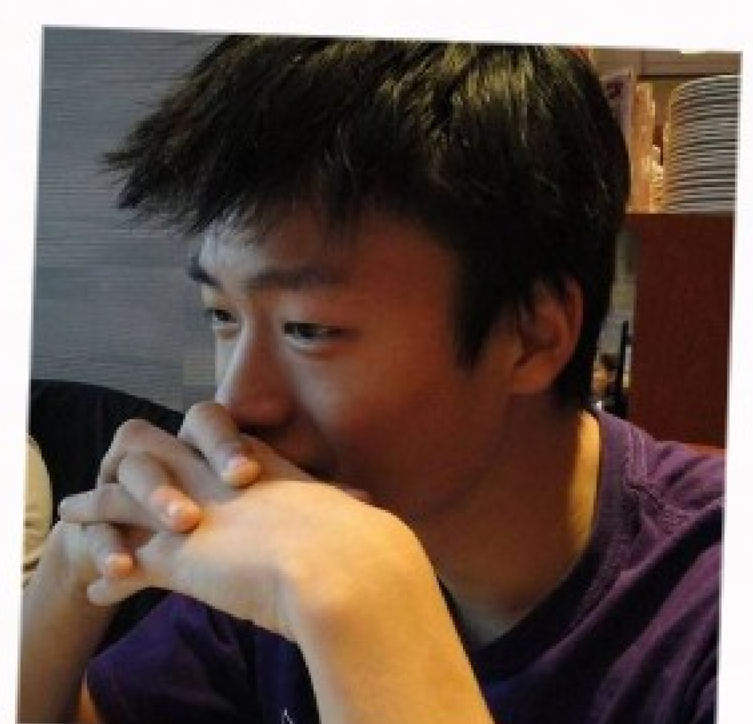
Improved Algorithms for Edge Colouring in the W-Streaming Model

Paul Liu



Moses Charikar

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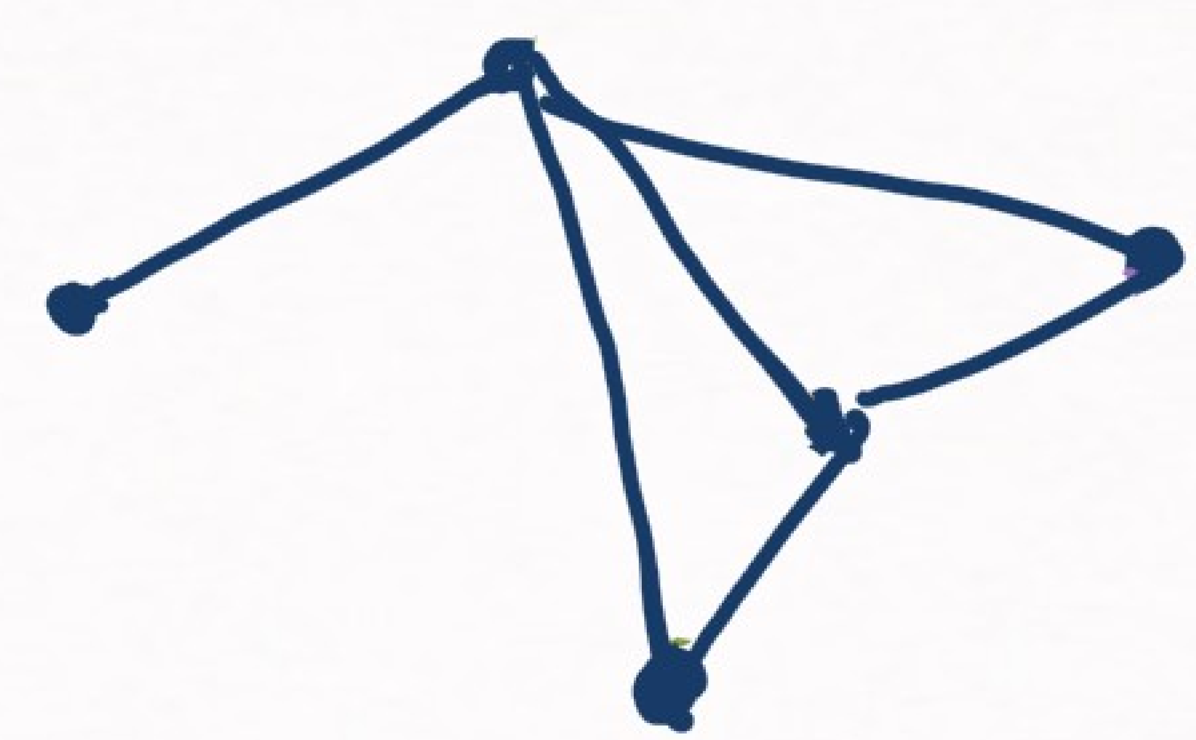

