we like, so let's say $\delta \leq 1$. So $x \geq 1$, implying $2x \geq 2$, implying $|2x| \geq 2$, implying:

$$|f(x) - \frac{1}{2}| < \frac{|2-x|}{|2x|} < \frac{\delta}{|2x|} \leq \frac{\delta}{2}$$

4.2 The scheme

Assume we have some fixed value $\epsilon > 0$. We know that if $|x-2| < \delta$, $|f(x) - \frac{1}{2}| < \frac{\delta}{2}$. So intuitively, set $\epsilon = \frac{\delta}{2}$ and solve for δ; the result is $\delta = 2\epsilon$. But during our fishing expedition, we assumed that $\delta \leq 1$, so let's make sure of that and set $\delta = \min(2\epsilon, 1)$.

4.3 The final proof

Take any $\epsilon > 0$. Choose $\delta = \min(2\epsilon, 1)$; we claim that for $0 < |x-2| < \delta$, $|f(x) - \frac{1}{2}| < \epsilon$. To see this, observe that:

$$|f(x) - \frac{1}{2}| = |\frac{1}{x} - \frac{1}{2}| = |\frac{2-x}{2x}| = \frac{|2-x|}{|2x|} < \frac{\delta}{|2x|}$$

where x is in the interval $(2-\delta, 2+\delta)$, so the minimum possible value of x is $2-1=1$ and the minimum possible value of the denominator is 2. So we may conclude:

$$|f(x)-\frac{1}{2}| < \frac{\delta}{2} \leq \frac{2\epsilon}{2} = \epsilon$$

This concludes the proof. But it wouldn't be fair to end without a nod to our friends Player and Hater. Under the game interpretation, what we really did in step 2 was construct a winning strategy for Player. No matter what ϵ Hater gives Player, Player replies with $\delta = \min(2\epsilon, 1)$. Then, whatever x Hater chooses in $(2-\delta, 2+\delta)$, our inequalities in step 3 show that $|f(x)-\frac{1}{2}| < \epsilon$ and Player wins.