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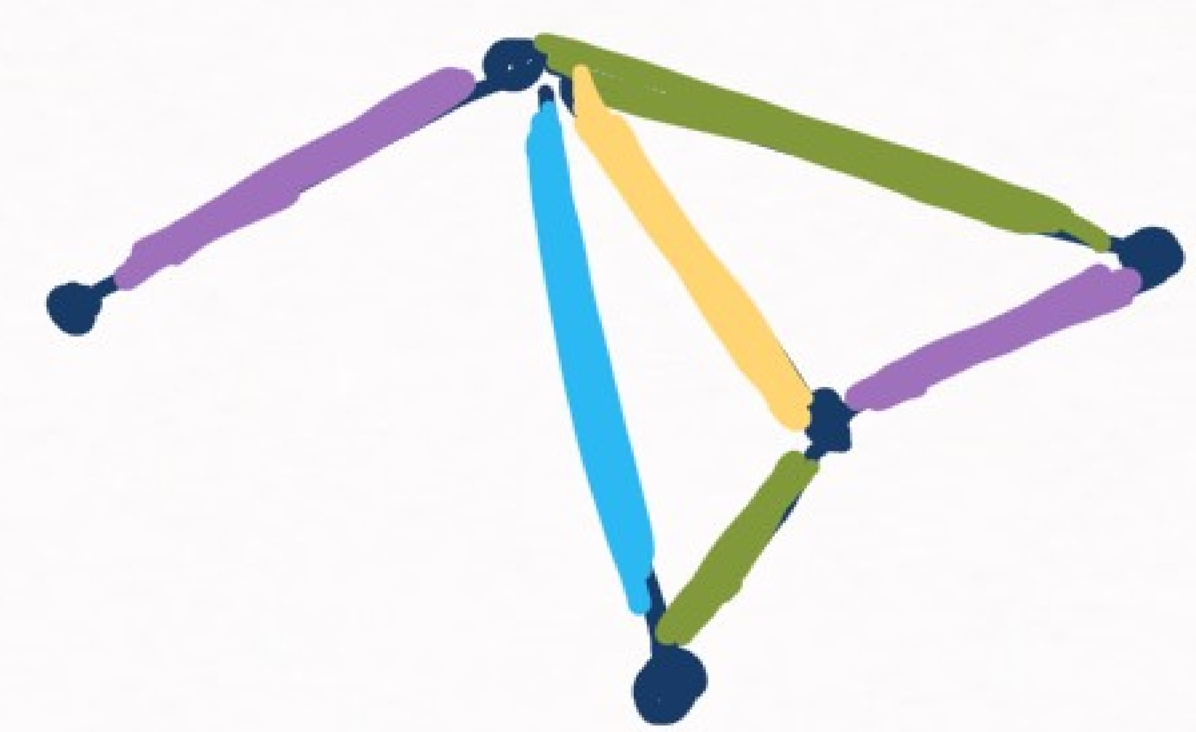

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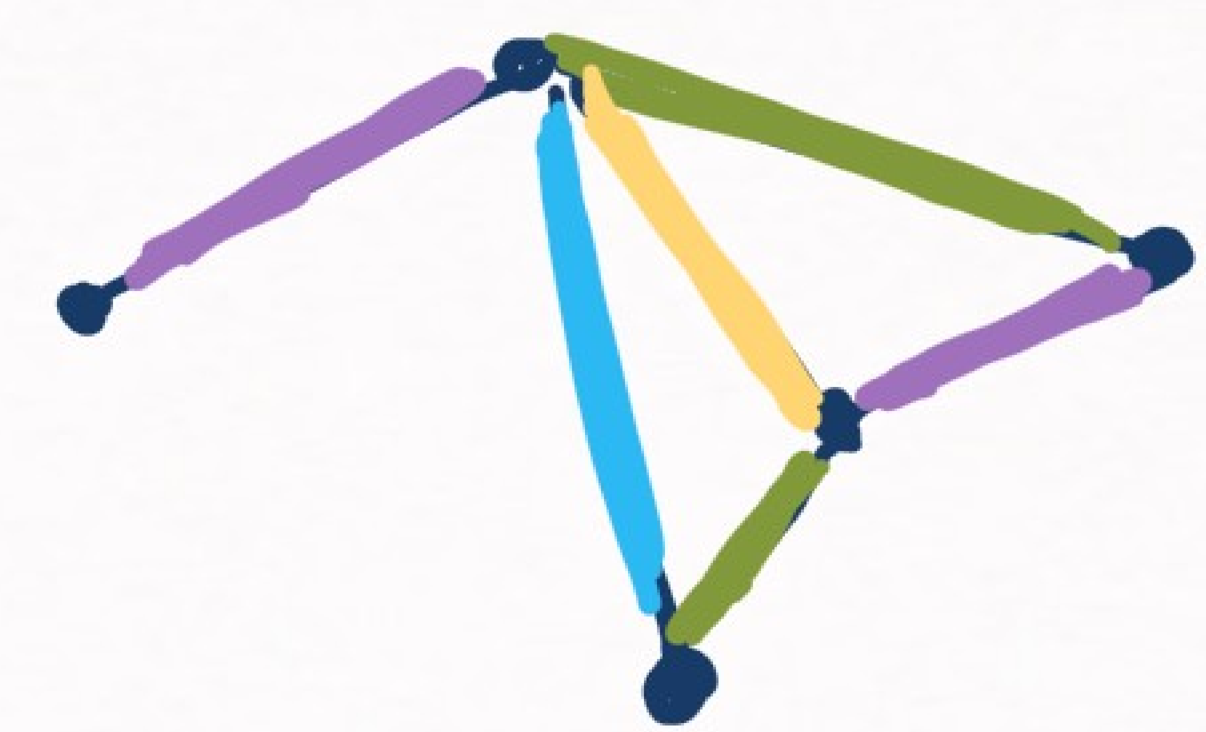
- $\Delta+1$ always possible! (Vizing) (Δ = max deg.)
- Δ required; NP-Complete to determine Δ vs. $\Delta+1$



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- Up to $O(n^2)$ edges streamed in
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- $O(n \operatorname{polylog} n)$ memory for alg.
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- Two regimes: vertex arrival (all edges adj. to a vertex arrive at once)

  edge-arrival (edges arrive one at a time)

our regime

Related Work: Online Model

- When $\Delta = \omega(\log n)$, $(1+o(1))\Delta$ is possible:
  - Adversarial vertex arrivals [Cohen, Peng, Wajc '19]
  - Random edge arrivals [Bhattacharya, Grandoni, Wajc '20]
    SODA 21

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  - All of these algs use too much memory for W-streaming

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[Our Results]	– $(1+o(1))\Delta$ colours (uploaded talk; ★animated★ slides!)	– $(1+o(1))\frac{\Delta^2}{s}$ colours * in $\tilde{O}(ns)$ space (all other algs use $\tilde{O}(n)$ space) Today's talk

A simple algorithm for adversarial Arrivals

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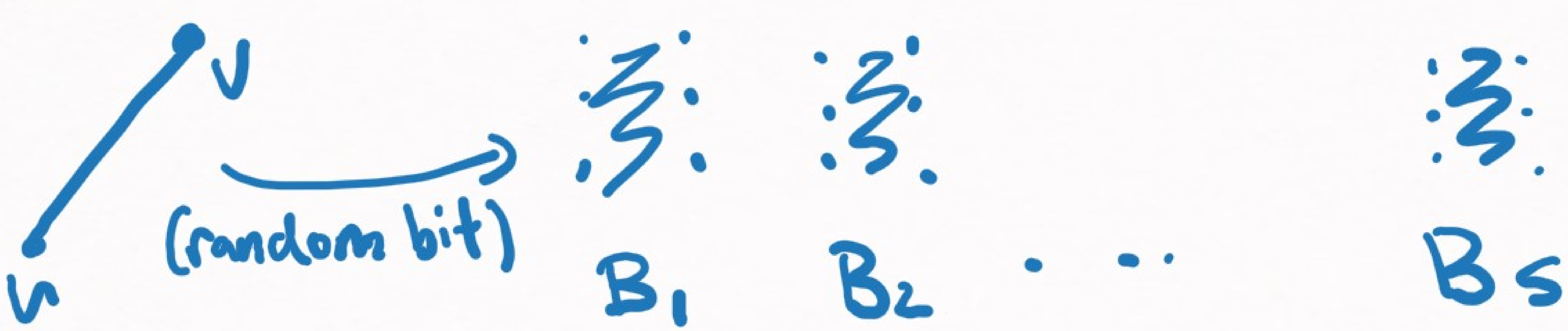

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- Choice of $36 \log n$ ensures at least one mismatched bit

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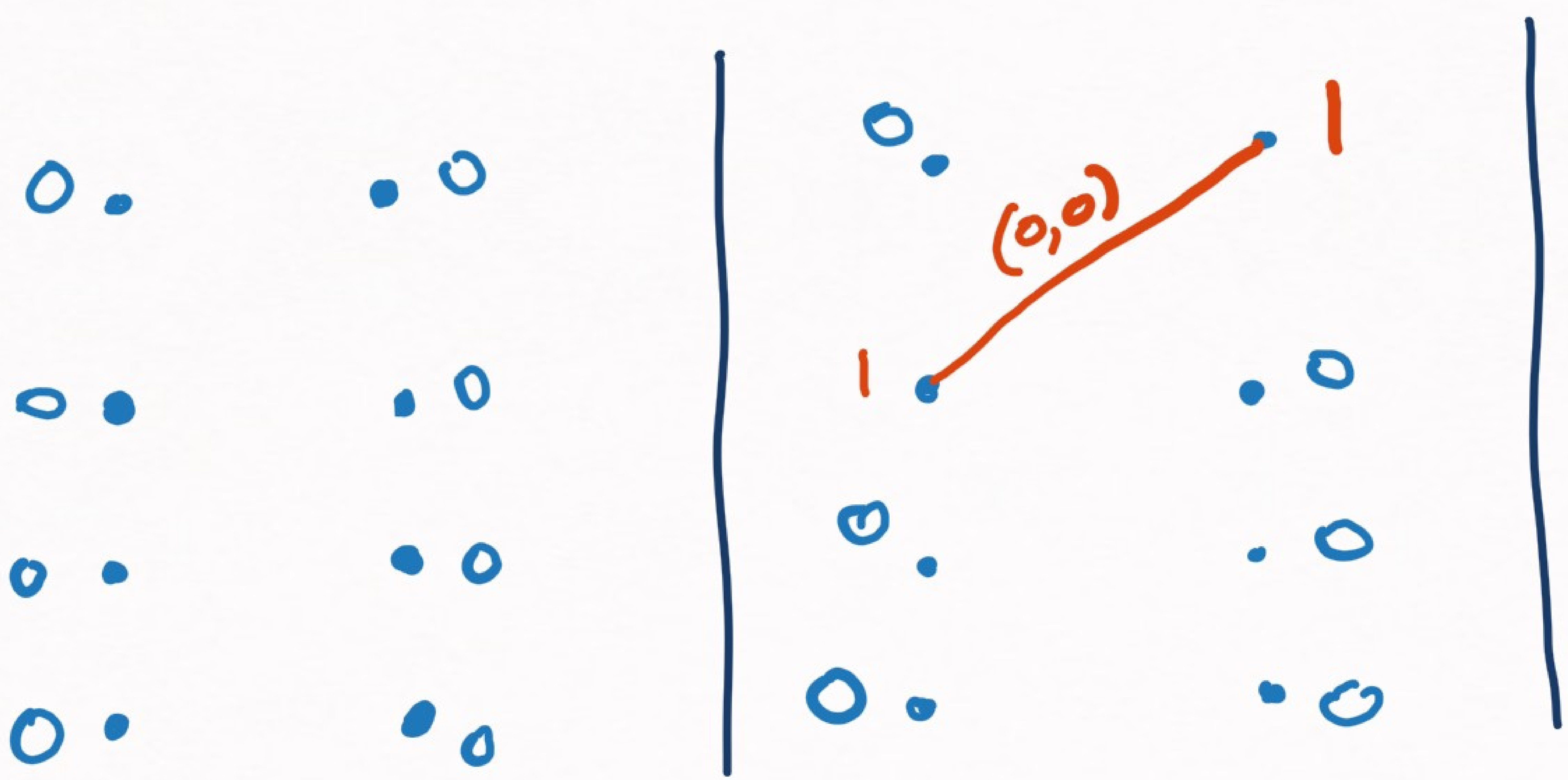

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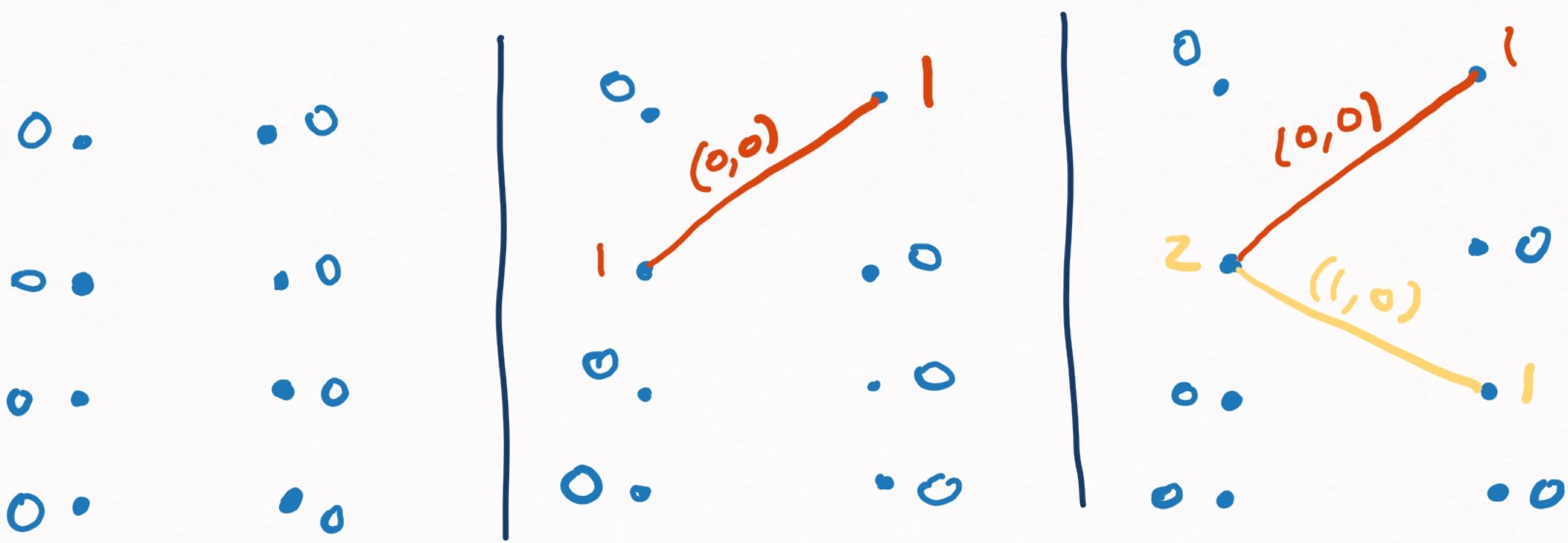

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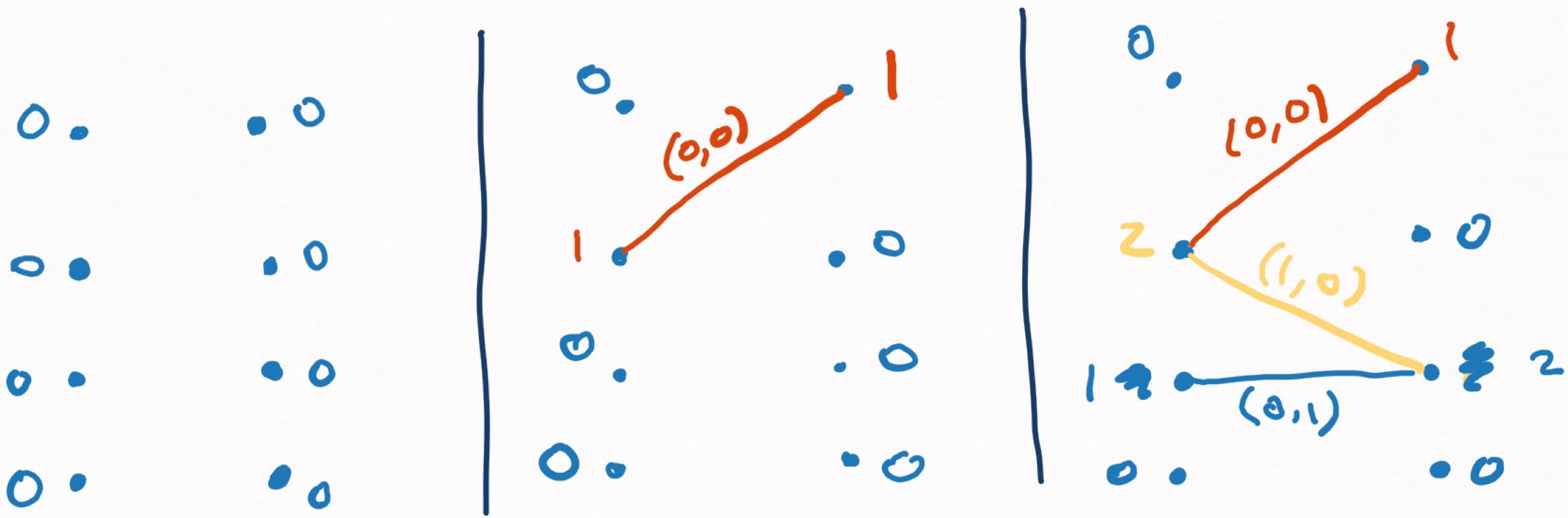

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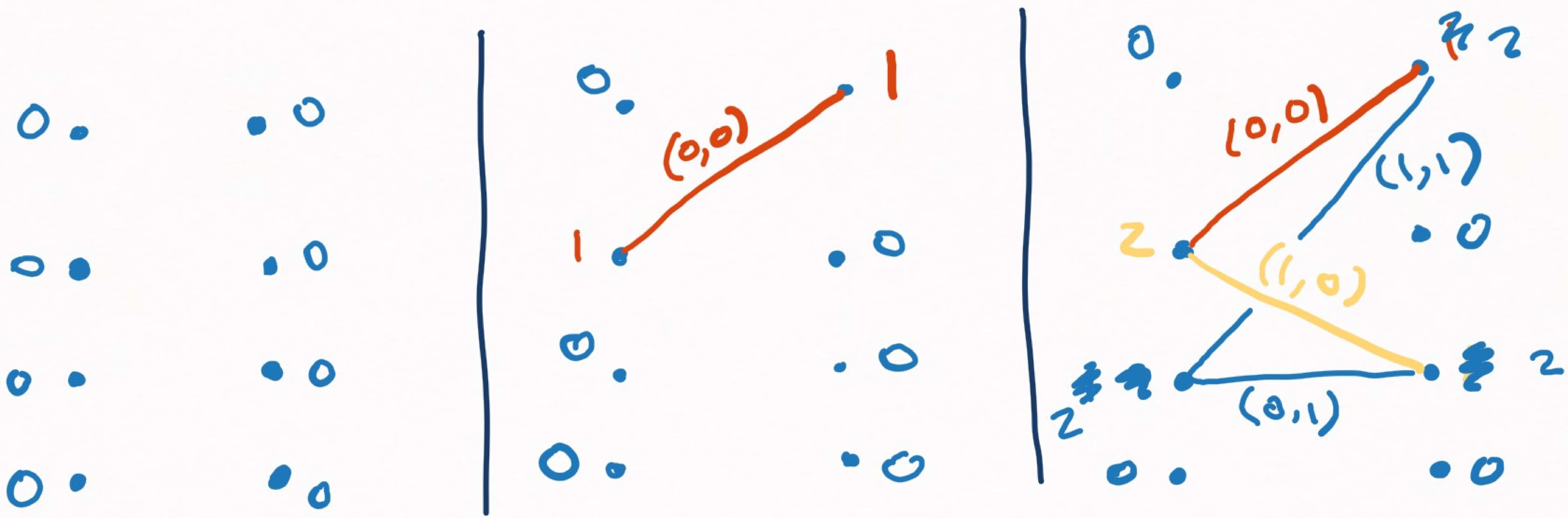

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- Worst case: each counter goes up to max deg B_i ($\simeq \Delta/s$)

  $\hookrightarrow$ (c_u, c_v) has $\left(\frac{\Delta}{s}\right)^2$ possibilities.

- Achievable:

Works for any $p, q \leq \max \deg B_i \simeq \frac{\Delta}{s}$.

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Wrapping Up

$$\text{Total \# colours} : \quad \underbrace{S}_{\text{\# of } B_i} \cdot \Big(\underbrace{(1+o(1))\frac{\Delta}{S}}_{\text{max deg per } B_i} \Big)^{2}$$

colours used by alg.

$$= \frac{\Delta^2}{S}\,(1+o(1)).$$

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(Worst case can be achieved by streaming poly(Δs) copies of star-shaped graphs


)

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THANK YOU!